HEIGHT BOUND AND PREPERIODIC POINTS
FOR JOINTLY REGULAR FAMILIES OF RATIONAL MAPS

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ABSTRACT. Silverman [14] proved a height inequality for a jointly regular family of rational maps and the author [10] improved it for a jointly regular pair. In this paper, we provide the same improvement for a jointly regular family: let \( h : \mathbb{P}^n \rightarrow \mathbb{R} \) be the logarithmic absolute height on the projective space, let \( r(f) \) be the \( D \)-ratio of a rational map \( f \) which is defined in [10] and let \( \{f_1, \ldots, f_k \mid f_i : \mathbb{A}^n \rightarrow \mathbb{A}^n \} \) be a finite set of polynomial maps which is defined over a number field \( K \). If the intersection of the indeterminacy loci of \( f_1, \ldots, f_k \) is empty, then there is a constant \( C \) such that

\[
\sum_{i=1}^{k} \frac{1}{\deg f_i} h(f_i(P)) > \left(1 + \frac{1}{r}\right) f(P) - C \quad \text{for all } P \in \mathbb{A}^n
\]

where \( r = \max_{i=1,\ldots,k} (r(f_i)) \).

1. Introduction

Let \( K \) be a number field and let \( h : \mathbb{P}^n_K \rightarrow \mathbb{R} \) be the logarithmic absolute height function on the projective space. If \( f : \mathbb{P}^n_K \rightarrow \mathbb{P}^n_K \) is a morphism defined over a number field \( K \), then we can make a good estimate of \( h(P) \) with \( h(f(P)) \). Define the degree of \( f \) to be the number induced by the linear operator \( f^* \) on \( \text{Pic}(\mathbb{P}^n) = \mathbb{Z} \):

\[
f^*H = \deg f \cdot H \quad \text{on} \quad \text{Pic}(\mathbb{P}^n).
\]

Then, the functorial property of the Weil height machine will prove the Northcott’s theorem. The author refers [16, Theorem B.3.2] to the reader for the details of the Weil height machine.

**Theorem 1.1** (Northcott [12]). If \( f : \mathbb{P}^n_K \rightarrow \mathbb{P}^n_K \) is a morphism defined over a number field \( K \), then there are two constants \( C_1 \) and \( C_2 \), which are independent of point \( P \), such that

\[
\frac{1}{\deg f} h(f(P)) + C_1 > h(P) > \frac{1}{\deg f} h(f(P)) - C_2
\]

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for all \( P \in \mathbb{P}_K^n \).

If \( f \) is not a morphism but a rational map, then the functoriality of the Weil height machine breaks down: the two height functions \( h_{\mathcal{T}}(P) \) and \( h_H(f(P)) \) are not equivalent. Hence, Northcott’s Theorem is not valid for rational maps (However, we still have \( h(P) > \frac{1}{\deg f} h(f(P)) - C_2 \) by the triangular inequality. See [16, Proposition B.7.1]).

For example, consider a polynomial map, one of popular objects in complex dynamics. Define
\[
f := (f_1, \ldots, f_n) : \mathbb{A}^n_K \to \mathbb{A}^n_K,
\]
where \( f_1, \ldots, f_n \) are homogeneous polynomials of degree \( d \). We may consider \( f \) as a rational map on \( \mathbb{P}_K^n \): let \( P \in \mathbb{P}_K^n \). We define \( f(P) \) to be the following limit value if it exists.
\[
f(P) := \lim_{Q \to P} f(Q).
\]
We call it the meromorphic extension of \( f \). In general, the meromorphic extension of \( f \) is not a morphism. So, we need other way to find an upper bound of \( h(P) \).

Silverman suggested a way of constructing an upper bound of \( h(P) \) when we have a special family of polynomial maps.

**Definition 1.2.** Let \( S = \{f_1, \ldots, f_k \mid f_i : \mathbb{P}_K^n \dashrightarrow \mathbb{P}_K^n \} \) be a finite set of rational maps defined over a number field \( K \) and let \( I(f) \) be the indeterminacy locus of \( f \). We say that \( S \) is *jointly regular* when
\[
\bigcap_{i=1}^k I(f_i) = \emptyset.
\]
We also say that a finite set of polynomial maps \( S' = \{g_1, \ldots, g_k \mid g_i : \mathbb{A}_K^n \to \mathbb{A}_K^n \} \) is *jointly regular* if the set of rational maps
\[
S = \{f_i : \mathbb{P}_K^n \dashrightarrow \mathbb{P}_K^n \mid f_i \text{ is the meromorphic extension of } g_i \in S' \}
\]
is jointly regular.

**Theorem 1.3** ([14], Theorem 3). Let \( \{f_1, \ldots, f_k \mid f_i : \mathbb{A}_K^n \to \mathbb{A}_K^n \} \) be a jointly regular family of polynomial maps defined over a number field \( K \). Then, there is a constant \( C \) satisfying
\[
\sum_{i=1}^k \frac{1}{\deg f_i} h(f_i(P)) > h(P) - C
\]
for all \( P \in \mathbb{A}_K^n \).

In this paper, we will improve Theorem 1.3 using the \( D \)-ratio. The \( D \)-ratio requires new concepts to be defined so that we will state the main theorem without the definition of the \( D \)-ratio first and will introduce the \( D \)-ratio in Definition 2.12 later.
Theorem 1.4. Let $H$ be a hyperplane of $\mathbb{P}^n$, let $\mathbb{A}^n = \mathbb{P}^n \setminus H$, let $S = \{f_1, \ldots, f_k \mid f_i : \mathbb{A}^n_K \to \mathbb{A}^n_K\}$ be a jointly regular family of polynomial maps defined over a number field $K$, let $r(f)$ be the $D$-ratio of $f$ and let $r = \max_{i=1, \ldots, k}(r(f_i))$. Suppose that $S$ has at least two elements. Then, there is a constant $C$ satisfying
\[
\sum_{l=1}^{k} \frac{1}{\deg f_l} h(f_l(P)) > \left(1 + \frac{1}{r}\right) h(P) - C
\]
for all $P \in \mathbb{A}_K^n$.

This theorem improves Silverman’s result for preperiodic points [14, Theorem 4], which is exactly same with Theorem 1.5 except the description of $\delta_S$.

Theorem 1.5. Let $S = \{f_1, \ldots, f_k \mid f_i : \mathbb{A}^n_K \to \mathbb{A}^n_K\}$ be a jointly regular family of polynomial maps, let $f(f_l)$ be the $D$-ratio of $f_l$ and let $\Phi$ be the monoid of polynomial maps generated by $S$. Define
\[
\delta_S := \left(1 + \frac{1}{1/r}\right) \sum_{l=1}^{k} \frac{1}{\deg f_l},
\]
where $r = \max_{i=1, \ldots, k}(r(f_i))$.

If $\delta_S < 1$, then
\[
\text{Preper}(\Phi) := \bigcap_{f \in \Phi} \text{Preper}(f) \subset \mathbb{A}_K^n
\]
is a set of bounded height.

From now on, we will let $K$ be a number field, let $H$ be a hyperplane on $\mathbb{P}^n$ and let $\mathbb{A}^n = \mathbb{P}^n \setminus H$ be an affine space. We also let $f : \mathbb{A}^n \to \mathbb{A}^n$ be a polynomial map and let $I(f)$ be the indeterminacy locus of $f$ unless stated otherwise.

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2. Preliminaries

We need two main ingredients, the resolution of indeterminacy and the $D$-ratio of polynomial maps. For details, the author refers [1] and [3, II.7] for blowups and the resolution of indeterminacy, and [10] for the $D$-ratio.

2.1. Blowup and the resolution of indeterminacy

We have the general theorem of the resolution of indeterminacy, which is a corollary of the theorem of the resolution of singularity.
Theorem 2.1 (Resolution of indeterminacy). Let $f : V \to W$ be a rational map between proper varieties such that $V$ is nonsingular. Then there is a proper nonsingular variety $\hat{V}$ with a birational morphism $\pi : \hat{V} \to V$ such that $\phi = f \circ \pi : \hat{V} \to W$ is a morphism:

$$
\begin{array}{ccc}
V & \xrightarrow{\pi} & \hat{V} \\
\phi \downarrow & & \downarrow \\
V & = & W
\end{array}
$$

For notational convenience, we will define the following.

**Definition 2.2.** Let $f : \mathbb{P}^n \to \mathbb{P}^n$ be a rational map and let $V$ be a blowup of $\mathbb{P}^n$ with a birational morphism $\pi : V \to \mathbb{P}^n$. We say that a pair $(V, \pi)$ is a resolution of indeterminacy of $f$ if $f \circ \pi : V \to \mathbb{P}^n$ is extended to a morphism. And we call the extended morphism $\phi := f \circ \pi$ a resolved morphism of $f$.

Using Hironaka’s Theorem (Theorem 2.5), we will observe the relation between the resolution of indeterminacy and the indeterminacy locus of $f$.

**Definition 2.3.** Let $\pi : V \to \mathbb{P}^n$ be a birational morphism. Then, we say that a closed subscheme $\mathfrak{I}$ of $\mathbb{P}^n$ is the center scheme of $\pi$ if the ideal sheaf $\mathcal{S}$ corresponding to $\mathfrak{I}$ generates $V$:

$$V = \text{Proj} \left( \bigoplus_{d \geq 0} \mathcal{S}^d \right).$$

**Definition 2.4.** Let $\pi : W \to V$ be a birational morphism. We say that $\pi$ is a monoidal transformation if its center scheme is a smooth irreducible subvariety of $V$. We say that $W$ is a successive blowup of $V$ if the corresponding birational map $\pi : W \to V$ is a composition of monoidal transformations.

**Theorem 2.5** (Hironaka). Let $f : X \to Y$ be a rational map between proper varieties such that $V$ is nonsingular. Then, there is a finite sequence of proper varieties $X_0, \ldots, X_r$ such that

1. $X_0 = X$,
2. $\rho_i : X_i \to X_{i-1}$ is a monoidal transformation,
3. If $T_i$ is the center scheme of $\pi_i$, then $\rho_0 \circ \cdots \circ \rho_i(T_i) \subset I(f)$ on $X$,
4. $f$ is extended to a morphism $\tilde{f} : X_r \to Y$ on $X_r$,
5. Consider the composition of all monoidal transformation $\rho : X_m \to X$. Then, the underlying subvariety of the center scheme $T$ of $\rho$, a subvariety made by the zero set of the ideal sheaf corresponding to $T$, is exactly $I(f)$.
Proof. See [4, Question (E) and Main Theorem II]. \qed

In \S \ref{sec:2.2}, we will find a basis of $\text{Pic}(V)$ when $(V, \pi)$ is a resolution of indeterminacy. Especially, we need a basis consisting of irreducible divisors. However, pullbacks of divisors may not be irreducible because of the exceptional part. So, we define the proper transformation, which is usually irreducible.

**Definition 2.6.** Let $\pi : \tilde{V} \to V$ be a birational morphism with center scheme $\mathcal{I}$ and let $D$ be an irreducible divisor on $V$. We define the proper transformation of $D$ by $\pi$ to be

$$\pi^\#D = \pi^{-1}(D \cap U),$$

where $U = V \setminus Z(\mathcal{I})$ and $Z(\mathcal{I})$ is the underlying subvariety made by the zero set of the ideal corresponding to $\mathcal{I}$.

**2.2. The $\mathbb{A}^n$-effectiveness and the $D$-ratio**

The main question of this paper is to find an upper bound of $\height(P)$ using $h(f_l(P))$ for jointly regular family $f_1; \ldots; f_k\colon \mathbb{A}^n \to \mathbb{A}^n$. So, we will consider $\mathbb{A}^n$ as a dense open subset of $\mathbb{P}^n$ and fix the hyperplane $H = \mathbb{P}^n \setminus \mathbb{A}^n$ to find a basis of the Picard group of a blowup of $\mathbb{P}^n$ and use a special kind of divisors on a blowup of $\mathbb{P}^n$ to measure the height values of $P \in \mathbb{A}^n$. First of all, we need to clarify how to get such basis of $\text{Pic}(V)$.

**Proposition 2.7.** Let $V$ be a successive blowup of $\mathbb{P}^n$ with a birational morphism $\pi : V \to \mathbb{P}^n$; there are monoidal transformations $\pi_i : V_i \to V_{i-1}$ such that $V_r = V$ and $V_0 = \mathbb{P}^n$. Let $H$ be a hyperplane on $\mathbb{P}^n$, let $F_i$ be the exceptional divisor of the blowup $\pi_i : V_i \to V_{i-1}$, let $\rho_i = \pi_{i+1} \circ \cdots \circ \pi_r$ and let $E_i = \rho_i^\#F_i$. Then, $\text{Pic}(V)$ is a free $\mathbb{Z}$-module with a basis

$$\{H_V = \pi^\#H, E_1, \ldots, E_r\}.$$

Proof. [3, Exer.II.7.9] shows that

$$\text{Pic}(\tilde{X}) \simeq \text{Pic}(X) \oplus \mathbb{Z}$$

if $\pi : \tilde{X} \to X$ is a monoidal transformation. More precisely,

$$\text{Pic}(\tilde{X}) = \{\pi^\#D + nE \mid D \in \text{Pic}(X)\},$$

where $E$ is the exceptional divisor of $\pi$ on $\tilde{X}$. Suppose that $X = V_{i-1}$ and $\tilde{X} = V_i$ and get the desired result. \qed

Now, we define the special kind of divisors, the $\mathbb{A}^n$-effective divisors.

**Definition 2.8.** Let $V$ be a successive blowup of $\mathbb{P}^n$ with a birational morphism $\pi : V \to \mathbb{P}^n$, let $H$ be a fixed hyperplane of $\mathbb{P}^n$ and let $\text{Pic}_Q(V) = \mathbb{Q}H_V \oplus \mathbb{Q}E_1 \oplus \cdots \oplus \mathbb{Q}E_r$

with the basis described in Proposition 2.7. We define the $\mathbb{A}^n$-effective cone to be

$$\text{AFE}(V) := \mathbb{Q}^{\geq 0}H_V \oplus \mathbb{Q}^{\geq 0}E_1 \oplus \cdots \oplus \mathbb{Q}^{\geq 0}E_r.$$
where $\mathbb{Q} \geq 0$ is the set of nonnegative rational numbers. We say a divisor $D$ of $V$ is $\mathbb{A}^n$-effective if the linear equivalence class of $D$ is contained in $\text{AFE}(V)$ and denote it by 

$$ D \succ 0. $$

Moreover, on $\text{Pic}_Q(V)$, we write

$$ D_1 \succ D_2 $$

if $D_1 - D_2$ is $\mathbb{A}^n$-effective.

The next proposition will explain why we define the “$\mathbb{A}^n$-effectiveness”. Namely, the height functions corresponding to $\mathbb{A}^n$-effective divisors will have nice properties on $\mathbb{A}^n$.

**Proposition 2.9.** Let $V$ be a successive blowup of $\mathbb{P}^n$ with a birational morphism $\pi : V \to \mathbb{P}^n$ and let $D, D_i$ be divisors on $V$.

1. (Effectiveness) If $D$ is $\mathbb{A}^n$-effective, then $D$ is effective.
2. (Boundedness) If $D$ is $\mathbb{A}^n$-effective, then $h_D(P)$ is bounded below on $V \setminus (H_V \cup (\bigcup_{i=1}^r E_i))$.
3. (Transitivity) If $D_1 \succ D_2$ and $D_2 \succ D_3$, then $D_1 \succ D_3$.
4. (Funtoriality) If $\rho : W \to V$ is a monoidal transformation and $D_1 \succ D_2$, then $\rho^*D_1 \succ \rho^*D_2$.

**Proof.** See [10, Proposition 3.3].

In Section 1, we introduce the main theorem without the definition of the $D$-ratio because it requires the $\mathbb{A}^n$-effectiveness. Now, we are ready to define the $D$-ratio, one of main ingredients of this paper.

**Definition 2.10.** Let $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a rational map such that $I(f) \subset H$, let $(V, \pi_V)$ be a resolution of indeterminacy of $f$ and let $\phi_V$ be a resolved morphism so that the following diagram commutes:

$$
\begin{array}{ccc}
V & \xrightarrow{\phi_V} & \mathbb{P}^n \\
\pi_V \downarrow & & \downarrow f \\
\mathbb{P}^n & \xrightarrow{f} & \mathbb{P}^n
\end{array}
$$

Suppose that

$$
\pi_V^*H = a_0H_V + \sum_{i=1}^r a_iE_i \quad \text{and} \quad \phi_V^*H = b_0H_V + \sum_{i=1}^r b_iE_i,
$$

where $a_i, b_i$ are nonnegative integers. If $b_i \neq 0$ for all $i$ satisfying $a_i \neq 0$, we define the $D$-ratio of $\phi_V$ to be

$$
r(\phi_V) = \deg \phi_V \cdot \max_{a_i \neq 0} \left( \frac{a_i}{b_i} \right). $$
If there is an index $i$ satisfying $a_i \neq 0$ and $b_i = 0$, define

$$r(\phi_V) = \infty.$$  

The readers might concern if the $D$-ratio is only defined for resolved morphisms. The following lemma will allow us to define the $D$-ratio for the rational maps.

**Lemma 2.11.** Let $(V, \pi_V)$ and $(W, \pi_V)$ be resolutions of indeterminacy of $f$ with resolved morphisms $\phi_V = f \circ \pi_V$ and $\phi_W = f \circ \pi_W$ respectively:

$$\begin{array}{c}
\xymatrix{ W 
\ar[r]_{\phi_W} & V 
\ar[l]_{\phi_V} \\
\mathbb{P}^n 
\ar[r]_{f} & \mathbb{P}^n
} \end{array}$$

Then, we have

$$r(\phi_V) = r(\phi_W).$$

**Proof.** See [10, Lemma 4.3].

**Definition 2.12.** Let $f : \mathbb{P}^n \rightarrow \mathbb{P}^n$ be a rational map with $I(f) \subset H$. Then, we define the $D$-ratio of $f$ to be

$$r(f) = r(\phi_V)$$

for any resolution of indeterminacy $(V, \pi_V)$ of $f$ with resolved morphism $\phi_V$.

**Proposition 2.13.** Let $f, g : \mathbb{P}^n \rightarrow \mathbb{P}^n$ be rational maps such that $I(f), I(g) \subset H$. Then,

1. $r(f) = 1$ if and only if $f$ is a morphism.
2. $r(f) \in [1, \infty]$.
3. $\frac{r(f)}{\deg f} \cdot \frac{r(g)}{\deg g} \geq \frac{r(g \circ f)}{\deg (g \circ f)}$.

**Proof.** See [10, Proposition 4.5, Theorem 5.2].

**Example 2.14.** Let $f : \mathbb{A}^n \rightarrow \mathbb{A}^n$ be a polynomial automorphism with the inverse map $f^{-1} : \mathbb{A}^n \rightarrow \mathbb{A}^n$. Then, $r(f) = \deg f \times \deg f^{-1}$ (For details, see [9]). For example, a Hénon map

$$f_H(x, y, z) = (z, x + z^2, y + x^2)$$

is a regular polynomial automorphism with the inverse map

$$f_H^{-1}(x, y, z) = (y - x^2, z - (y - x^2)^2, x).$$

Thus,

$$r(f_H) = r(f_H^{-1}) = \deg f_H \times \deg f_H^{-1} = 2 \times 4 = 8.$$
Example 2.15. Let $f[x, y, z] = [x^2, yz, z^2]$. Then, the indeterminacy locus of $f$ consists of one point $P = [0, 1, 0]$. Then, the blowup $V$ along closed scheme corresponding ideal sheaf $(z, x^2)$ will resolve indeterminacy, which is a successive blowup along $P$ and $H^\# \cap F_1$, where $F_1$ is the exceptional divisor of the first blowup.

Let $E_1$ be the proper transformation of $F_1$ and let $E_2$ be the exceptional divisor of the second blowup:

$$
\begin{array}{c}
H^\#
\end{array}
\leftarrow
\begin{array}{c}
E_1
\end{array}
\begin{array}{c}
H^\#
\end{array}
\begin{array}{c}
F_1
\end{array}
\begin{array}{c}
E_2
\end{array}
$$

Then, the following intersection numbers are easily calculated:

$E_2^2 = -1, E_1^2 = -2, (H^\#)^2 = -1, H^\# \cdot E_1 = 0$ and $H^\# \cdot E_2 = E_1 \cdot E_2 = 1$.

Furthermore, by the projection formula and the exact calculation of $\phi_*$, we get

$$
\begin{align*}
H^\# \cdot \phi^* H &= \phi_* H^\# \cdot H = 0,
E_1 \cdot \phi^* H &= \phi_* E_1 \cdot H = 0,
E_2 \cdot \phi^* H &= \phi_* E_2 \cdot H = 1.
\end{align*}
$$

Since $\text{Pic}(V) = \langle H^\#, E_1, E_2 \rangle$, we may assume that

$$
\phi^* H = aH^\# + bE_1 + cE_2
$$

for some integers $a, b$ and $c$. Then, by previous facts,

$$
\phi^* H \cdot H^\# = -a + c = 0, \quad \phi^* H \cdot E_1 = a - 2b = 0.
$$

Therefore,

$$
\phi^* H = 2H^\# + E_1 + 2E_2, \quad \pi^* H = H^\# + E_1 + 2E_2
$$

and hence

$$
r(f) = 2 \times 1 = 2.
$$

3. Jointly regular families of rational maps

Proof of Theorem 1.4. For notational convenience, let

- $d_l = \deg f_l$,
- $r_l = r(f_l)$,
- $(V_l, \pi_l)$ be a resolution of indeterminacy of $f_l$ constructed by Theorem 2.5: assume that $\pi_l$ is a composition of monoidal transformations and $\{\pi^\#_l H = H_{V_l}, E_{l1}, \ldots, E_{ls_l}\}$ is the basis of $\text{Pic}(V_l)$ given by Proposition 2.7.
\[ \phi_l \text{ be the resolved morphism of } f_l \text{ on } V_l. \]

\[ \pi_l^* H = a_0 H_{V_l} + \sum_{i=1}^{s_l} a_i E_{l_i} \quad \text{and} \quad \phi_l^* H = b_0 H_{V_l} + \sum_{i=1}^{s_l} b_i E_{l_i} \]

in \( \text{Pic}(V_l) = \mathbb{Z}\pi_l^* H \oplus \mathbb{Z}E_{l_1} \oplus \cdots \oplus \mathbb{Z}E_{l_{s_l}}. \)

We can easily check that \( a_0 = 1 \) and \( b_0 = d_l \) from \( \pi_l^* H = H \) and \( \pi_l \cdot \phi_l^* H = \deg \phi_l \cdot H. \) For details, see [10, Proposition 4.5(2)].

Let \( T_l \) be the center scheme of blowup for \( V_l \) and let \( W \) be the blowup of \( \mathbb{P}^n \) whose center scheme is \( \sum T_l. \) Then, \( W \) is a blowup of \( V_l \) for all \( l. \) Furthermore, since the underlying set of \( T_l \) is exactly \( I(f_l), \) the underlying set of \( \sum T_l = \cup I(f_l). \) Let \( \rho_l : W \to V_l, \pi_W \) be a composition of monoidal transformations:

\[
\begin{array}{ccc}
\mathbb{P}^n & \xrightarrow{\pi_W} & W \\
\downarrow{f_l} & & \downarrow{\rho_l} \\
V_l & \xrightarrow{\phi_l} & V_l'
\end{array}
\]

Then, still \( W \) is a blowup of \( \mathbb{P}^n \) and hence \( \text{Pic}(W) \) is generated by \( \pi_W^* H \) and the irreducible components \( F_j \) of the exceptional divisor:

\[ \text{Pic}(W) = \mathbb{Z}\pi_W^* H \oplus \mathbb{Z}F_1 \oplus \cdots \oplus \mathbb{Z}F_s. \]

Thus, we can represent \( \pi_W^* H \) as follows.

\[ \pi_W^* H = \pi_W^* H + \sum_{j=1}^{s} a_j F_j. \]

To describe \( \phi_l^* H \) precisely, define

\[ I_j = \{ 1 \leq j \leq s \mid \pi_W(F_j) \subset I(f_l) \} \quad \text{and} \quad I_j^c = \{ 1 \leq j \leq s \mid \pi_W(F_j) \not\subset I(f_l) \}. \]

By definition, it is clear that \( I_l \cup I_j^c = \{ 1, \ldots, s \} \) and \( I_l \cap I_j^c = \emptyset. \)

Thus, we can say

\[ \phi_l^* H = d_l \pi_W^* H + \sum_{j=1}^{s} \beta_l F_j = d_l \pi_W^* H + \sum_{j \in I_l} \beta_{l_j} F_j + \sum_{j \in I_l^c} \beta_{l_j} F_j. \]

We have the following lemmas which clarify the relation between coefficients of \( F_j \)'s.
Lemma 3.1. \[ \bigcup_{l=1}^{k} I_l = \bigcup_{l=1}^{k} I^c_l = \{1, \ldots, s\}. \]

Proof. \( \bigcup_{l=1}^{k} I_l = \{1, \ldots, s\} \) is clear; because the underlying set of the center scheme of \( W \) is \( \cup I(f_l) \), \( \cup \pi_W(F_j) = \pi_W(\cup F_j) = \cup I(f_l) \).

Suppose \( \bigcup_{l=1}^{k} I^c_l \subset \{1, \ldots, s\} \). Then, there is an index \( l_0 \) satisfying \( \pi_W(F_{l_0}) \subset I(f_l) \) for all \( l \). This implies \( \pi_W(F_{l_0}) \subset I(f_l) \) for all \( l \) and hence \( \emptyset \neq \pi_W(F_{l_0}) \subset \bigcap I(f_l) \) which contradicts to the assumption that \( S \) is jointly regular. \( \square \)

Lemma 3.2. Let \( \alpha_j \) and \( \beta_{ij} \) be the coefficients of \( F_j \) in \( \pi^*_V H \) and \( \hat{\phi}^*_j H \) respectively. Then,
\[ d_l \frac{\alpha_j}{\beta_{ij}} \leq r_l. \]

Especially, if \( j \in I^c_l \), then
\[ d_l \alpha_j = \beta_{ij}. \]

Proof. By definition of the \( D \)-ratio, the first inequality is clear:
\[ r_l = d_l \cdot \max_i \left( \frac{\alpha_i}{\beta_{ii}} \right) \geq d_l \cdot \frac{\alpha_j}{\beta_{ij}}. \]

Now, suppose that
\[ \rho^*_l \pi^*_l H = \kappa_{l0} \pi^*_W H + \sum_{j=1}^{s} \kappa_{lj} F_j = \kappa_{lj} \pi^*_W H + \sum_{j \in I^c_l} \kappa_{lj} F_j, \]
\[ \rho^*_l E_l = \lambda_{l0} \pi^*_W H + \sum_{j=1}^{s} \lambda_{lij} F_j = \lambda_{l0} \pi^*_W H + \sum_{j \in I^c_l} \lambda_{lij} F_j + \sum_{j \in I_l} \lambda_{lij} F_j. \]

We can easily calculate some of \( \kappa_{lj}, \lambda_{lij} \):

(a) \( \lambda_{l0} = 0 \) for all \( i = 1, \ldots, s_l \).

Because \( \pi_W(\rho^*_l E_i) \subset \cup I(f_l) \), we get \( (\pi_W)_* (\rho^*_l E_i) = 0 \). On the other hand, \( \pi_W \) eliminates all \( F_j \) so that
\[ \pi_W \left( \lambda_{l0} \pi^*_W H + \sum_{j=1}^{s} \lambda_{lij} F_j \right) = \lambda_{l0} H. \]

Hence, \( \lambda_{l0} = 0 \).

(b) \( \kappa_{l0} = 1 \).

We have
\[ \pi_W \left( \rho^*_l \pi^*_l H \right) = H \]

because \( \pi_W \) is one-to-one outside of the center of blowup of \( W \). Therefore,
\[ \pi_W \left( \rho^*_l \pi^*_l H \right) = \pi_W \left( \pi^*_W H - \sum_{j=1}^{s} a_{ij} \rho^*_l E_{li} \right) = H. \]
On the other hand, $\pi_*$ eliminates all $F_j$ so that
\[
\pi_{W*} \left( \kappa_{i0} \pi_W^# H + \sum_{j=1}^{s} \kappa_{ij} F_j \right) = \kappa_{i0} H.
\]

Hence, $\kappa_{i0} = 1$.

(c) $\lambda_{ij} = 0$ for all $j \in I_0^c$.

Because $\pi_l(E_i) \subseteq I(f_l)$ and $\pi_W(F_j) \not\subseteq I(f_l)$ for any $j \in I_0^c$, the multiplicity of $\rho_l(F_j)$ on $E_i$ is zero and hence $\gamma_{lij} = 0$. Thus, we can say
\[
\rho_l^* E_i = \sum_{j \in I_0} \lambda_{lij} F_j.
\]

Let’s complete the proof of Lemma 3.2. Since $\tilde{\phi}_l = \rho_l \circ \phi_l$ and $\pi_W = \rho_l \circ \pi_l$, we can use (a), (b) and (c) to get the description of $\pi_{W*}$ and $\tilde{\phi}_l^* H$:
\[
\pi_l^# H = \rho_l^* \pi_l^* H
\]
\[
= \rho_l^* \left( \pi_l^# H + \sum_{i=1}^{s_l} a_{li} E_{li} \right)
\]
\[
= \left( \kappa_{i0} \pi_W^# H + \sum_{j \in I_l} \kappa_{ij} F_j + \sum_{j \in I_l^c} \kappa_{ij} F_j \right)
+ \sum_{i=1}^{s_l} a_{li} \left( \lambda_{i0} \pi_W^# H + \sum_{j \in I_l} \lambda_{ij} F_j + \sum_{j \in I_l^c} \lambda_{ij} F_j \right)
\]
\[
= \pi_W^# H + \sum_{j \in I_l} \kappa_{ij} F_j + \sum_{j \in I_l^c} \kappa_{ij} F_j + \sum_{i=1}^{s_l} a_{li} \left( \sum_{j \in I_l} \lambda_{ij} F_j \right)
\]
\[
= \pi_W^# H + \sum_{j \in I_l} \kappa_{ij} F_j + \sum_{j \in I_l^{c}} \left( \sum_{i=0}^{s_l} \kappa_{ij} + a_{li} \lambda_{ij} \right) F_j
\]
and
\[
\tilde{\phi}_l^* H = \rho_l^* \phi_l^* H
\]
\[
= \rho_l^* \left( d_l \pi_l^# H + \sum_{i=1}^{s_l} b_{li} E_{li} \right)
\]
\[
= d_l \left( \kappa_{i0} \pi_W^# H + \sum_{j \in I_l} \kappa_{ij} F_j + \sum_{j \in I_l^c} \kappa_{ij} F_j \right)
+ \sum_{i=1}^{s_l} b_{li} \left( \lambda_{i0} \pi_W^# H + \sum_{j \in I_l} \lambda_{ij} F_j + \sum_{j \in I_l^c} \lambda_{ij} F_j \right)
\]
\[dl_0 @ W + \sum_{j \in I} c I_l j (d_i \lambda_{ij} b_{ij}) F_j.\]

Therefore,
\[d_l \alpha_j = d_l \sum_{j \in I} \kappa_{ij} = \beta_j \text{ for all } j \in I.\]

□

We now complete the proof of Theorem 1.4. Let \(r = \max_{l=1, \ldots, k} r_l.\) Note that
\[p_0^n > q_0^n + \sum_{j=1}^s q_j F_j\]
if \(p_j \geq q_j\) for all \(j = 0, \ldots, s.\) Thus,
\[\sum_{l=1}^k \frac{1}{d_l} \phi_l H = \sum_{l=1}^k \left( \pi_W^n H + \sum_{j \in I_l} \left( \frac{\beta_j}{d_l} F_j \right) + \sum_{j \in I_l} \left( \frac{\alpha_j}{r_l} F_j \right) \right)\]
\[\geq k \pi_W^n H + \sum_{l=1}^k \sum_{j \in I_l} \alpha_j F_j + \sum_{l=1}^k \left( \sum_{j \in I_l} \alpha_j F_j \right) \left( r \geq r_l \right)\]
\[\geq k \pi_W^n H + \sum_{j=1}^s \alpha_j F_j + \frac{1}{r} \sum_{j=1}^s \alpha_j F_j \left( r \geq r_l \right)\]
and hence
\[D = \sum_{l=1}^k \frac{1}{d_l} \phi_l H - \left( 1 + \frac{1}{r} \right) \pi_W^n H\]
is an \(A^n\)-effective divisor.

So, by Proposition 2.9, \(h_D\) is bounded below on \(A^n.\) Therefore, there is a constant \(C\) such that
\[h_D(Q) = \sum_{l=1}^k \frac{1}{d_l} \phi_l H(Q) - \left( 1 + \frac{1}{r} \right) h_{\pi_W^n H}(Q) > C.\]
for all \( Q \in \pi^{-1}_W(\mathbb{A}^n)(\overline{K}) \). Finally, for \( P = \pi_W(Q) \), we have \( \tilde{\varphi}(Q) = f(P) \) and hence we obtain
\[
\sum_{l=1}^k \frac{1}{d_l} h_H(P) - \left(1 + \frac{1}{r}\right) h_H(P) > C.
\]

\[\square\]

**Example 3.3.** Let
\[f_1 = (z, y + z^2, x + (y + z^2)^2), \quad f_2 = (x, y^2, z), \quad \text{and} \quad f_3 = (x^3, x + y, y + z^2).
\]
Their indeterminacy loci in \( \mathbb{P}^3 \) are
\[I(f_1) = \{[x, y, 0, 0]\}, \quad I(f_2) = \{[x, 0, z, 0]\}, \quad \text{and} \quad I(f_3) = \{[0, y, z, 0]\}.
\]
Then, the \( r(f_1) = 8 \), \( r(f_2) = 2 \) and \( r(f_3) = 3/2 \) (For details of the \( D \)-ratio calculation, see [10]). Therefore,
\[h((z, y + z^2, x + (y + z^2)^2)) + h((x, y^2, z)) + h((x^3, x + y, y + z^2)) \geq \left(1 + \frac{1}{8}\right) h((x, y, z)) - C
\]
for some constant \( C \).

**Corollary 3.4.** Let \( S \) be a jointly regular set of affine morphisms. Then,
\[
\kappa(S) := \liminf_{P \in \mathbb{A}^n(\overline{K})} \frac{1}{h(P) + \infty} \sum_{f \in S} \frac{1}{\deg f} h(f(P)) = 1 + \frac{1}{r},
\]
where \( r = \max_{f \in S} r(f) \).

**Remark 3.5.** Corollary 3.4 may not be the exact limit infimum values. For example, if there is a subset \( S' \subset S \) such that \( S' \) is still jointly regular and \( \max_{f \in S'} r(f) < \max_{f \in S} r(f) \), then
\[
\kappa(S) > 1 + \frac{1}{r'} > 1 + \frac{1}{r}.
\]

**Example 3.6.** We have the following examples for \( \kappa(S) = 1 + \min_{f \in S} \left(\frac{1}{r(f)}\right) \).

1. \( S = \{f, g\} \) where \( f, g \) are morphisms.
   If \( f, g \) are morphisms, then \( r(f) = r(g) = 1 \). Therefore,
   \[
   \frac{1}{\deg f} h(f(P)) + \frac{1}{\deg g} h(g(P)) = h(P) + h(P) + O(1).
   \]

2. \( S = \{f, f^{-1}\} \) where \( f \) is a regular affine automorphism and \( f^{-1} \) is the inverse of \( f \).
   It is proved by Kawaguchi. See [6].
4. An application to arithmetic dynamics

The purpose of this section is to prove Theorem 1.5. This result is a generalization of [14, Section 4]. The proof is almost the same except one: the only difference is that we have an improved height inequality for a jointly regular family.

Fix an integer \( m \geq 1 \) and let \( S = \{ f_1, \ldots, f_k : \mathbb{A}^n_K \to \mathbb{A}^n_K \} \) be a jointly regular family defined over a number field \( K \). For each \( m \geq 0 \), let \( W_m \) be the collection of ordered \( m \)-tuples chosen from \( \{ 1, \ldots, k \} \),

\[
W_m = \{ (i_1, \ldots, i_m) \mid i_j \in \{1, \ldots, k\} \}
\]

and let

\[
W_* = \bigcup_{m \geq 0} W_m.
\]

Thus \( W_* \) is the collection of words of \( r \) symbols.

For any \( I = (i_1, \ldots, i_m) \in W_m \), let \( f_I \) denote the composition of corresponding polynomial maps in \( S \):

\[
f_I : = f_{i_1} \circ \cdots \circ f_{i_m}.
\]

**Definition 4.1.** We denote the monoid of rational maps generated by \( S = \{ f_1, \ldots, f_k \} \) under composition by

\[
\Phi_S = \Phi := \{ \phi = f_I \mid I \in W_* \}.
\]

Let \( P \in \mathbb{A}^n \). The \( \Phi \)-orbit of \( P \) is defined to be

\[
\Phi(P) = \{ \phi(P) \mid \phi \in \Phi \}.
\]

The set of (strongly) \( \Phi \)-preperiodic points is the set

\[
\text{Preper}(\Phi) = \{ P \in \mathbb{A}^n \mid \Phi(P) \text{ is finite} \}.
\]

**Proof of Theorem 1.5.** By Theorem 1.4, we have a constant \( C \) such that

\[
0 \leq \left( \frac{1}{1 + \gamma} \right) \sum_{l=1}^k \frac{1}{d_l} h(f_l(Q)) - h(Q) + C \quad \text{for all } Q \in \mathbb{A}^n.
\]

Note that if \( r = \infty \), then \( \left( \frac{1}{1 + \gamma} \right) = 1 \), then it is done because of [14, Theorem 4]. Thus, we may assume that \( r \) is finite.

We define a map \( \mu : W_* \to \mathbb{Q} \) by the following rule:

\[
\mu_I : = \mu_{i_1, \ldots, i_m} = \prod_{l=1}^k d_l^{p_{I,l}},
\]

where \( p_{I,l} = - \lfloor t \mid i_l = l \rfloor \). Then, by definition of \( \delta_S \) and \( \mu_I \), the following is true:

\[
\delta_S^m = \left( \frac{r}{r+1} \right) \left( \sum_{l=1}^k \frac{1}{d_l} \right)^m = \left( \frac{r}{r+1} \right) \sum_{I \in W_m} \frac{1}{\deg f_{i_1} \cdots \deg f_{i_m}}
\]
Let \( P \in \mathbb{A}^n(\overline{\mathbb{Q}}) \). Then, (1) holds for \( f_I(P) \) for all \( I \in W_m \):

\[
0 \leq \left( \frac{r}{r+1} \right)^m \sum_{I \in W_m} \mu_I.
\]

Hence

\[
0 \leq \sum_{m=0}^{M-1} \sum_{I \in W_m} \mu_I \left( \frac{r}{r+1} \right)^m \left[ \sum_{l=1}^{k} \frac{1}{d_l} h(f_l(f_I(P))) \right] - \left( 1 + \frac{1}{r} \right) h(f_I(P)) + C.
\]

The main difficulty of the inequality is to figure out the constant term. From the definition of \( S \), we have

\[
\sum_{m=0}^{M} \sum_{I \in W_m} \mu_I \left( \frac{r}{r+1} \right)^m \sum_{l=1}^{k} \frac{1}{d_l} h(f_l(f_I(P))) = \sum_{m=0}^{M} \frac{\delta_S^m}{1 - \delta_S}.
\]

Now, do the telescoping sum and most terms in (2) will be canceled:

\[
\left( \sum_{m=0}^{M-1} \sum_{I \in W_m} \mu_I \left( \frac{r}{r+1} \right)^m \sum_{l=1}^{k} \frac{1}{d_l} h(f_l(f_I(P))) \right) - \left( \sum_{m=0}^{M} \sum_{I \in W_m} \mu_I h(f_I(P)) \right) = 0.
\]

Therefore, the remaining terms in (2) are the first term when \( m = M \) and the last term when \( m = 0 \). Thus, we get

\[
0 \leq \sum_{l \in W_M} \left( \frac{r}{r+1} \right)^M \mu_I \sum_{l=1}^{K} \frac{1}{d_l} h(f_l(f_I(P))) + \left( \frac{r}{r+1} \right)^M \mu_I C - h(P) + \sum_{l \in W_M} \left( \frac{r}{r+1} \right)^M \mu_I C
\]

\[
\leq \sum_{l \in W_M} \left( \frac{r}{r+1} \right)^M \mu_I \sum_{l=1}^{k} \frac{1}{d_l} h(f_l(f_I(P))) - h(P) + \frac{1}{1 - \delta_S} C.
\]

Let \( P \) be a \( \Phi \)-periodic point and define the height of the images of \( P \) by the monoid \( \Phi \) to be

\[
h(\Phi(P)) = \sup_{R \in \Phi(P)} h(R).
\]
Since
\[
\sum_{I \in W_M} \left( \frac{r}{r+1} \right)^M \mu_I \sum_{l=1}^k \frac{1}{d_l} = \left( \frac{r}{r+1} \right)^M \sum_{I \in W_{M+1}} \mu_I = \left( 1 + \frac{1}{r} \right) \delta_S^{M+1}
\]
and
\[
h(\Phi(P)) \geq h(g(P)) \text{ for all } g \in \Phi,
\]
we get
\[
h(P) \leq \left[ \sum_{I \in W_M} \left( \frac{r}{r+1} \right)^M \mu_I \sum_{l=1}^k \frac{1}{d_l} \right] h(\Phi(P)) + \frac{1}{1 - \delta_S} C
\]
\[\leq \left( 1 + \frac{1}{r} \right) \delta_S^{M+1} h(\Phi(P)) + \frac{1}{1 - \delta_S} C.
\]
By assumption, $\delta_S < 1$ and $h(\Phi(P))$ is finite, so letting $M \to \infty$ shows that $h(P)$ is bounded by a constant that depends only on $S$. \[\square\]

References

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