SPATIAL DECAY BOUNDS OF SOLUTIONS TO THE
NAVIER-STOKES EQUATIONS FOR TRANSIENT
COMPRESSIBLE VISCOUS FLOW

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Abstract. In this paper, spatial decay estimates for the time dependent compressible viscous isentropic flow in a semi-infinite three dimensional pipe are derived. An upper bound for the total energy in terms of the initial boundary data is obtained as well. The results established in this paper may be viewed as a version of Saint-Venant’s principle in transient compressible Navier-Stokes flow.

1. Introduction

The asymptotic behavior of the solution or norms of the solution of initial-boundary value problem for the heat equation, fluid equation and more general parabolic equations has been studied for many years. Estimates of Saint-Venant type which exhibit exponential spatial decay in transient heat conduction problems appear to have originated with Boley, who shows the spatial decay of end effect at any time $t$ in transient problems is faster than or at least equal to that the steady case [3]. Many authors have contributed to the literature on Saint-Venant type estimates for parabolic and elliptic problems as well as for the more general Phragmén-Lindelöf type principles which result in an alternative of growth or decay (see [1], [2], [8], [10], [11], [12], [13], [14]). The survey article by Horgan and Knowles [9] and the updates by Horgan [6, 7] discuss the history, rationale and the importance of such estimates and contain an extensive list of pertinent references.

A number of papers in the literature have dealt with the flow of incompressible viscous fluid which is governed by steady or transient Navier-Stokes...
equations in a semi-infinite channel or pipe see e.g. ([1], [2], [8], [11], [13], [14], [24]). These pipe or channel flow results may be regarded as Saint-Venant type decay estimates. In fact, the first paper to point out this connection is that of Horgan and Wheeler [11]. For other results of Saint-Venant type, see [8], [13], [14], [22], [23]. Of interest also are the papers [9], [6] and [7] therein. However, it seems that, up to now, few analogies concerning compressible viscous Navier-Stokes flow equations have been seen in the literature. Recently, the authors discuss the boundary value problem of steady compressible Navier-Stokes flow in a semi-infinite pipe and establish the Phragmén-Lindelöf type alternative [12], which illustrates the solutions for steady compressible viscous Navier-Stokes flow problem either grow or decay exponentially as the distance from the entry section tends to infinity.

In this paper, we consider the transient compressible viscous Navier-Stokes flow problem with a motivation to establish the spatial decay estimates analogous to those obtained for the steady state case. In Section 2, we formulate the basic initial-boundary value problem which provides the framework for our investigation. Section 3 is devoted to deriving a basic differential inequality that leads directly to an exponential decay estimate for the solution. Finally, in Section 4, to make the estimate explicit, we derive an upper bound for the total energy in terms of the prescribed initial boundary data.

2. Formulation of the problem

In this paper, we are concerned with nonstationary compressible viscous isentropic flow in a three dimensional cylindrical pipe. The fluid motion is described in the following form by the conservation law of mass and momentum

\begin{align}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P(\rho) - \mu \Delta u - (\mu + \lambda) \nabla (\text{div}u) &= \rho f,
\end{align}

where \( u = (u_1, u_2, u_3) \), \( \rho \), \( P(\rho) \) represent respectively the fluid velocity, density and pressure, \( f(x) = (f_1(x), f_2(x), f_3(x)) \) is the external force, constants \( \mu, \lambda \) are viscous coefficients satisfying

\[ \mu > 0, \quad 3\lambda + 2\mu \geq 0 \]

by physical requests.

We are interested only in the isentropic case where the pressure is given by

\[ P(\rho) = a \rho^\gamma, \]

with a positive constant \( a \) and \( \gamma > 1 \). This is important from the physical point of view since air has the adiabatic constant \( \gamma = \frac{7}{5} \).

In our problem, domain \( R \) is defined as

\[ R = \{(x_1, x_2, x_3) | (x_1, x_2) \in D, x_3 > 0\}, \]
the arbitrary cross section $D$ being a bounded simply-connected region in $(x_1,x_2)$-plane with piecewise smooth boundary $\partial D$. We also use the notation 
\[ R_z = \{(x_1,x_2,x_3)|(x_1,x_2) \in D, x_3 \geq z > 0\} \]
for fixed $z$. Thus in particular $R_0 \equiv R$. Throughout this paper, the usual summation convection is employed with repeated Latin subscripts summed from 1 to 3 and repeated Greek subscripts summed from 1 to 2. The comma is used to indicate partial differentiation, i.e., $u_{i,j} = \frac{\partial u_i}{\partial x_j}$.

In recent years, the local or global (in time) solvability to the various initial boundary value problem for the Navier-Stokes equation for nonstationary compressible viscous fluid has been established by many authors (see, e.g. [16], [17], [18], [19], [25]). The most general results are 
\[ u \in C((0,T],H^3), \rho \in L^2 \cap L^\infty \]
for the interior and exterior problems. It should stress that in most case “small data”, i.e., small external potential forces are in request [4], [5].

Recently, in a separate paper [12], the authors have proved the Phragmén-Lindelöf alternative for corresponding stationary compressible viscous flow. Assume $(\tilde{u}, \tilde{\rho})$ is the classical solution of the corresponding stationary compressible viscous flow governed by 
\begin{equation}
\begin{aligned}
(\rho \tilde{u}_i)_i &= 0, \\
-\mu \Delta \tilde{u}_i + (\mu + \lambda) u_{j,ji} + \rho u_j u_{i,j} + a(\rho^\gamma)_{,i} &= \rho f_i \quad \text{in } R, \\
u_i|_{\partial D} &= 0, \\
u_i(x_1,x_2,0) &= h_i(x_1,x_2) \quad \text{in } D.
\end{aligned}
\end{equation}

Then, the $H^1$ norm of $\tilde{u}$ either grows or decays exponentially as the distance from the entry section tends to infinity, specially, in decay case, we have 
\begin{equation}
\lambda \int_{R_z} \tilde{u}_{i,j} \tilde{u}_{i,j} dx + (\mu + \lambda) \int_{R_z} (\tilde{u}_{i,i})^2 dx \leq \tilde{\theta} e^{-wz},
\end{equation}
where $\tilde{\theta}$ and $w$ are computable constants.

Now, we take a constant $\rho_\infty > 0$, and consider a stationary solution $(\tilde{u}, \tilde{\rho})$ satisfying 
\begin{equation}
(\tilde{u}, \tilde{\rho}) \rightarrow (0, \rho_\infty) \quad \text{as } |x| \rightarrow \infty.
\end{equation}

In addition, assume that there exists a potential function $\phi$ which decays exponentially in a suitable sense, such that 
\begin{equation}
-\nabla \phi = f.
\end{equation}

Then, by (2.3), the potential function $\phi$ may be determined by the equation below 
\begin{equation}
a(\tilde{\rho}^\gamma)_{,i} = -\tilde{\rho} \phi_{,i}.
\end{equation}
Expanding $\mathbb{R}$ to full space $\mathbb{R}^3$, integrating (2.9) leads to

$$
\tilde{\rho}(x) = \left[\rho_{\infty}^{\gamma-1} - \frac{\gamma - 1}{a\gamma}\phi(x)\right]^{\frac{1}{\gamma - 1}}.
$$

Hence, in order to avoid the vacuum state, we must expect

$$
\sup \phi < \frac{a\gamma}{\gamma - 1}\rho_{\infty}^{\gamma-1}, \quad \gamma > 1.
$$

It is easy to see that if $\gamma$ is close enough to 1 for given $\phi(x)$, then (2.11) holds.

Let us describe more precisely the problem. Assume $(u, \rho)$ satisfying

$$
0 < \rho < \inf \rho(x, t) < \sup \rho(x, t) < \rho_f
$$

is the classical solution of the following initial boundary value problem

$$
\begin{cases}
\rho_t + (\rho u_i)_i = 0, \\
\rho u_{i,t} + \rho u_j u_{i,j} + \rho \left(\frac{P'\phi}{\rho} - \frac{(P\tilde{\rho})_\theta}{\rho}\right) - \mu \Delta u_i - (\mu + \lambda) u_{j;i} = 0
\end{cases}
$$

in $\mathbb{R} \times \{t > 0\}$,

with initial boundary value conditions

$$
\begin{align*}
u_i(x, 0) &= 0 \text{ in } \mathbb{R}, \\
\rho(x, 0) &= \rho_0(x) = \tilde{\rho}(x) \text{ in } \mathbb{R}, \\
u_i(x, t) &= 0 \text{ on } \partial D \times (0, \infty) \times \{t > 0\}, \\
u_i(x_1, x_2, 0, t) &= h_i(x_1, x_2, t) \text{ in } D \times \{t > 0\}, \\
u_i, u_{i,j}, u_{i,t} &= o(x_3^{-1}) \text{ uniformly in } x_1, x_2, t \text{ as } x_3 \to \infty,
\end{align*}
$$

where $h_i(x_1, x_2, t)$ ($i = 1, 2, 3$) are assumed to satisfy the compatibility.

**Remark 1.** (i) According to the embedding theorem of Sobolev space, the classical solution of problem (2.12)-(2.16) is available;

(ii) A priori asymptotical decay assumption (2.17) is for convenience, if one intends to establish the Phragmén-Lindelöf alternative for the problem, assumption (2.17) can be omitted;

(iii) $\rho_f, \rho$ are positive constants, in fact, the boundedness of density is natural from the physical viewpoint.

Moreover, we prepare two auxiliary inequalities which are frequently used later.

(i) Poincaré’s inequality for Dirichlet integrable function $\phi$ which vanishes on $\partial D$, namely

$$
\int_D \phi^2 dA \leq \frac{1}{\lambda_1} \int_D \phi \cdot \phi \cdot dA,
$$

where $\lambda_1$ is the first eigenvalue of the Laplace-Beltrami operator.
where \( \lambda_1 \) is the smallest eigenvalue in the problem
\[
\begin{align*}
\psi_{,\alpha\alpha} + \lambda_1 \psi &= 0 \quad \text{in} \ D, \\
\psi &= 0 \quad \text{on} \ \partial D.
\end{align*}
\]
Lower bounds for \( \lambda_1 \) are well known.

(ii) Sobolev inequality which holds for \( \phi \in C^1_0(D) \)
\[
\int_D \phi^4 dA \leq \frac{1}{2} \int_D \phi^2 dA \int_D \phi_{,\alpha} \phi_{,\alpha} dA,
\]
a derivation of (2.19) can be found in Serrin [21] and Payne [20].

3. Spatial decay estimate

This section is devoted to establishing spatial decay estimate for energy integral defined by solution \((u, \rho)\).

An application of the divergence theorem together with (2.12)-(2.17) allows us to obtain, for \( t > 0, z > 0 \),
\[
0 = \int_0^t \int_{D_z} \left[ \rho_{i3} u_{i,j} + \rho (\frac{P(\rho)}{\rho})_{,i} - (\frac{P(\tilde{\rho})}{\tilde{\rho}})_{,i} \right] u_{i,j} dA d\tau
\]
\[
= \frac{1}{2} \int_{R_z} \rho u_{i,j} dx - \frac{1}{2} \int_0^t \int_{D_z} \rho u_{i,j} u_{j} dA d\tau
\]
\[
= \frac{1}{2} \int_0^t \int_{R_z} \rho u_{i,j} dx - \frac{1}{2} \int_0^t \int_{D_z} \rho u_{i,j} u_{j} dA d\tau
\]
\[
+ \int_0^t \int_{R_z} \rho \int_{\tilde{\rho}}^{\rho} \frac{P'(s)}{s} ds dr d\tau + \mu \int_0^t \int_{D_z} u_{i,j} u_{i,j} dA d\tau
\]
\[
+ (\mu + \lambda) \int_0^t \int_{D_z} u_{i,j} u_{i,j} dA d\tau,
\]
where \( D_z \) indicates that the integral is taken over \( D \) in the plane \( x_3 = z \).

We denote
\[
G(\rho) = \int_{\tilde{\rho}}^{\rho} \int_{\tilde{\rho}}^{\rho} \frac{P'(s)}{s} ds dr d\tau,
\]
thus
\[
G'(\rho) = \int_{\tilde{\rho}}^{\rho} \frac{P'(s)}{s} ds,
\]
\[
G(\tilde{\rho}) = G'(\tilde{\rho}) = 0.
\]
Furthermore, it is readily to prove
\[
G(\rho) = a_\gamma \int_{\tilde{\rho}}^{\rho} s^{-2} ds dr d\tau \leq \frac{a_\gamma \pi \gamma}{2 \rho^2} (\rho - \tilde{\rho})^2 := a_\gamma c(\rho - \tilde{\rho})^2.
\]
\[ G(\rho) \geq \frac{a_\gamma \rho^\gamma}{2\pi^2} (\rho - \bar{\rho})^2 := a_\gamma c (\rho - \bar{\rho})^2, \]

where and in what follows, \( c \) denotes a computable constant which may be different from line by line.

Thus, we obtain

\[ k_1 (\rho - \bar{\rho})^2 \geq G(\rho) \geq c_1 (\rho - \bar{\rho})^2, \]

where \( c_1 \) and \( k_1 \) are positive constants.

In light to (3.1), (3.2), (3.4), we define

\[ E(z;t) = \frac{1}{2} \int_{R_z} \rho u_i u_i dx_{|r=t} + \int_{R_z} G(\rho) dx_{|r=t} \]
\[ = \frac{1}{2} \int_0^t \int_{R_z} \rho u_i u_i dA d\tau + \int_0^t \int_{R_z} G(\rho) dA d\tau \]
\[ \quad - \mu \int_0^t \int_{D_z} \rho u_i u_i dA d\tau - (\mu + \lambda) \int_0^t \int_{D_z} u_i u_i dA d\tau. \]

A differentiation of (3.8) with respect to \( z \) leads to

\[ \frac{\partial E(z,t)}{\partial z} = - \frac{1}{2} \int_{D_z} \rho u_i u_i dA_{|r=t} - \int_{D_z} G(\rho) dA_{|r=t} \]
\[ \quad - \mu \int_0^t \int_{D_z} u_i u_i dA d\tau + (\mu + \lambda) \int_0^t \int_{D_z} u_i u_i dA d\tau \]
\[ \leq 0. \]

Following the method proposed by Lin and Payne [15]: For \( t > 0 \), let \( \bar{t} \) be the value of time between 0 and \( t \), at which

\[ m(z,\bar{t}) = \frac{1}{2} \int_{D_z} \rho u_i u_i dA_{|r=\bar{t}} + \int_{D_z} G(\rho) dA_{|r=\bar{t}} \]

gains its maximum value, i.e.,

\[ m(z,\bar{t}) = \max_{0 \leq \tau \leq t} \left\{ \frac{1}{2} \int_{D_z} \rho u_i u_i dA + \int_{D_z} G(\rho) dA \right\}. \]

We now define

\[ t^* = \lim_{z \to \infty} \left\{ \sup_{0 \leq z \leq Z} \bar{t}(z) \right\}, \]

and write

\[ E(z,t^*) = \frac{1}{2} \int_{R_z} \rho u_i u_i dx_{|r=t^*} + \int_{R_z} G(\rho) dx_{|r=t^*}. \]
\[
+ \mu \int_0^{t^*} \int_{\mathbb{R}^n} u_{i,j} u_{i,j} \, dx \, d\tau + (\mu + \lambda) \int_0^{t^*} \int_{\mathbb{R}^n} (u_1)^2 \, dx \, d\tau
\]
\[
= \frac{1}{2} \int_0^{t^*} \int_{D_1} \rho u_3 u_i u_i \, dA \, d\tau + \frac{1}{2} \int_0^{t^*} \int_{D_2} \rho u_3 P' (s) \, ds \, dA \, d\tau
\]
\[
- \mu \int_0^{t^*} \int_{D_3} u_{i,3} u_i u_i \, dA \, d\tau - (\mu + \lambda) \int_0^{t^*} \int_{D_4} u_{i,3} u_i \, dA \, d\tau.
\]

Following the same procedure as in deriving \( t^* \), we now define \( t_1^* \), \( t_2^* \), \( t_3^* \) by

\[
m_1(z, t_1^*) = \max_{0 \leq \tau \leq t} \left[ \frac{1}{2} \int_{D_1} \rho u_i u_i \, dA \right],
\]
\[
m_2(z, t_2^*) = \max_{0 \leq \tau \leq t} \left[ \int_{D_2} G(\rho) \, dA \right],
\]
\[
m_3(z, t_3^*) = \max_{0 \leq \tau \leq t} \left[ \frac{1}{2} \int_{D_3} u_i u_i \, dA \right].
\]

In light of \( \rho \leq \rho_0 \leq \rho^* \), we get

\[
m_3(z, t_3^*) \leq \frac{1}{p} m_1(z, t_3^*) \leq \frac{1}{p} m_1(z, t_1^*).
\]

From the definition of \( m_1(z, t_1^*) \), \( m_2(z, t_2^*) \) and \( m(z, t^*) \), we can easily get

\[
m_1(z, t_1^*) \leq m(z, t_1^*) \leq m(z, t^*),
\]
\[
m_2(z, t_2^*) \leq m(z, t_2^*) \leq m(z, t^*).
\]

We next commence to establish a differential inequality which will deduce our spatial decay results. By using the Schwarz’s inequality and inequality (2.18), we find

\[
\left| \mu \int_0^{t^*} \int_{D_1} u_{i,3} u_i u_i \, dA \, d\tau \right| 
\leq \frac{\mu}{2 \sqrt{\lambda_1}} \int_0^{t^*} \int_{D_1} u_{i,j} u_{i,j} \, dA \, d\tau.
\]

Similarly, we obtain

\[
\left| (\mu + \lambda) \int_0^{t^*} \int_{D_2} u_{i,3} u_i u_i \, dA \, d\tau \right|
\leq \frac{1}{2 \sqrt{\lambda_1}} \frac{\mu}{\mu + \lambda} (\mu + \lambda)^2 \int_0^{t^*} \int_{D_2} (u_1)^2 \, dx \, d\tau \, dA \, d\tau + \frac{1}{2 \sqrt{\lambda_1}} (\mu + \lambda) \int_0^{t^*} \int_{D_2} (u_1)^2 \, dx \, d\tau \, dA \, d\tau.
\]
Making use of (2.18), (2.19) and using (2.15), (2.16), we obtain

\begin{equation}
\left| \frac{1}{2} \int_{D_2} \int_{t_1}^{t_2} \rho u_3 u_i dAd\tau \right|
\end{equation}

\begin{align*}
&\leq \frac{\bar{p}}{2} \int_{D_2} \int_{t_1}^{t_2} u_3^2 dAd\tau \cdot \int_{t_1}^{t_2} (u_i u_i)^2 dAd\tau \frac{1}{2} \\
&\leq \frac{\bar{p}}{2} \int_{D_2} \int_{t_1}^{t_2} u_3^2 dAd\tau \cdot \int_{t_1}^{t_2} \left[ \int_{D_2} u_i u_i dA \cdot \int_{D_2} u_{i,\alpha} u_{i,\alpha} dA \right] d\tau \frac{1}{2} \\
&\leq \frac{\bar{p}}{2 \sqrt{2}} \left( \int_{D_2} u_i u_i dA |_{\tau=t_2^1} \right)^{\frac{1}{2}} \cdot \left( \int_{t_1}^{t_2} \int_{D_2} u_3^2 dAd\tau \right)^{\frac{1}{2}} \cdot \left( \int_{t_1}^{t_2} \int_{D_2} u_{i,\alpha} u_{i,\alpha} dA \right)^{\frac{1}{2}} \\
&\leq \frac{\bar{p}}{2 \sqrt{2}} \left[ \frac{\partial E(z,t^*)}{\partial z} \right] \frac{1}{2},
\end{align*}

Since we know
\[ G'(\rho) = a \gamma \int_{\bar{\rho}} \rho^{\gamma-2} d\rho \leq \frac{a \gamma \bar{\rho}^{\gamma}}{\mu^2} (\rho - \bar{\rho}). \]

We observe that
\begin{align*}
\int_{0}^{t_2} \int_{D_2} \rho u_3 \int_{\bar{\rho}}^{\rho} \frac{P'(s)}{s} d\rho dA d\tau &\leq \int_{t_1}^{t_2} \int_{D_2} \rho u_3 G'(\rho) dAd\tau \\
&\leq c_2 \bar{p} \int_{t_1}^{t_2} \int_{D_2} (\rho - \bar{\rho})^2 dAd\tau \int_{t_1}^{t_2} \int_{D_2} u_3^2 dAd\tau \frac{1}{2},
\end{align*}

where \( c_2 = \frac{a \gamma \bar{\rho}^{\gamma}}{\mu^2} \).

Combining (3.9), (3.18), (3.22) and (3.7), we deduce

\begin{equation}
\int_{0}^{t_2} \int_{D_2} \rho u_3 \int_{\bar{\rho}}^{\rho} \frac{P'(s)}{s} d\rho dA d\tau \\
\leq \frac{c_3 \bar{p}}{c_1} \left( t^* \int_{D_2} G(\rho) dA |_{\tau=t_2^1} \cdot \int_{t_1}^{t_2} \int_{D_2} u_3^2 dAd\tau \right)^{\frac{1}{2}} \\
\leq \frac{c_3}{k^* \lambda_1} \left( t^* \int_{D_2} G(\rho) dA |_{\tau=t_2^1} \cdot \int_{t_1}^{t_2} \int_{D_2} u_3 u_{3,j} dAd\tau \right)^{\frac{1}{2}} \\
\leq \frac{c_3 t^* \frac{k^*}{k^* \lambda_1}}{\sqrt{\lambda_1}} \left( \frac{\partial E(z,t^*)}{\partial z} \right),
\end{equation}

where \( c_3 = \frac{c_2}{c_1} \).
A combination of (3.19)-(3.22) leads to

\[ E(z, t^*) \leq c_4 \left( -\frac{\partial E(z, t^*)}{\partial z} \right)^2 + c_5 \frac{\partial E(z, t^*)}{\partial z}, \]

with \( c_4 = \frac{7}{\sqrt{2\lambda_1 \lambda_2}}, \) \( c_5 = \frac{\mu}{\sqrt{\lambda_1 \lambda_2}} + \frac{1}{2} \sqrt{\frac{\lambda_1 + \lambda_2}{\lambda_1 \lambda_2}} + c_4 \frac{1}{\lambda_1}. \)

We now use the arithmetic-geometric mean inequality to the first term on the right hand side of (3.23) to obtain

\[ c_4 \left( -\frac{\partial E(z, t^*)}{\partial z} \right)^2 \leq \frac{c_4}{2} \left( -\frac{\partial E(z, t^*)}{\partial z} \right)^2 + \frac{c_4}{2} \left( -\frac{\partial E(z, t^*)}{\partial z} \right). \]

Combining (3.24) and (3.23), we are led to

\[ E(z, t^*) \leq \frac{c_4}{2} \left( -\frac{\partial E(z, t^*)}{\partial z} \right)^2 + \frac{c_4}{2} \left( -\frac{\partial E(z, t^*)}{\partial z} \right) \]

where \( c_6 = c_4, \) \( c_7 = c_5 + \frac{c_4}{2}, \) and solving for \( \frac{\partial E(z, t^*)}{\partial z} \) from (3.25) we find

\[ \frac{\partial E(z, t^*)}{\partial z} \leq - \left[ \left( \frac{E(z, t^*)}{c_6} + \frac{c_7}{4c_6} \right)^2 - \frac{c_7}{2c_6} \right]. \]

Now, setting

\[ \psi^2(z, t^*) = \frac{E(z, t^*)}{c_6} + \frac{c_7}{4c_6}, \quad \psi > 0. \]

We note that

\[ \psi(z, t^*) - \frac{c_7}{2c_6} > 0. \]

We rewrite (3.26) in terms of \( \psi \)

\[ \frac{2c_6 \psi \frac{\partial \psi}{\partial z}}{\psi - \frac{c_7}{2c_6}} \leq -1. \]

An integration of (3.29) from 0 to \( z \) leads to

\[ 2c_6 \left( \psi(z, t^*) - \psi(0, t^*) \right) + c_7 \ln(\psi(z, t^*) - \frac{c_7}{2c_6}) \leq -z + c_7 \ln(\psi(0, t^*) - \frac{c_7}{2c_6}). \]

From (3.30), we can get

\[ \psi(z, t^*) - \frac{c_7}{2c_6} \leq (\psi(0, t^*) - \frac{c_7}{2c_6}) \exp \left[ \frac{2c_6}{c_7} \psi(0, t^*) \right] e^{-\frac{z}{c_7}}. \]

By virtue of the fact

\[ \frac{E(z, t^*)}{c_6} + \frac{c_7}{4c_6} \frac{1}{2} - \frac{c_7}{2c_6} = \frac{E(z, t^*)}{c_6} \]

we can conclude

\[ \frac{E(z, t^*)}{c_6} + \frac{c_7}{4c_6} \frac{1}{2} + \frac{c_7}{2c_6} \geq \frac{E(z, t^*)}{c_6} + \frac{c_7}{4c_6} \frac{1}{2} + \frac{c_7}{2c_6}. \]
It follows that (3.31) yields the estimate
\begin{equation}
E(z, t^*) \leq Q e^{-\frac{z}{c_7}},
\end{equation}
where $Q$ is a positive quantity defined as
\begin{equation}
Q = E(0, t^*) \exp \left[ \frac{2c_6}{c_7} \left( \frac{E(0, t^*)}{c_6} + \frac{c_6^2}{4c_6^2} \right)^2 \right].
\end{equation}
Inequality (3.33) implies that
\begin{equation}
\lim_{z \to \infty} E(z, t^*) Q e^{\frac{z}{c_7}} \leq 1.
\end{equation}
An application of L'Hospital theorem in (3.35) then leads to for $z$ large enough
\begin{equation}
\frac{1}{2} \int_{D_z} \rho u_i u_i dA|_{\tau=t^*} + \int_{D_z} G(\rho) dA|_{\tau=t^*} + \mu \int_0^{t^*} \int_{D_z} u_{i,j} u_{i,j} dA d\tau + (\mu + \lambda) \int_0^{t^*} \int_{D_z} u_{i,j}^2 dA d\tau 
\end{equation}
\begin{equation}
\leq \frac{Q}{c_7} e^{-\frac{z}{c_7}}.
\end{equation}
From the definitions of $E(z, t^*)$ and $t^*$, we may conclude from (3.33) and (3.36)
\begin{equation}
\frac{1}{2} \int_{R_z} \rho u_i u_i dx|_{\tau=t^*} + \int_{R_z} G(\rho) dx|_{\tau=t^*} 
\end{equation}
\begin{equation}
\leq \frac{1}{2} \int_{R_z} \rho u_i u_i dx|_{\tau=t} + \int_{R_z} G(\rho) dx|_{\tau=t},
\end{equation}
and
\begin{equation}
\frac{1}{2} \int_{D_z} \rho u_i u_i dA|_{\tau=t} + \int_{D_z} G(\rho) dA|_{\tau=t} \leq \frac{Q}{c_7} e^{-\frac{z}{c_7}}
\end{equation}
where $Q$ is defined by (3.34).
We thus have proved:

**Theorem.** Assume that, for external force $\mathbf{f}$, and the density of the solution is assumed bounded below and above by some positive constants, there exists a potential function $\phi$, satisfying (2.7), (2.8). Let $(\mathbf{u}, \rho)$ be solution of initial-boundary value problem (2.12)-(2.17). Then for arbitrary $z > 0, t > 0$, inequalities (3.37) and (3.38) hold.
4. Upper bound for the total energy \( E(0, t^*) \)

In order to make our decay estimate results in Section 3 explicit, we now compute an upper bound for total energy \( E(0, t^*) \) in terms of prescribed data.

We start from (3.12) evaluated at \( z = 0 \),

\[
E(0, t^*) = \frac{1}{2} \int_0^{t^*} \int_{D_0} \rho u_3 u_i dA d\tau + \int_0^{t^*} \int_{D_0} \rho u_3 \int_{\tilde{\rho}}^{\rho} \frac{P'(s)}{8} ds dA d\tau - \mu \int_0^{t^*} \int_{D_0} u_{i,3} u_i dA d\tau - (\mu + \lambda) \int_0^{t^*} \int_{D_0} u_{i,j} u_3 dA d\tau.
\]

(4.1)

We now introduce the auxiliary functions

\[
(4.2)
\]

\[
\text{parameter } \sigma > 0 \text{ will be determined later.}
\]

Obviously, \( g_i \) satisfy the same boundary value conditions as \( u_i \), we find

\[
(4.3)
\]

\[
= \frac{1}{2} \int_0^{t^*} \int_{D_0} \rho g_3 g_i dA d\tau - \int_0^{t^*} \int_{R} (-\rho u_3 u_i + \mu u_i, g_i + (\mu + \lambda) u_{i,3} g_i) d\tau + \mu \int_0^{t^*} \int_{R} u_{i,3} g_i d\tau + (\mu + \lambda) \int_0^{t^*} \int_{R} u_{i,3} g_i d\tau
\]

\[
= \mu \int_0^{t^*} \int_{R} u_{i,3} g_i d\tau + (\mu + \lambda) \int_0^{t^*} \int_{R} u_{i,3} g_i d\tau + \int_0^{t^*} \int_{R} \rho u_i g_i d\tau - \int_0^{t^*} \int_{R} \rho u_i g_i d\tau + \int_0^{t^*} \int_{R} \rho u_i g_i d\tau
\]

\[
\int_0^{t^*} \int_{R} \rho (P(\rho))_{,i} - \frac{(P(\rho))_{,i}}{\rho} g_i d\tau + D_1(t),
\]

\[
where D_1(t) \text{ is a data term, i.e., in view of the definition of } g_i,
\]

\[
\left| - \frac{1}{2} \int_0^{t^*} \int_{D_0} \rho g_3 g_i dA d\tau \right| \leq \frac{\n}{4} \int_0^{t^*} \int_{D_0} h_{i,3} dA d\tau + \frac{\n}{4} \int_0^{t^*} \int_{D_0} (h_i h_i)^2 dA d\tau
\]

\[
= D_1(t).
\]
We observe that in (4.3)

\[(4.4)\]

\[I_1 = \int_0^t \int_R \rho (P(\rho) \cdot \frac{\partial \rho}{\partial \varphi}) \rho dx \, dt = \int_0^t \int_R \rho_0 (G'(\rho)) \rho dx \, dt \]

\[= - \int_0^t \int_{D_0} \rho \int_0^{g_3 \varepsilon (\rho)} \rho G'(\rho) dx \, dt - \int_0^t \int_R \rho (G'(\rho)) \rho dx \, dt - \int_0^t \int_R \rho (G'(\rho)) \rho dx \, dt \]

\[= - \int_0^t \int_{D_0} g_3 \varepsilon (\rho) \rho G'(\rho) dx \, dt + \int_0^t \int_{D_0} g_3 \varepsilon (\rho) \rho G'(\rho) dx \, dt \]

\[+ \int_0^t \int_R \rho (G(\rho) - \rho G'(\rho)) dx \, dt \]

\[= \int_0^t \int_{D_0} g_3 (G(\rho) - \rho G'(\rho)) dx \, dt + \int_0^t \int_R \rho (G(\rho) - \rho G'(\rho)) dx \, dt \]

\[\leq c_1 \int_0^t \int_{D_0} h_3^2 (\rho^2 - \rho^2) dx \, dt + c_1 \int_0^t \int_R (\rho^2 - \rho^2) dx \, dt + c_1 \int_0^t \int_R (\rho^2 - \rho^2) dx \, dt \]

\[+ c_1 \int_0^t \int_R g_{i,j} g_{i,j} dx \, dt.\]

In light of the boundedness of \(\rho\) and \(\hat{\rho}\), we only need want to bound

\[\int_0^t \int_{D_0} g_{i,j} g_{i,j} dx \, dt.\]

Using the Schwarz inequality

\[\int_R g_{i,j} g_{i,j} dx \leq 2 \int_R (g_{1,1}^2 + g_{2,2}^2 + g_{3,3}^2) dx \]

\[= 2 \int_0^\infty \int_{D_3} (h_{1,1}^2 + h_{2,2}^2) e^{-2\sigma x_3} dA dx_3 \]

\[+ 2 \int_0^\infty \int_{D_3} h_{3}^2 e^{-2\sigma x_3} (\rho^2) dA dx_3 \]

\[= \frac{1}{\sigma} \int_{D_0} (h_{1,1}^2 + h_{2,2}^2) dA + \sigma \int_{D_0} h_{3}^2 dA.\]

Hence, we obtain

\[(4.5)\]

\[I_1 \leq \text{Data},\]

where and in what follows, Data denotes the integrals in terms of prescribed functions.
On the other hand, we have

\[ E(0, t^*) = \frac{1}{2} \int_R \rho u_i u_i dx|_{\tau=t^*} + \int_R G(\rho) dx|_{\tau=t^*} + \mu \int_0^{t^*} \int_R u_{i,j} u_{i,j} dx d\tau \]

\[ + (\mu + \lambda) \int_0^{t^*} \int_R (u_i)^2 dx d\tau. \]

(4.6)

We next treat with the integrals on the right of (4.3) with an aim to seek the suitable upper bounds for those quantities.

Using the Schwarz’s inequality, we find that

\[ \mu \int_0^{t^*} \int_R u_{i,j} g_{i,j} dx d\tau \leq \frac{1}{8} \mu \int_0^{t^*} \int_R u_{i,j} u_{i,j} dx d\tau + 2\mu \int_0^{t^*} \int_R g_{i,j} g_{i,j} dx d\tau \]

\[ \leq \frac{1}{8} \mu \int_0^{t^*} \int_R u_{i,j} u_{i,j} dx d\tau + 2\mu \int_0^{t^*} \int_R g_{i,j} g_{i,j} dx d\tau \]

\[ \leq \frac{1}{8} \mu \int_0^{t^*} \int_R u_{i,j} u_{i,j} dx d\tau + \text{Data}, \]

(4.7)

\[ (\mu + \lambda) \int_0^{t^*} \int_R u_i g_i dx d\tau \leq \frac{1}{4} (\mu + \lambda) \int_0^{t^*} \int_R (u_i)^2 dx d\tau + \text{Data}, \]

(4.8)

\[ - \int_0^{t^*} \int_R \rho u_i g_i dx d\tau \]

\[ \leq \frac{\nu}{\sqrt{\lambda_1}} (\int_0^{t^*} \int_R u_i u_i dx d\tau \cdot \int_0^{t^*} \int_R g_i g_i dx d\tau) \frac{1}{2} \]

\[ \leq \frac{\mu}{8} \int_0^{t^*} \int_R u_{i,j} u_{i,j} dx d\tau + \frac{\mu^2}{16 \lambda_1} \int_0^{t^*} \int_R g_i g_i dx d\tau \]

\[ \leq \frac{\mu}{8} \int_0^{t^*} \int_R u_{i,j} u_{i,j} dx d\tau + \text{Data}, \]

(4.9)

\[ \int_R \rho u_i g_i dx|_{\tau=t^*} \]

\[ \leq \frac{1}{8} \int_R \rho u_i u_i dx|_{\tau=t^*} + 2 \int_R g_i g_i dx|_{\tau=t^*} \]

\[ \leq \frac{1}{8} \int_R \rho u_i u_i dx|_{\tau=t^*} + 2\rho \int_R g_i g_i dx|_{\tau=t^*} \]

\[ \leq \frac{1}{8} \int_R \rho u_i u_i dx|_{\tau=t^*} + 4\rho \int_0^{t^*} \int_R g_i g_i dx d\tau \]

\[ \leq \frac{1}{8} \int_R \rho u_i u_i dx|_{\tau=t^*} + 2\rho \int_0^{t^*} \int_R g_i g_i dx d\tau + 2\rho \int_0^{t^*} \int_R g_i g_i dx d\tau \]

(4.10)
It remains to treat the term which integrant contains \( g_{i,j} \), we find

\[
(4.11) \quad - \int_0^{t^*} \int_R \rho u_i u_j g_{i,j} dx \, d\tau \\
\leq \overline{p} \int_0^{t^*} \int_R |u_i u_j g_{i,j}| dx \, d\tau \\
= - \overline{p} \text{sgn}(u_i u_j g_{i,j}) \int_0^{t^*} \int_{D_n} g_{3 \alpha} g_{3 \alpha} dA \, d\tau \\
- \overline{p} \text{sgn}(u_i u_j g_{i,j}) \int_0^{t^*} \int_R (u_{i,j} u_{j,i} + u_{i,j} g_{i}) dx \, d\tau \\
\leq \overline{p} \int_0^{t^*} \int_{D_n} |g_{3 \alpha} g_{3 \alpha}| dA \, d\tau + \overline{p} \int_0^{t^*} \int_R |u_{i,j} u_{j,i}| dx \, d\tau \\
+ \overline{p} \int_0^{t^*} \int_R |u_{i,j} u_{j,i}| dx \, d\tau.
\]

We note that, by using the Sobolev’s inequality (2.19), and the Schwarz’s inequality,

\[
(4.12) \quad I_2 = \overline{p} \int_0^{t^*} \int_R |u_{i,j} u_{j,i}| dx \, d\tau \\
\leq \overline{p} \int_0^{t^*} \int_\infty^\infty (\int_{D_n} u_{i,j} u_{i,j} dA)^{\frac{1}{2}} \left( \int_{D_n} (u_{i,j} u_{i,j})^2 dA \right)^{\frac{1}{2}} \left( \int_{D_n} (g_{\alpha} g_{\alpha})^2 dA \right)^{\frac{1}{2}} d\eta \, d\tau \\
\leq \overline{p} \int_0^{t^*} \int_\infty^\infty (\int_{D_n} u_{i,j} u_{i,j} dA)^{\frac{1}{2}} \left( \frac{1}{2} \int_{D_n} u_{j,i} dA \cdot \int_{D_n} u_{j,\alpha} u_{j,\alpha} dA \right)^{\frac{1}{2}} \\
\cdot (\int_{D_n} (g_{\alpha} g_{\alpha})^2 dA)^{\frac{1}{2}} d\eta \, d\tau \\
\leq \frac{\overline{p}}{2^4} \int_0^{t^*} \left( \max_z \int_{D_n} u_{j,i} dA \right)^{\frac{1}{2}} \cdot \left( \int_0^{\infty} \int_{D_n} u_{i,j} u_{i,j} dA \, d\eta \right)^{\frac{1}{2}} \\
\cdot \left( \int_{D_n} (g_{\alpha} g_{\alpha})^2 dA \right)^{\frac{1}{2}} d\tau.
\]

Let we suppose \( \int_{D_n} u_{j,i} dA \) gains its maximum value at \( z = z_1 \),

\[
(4.13) \quad \int_{D_{z_1}} u_{j,i} dA = 2 \int_0^{z_1} \int_{D_n} u_{j,i} u_{j,\alpha} dA \, d\eta + \int_{D_n} u_{j,i} dA
\]
we get
\begin{equation}
(4.15)
\end{equation}

Inserting (4.13) into (4.12), and using \((a + b)^\frac{1}{2} \leq a^\frac{1}{2} + b^\frac{1}{2}\) for \(a > 0, b > 0\), we get
\begin{equation}
(4.14)
\end{equation}

\[
I_2 \leq \mathcal{P} \int_0^{\tau^*} \left( \int_0^\infty \int_{D_n} u_j u_j dAd\eta \right)^{\frac{1}{2}} \cdot \left( \int_0^\infty \int_{D_n} u_{j,q} u_{j,q} dAd\eta \right)^{\frac{1}{2}} \\
\cdot \left( \int_0^\infty \int_{D_n} u_{i,j} u_{i,j} dAd\eta \right)^{\frac{1}{2}} \cdot \left( \int_0^\infty \int_{D_n} (g_j g_j)^2 dAd\eta \right)^{\frac{1}{2}} d\tau \\
+ \text{Data} \times \frac{1}{2 \mathcal{P}} \int_0^{\tau^*} \left( \int_0^\infty \int_{D_n} u_{i,j} u_{i,j} dAd\eta \right)^{\frac{1}{2}} \cdot \left( \int_0^\infty \int_{D_n} u_{j,q} u_{j,q} dAd\eta \right)^{\frac{1}{2}} \\
\cdot \left( \int_0^\infty \int_{D_n} (g_j g_j)^2 dAd\eta \right)^{\frac{1}{2}} d\tau \\
\leq \mathcal{P} \left( \int_0^{\tau^*} \left( \int_0^\infty \int_{D_n} (g_j g_j)^2 dAd\eta \right)^2 d\tau \right)^{\frac{1}{2}} E(0, \tau^*) \\
+ \text{Data} \times \frac{1}{\mu^\frac{7}{2}} \left( \int_0^{\tau^*} \int_{D_n} u_{i,j} u_{i,j} dA d\tau \right)^{\frac{1}{2}} \cdot \left( \int_0^{\tau^*} \int_{D_n} (g_j g_j)^2 dAd\eta d\tau \right)^{\frac{1}{2}} \\
\leq \frac{\mathcal{P}}{\mu^\frac{7}{2}} \left( \int_0^{\tau^*} \left( \int_0^\infty \int_{D_n} (g_j g_j)^2 dAd\eta \right)^2 d\tau \right)^{\frac{1}{2}} E(0, \tau^*) \\
+ \text{Data} \times E(0, \tau^*)^\frac{1}{2} \cdot \left( \int_0^{\tau^*} \int_{D_n} (g_j g_j)^2 dAd\eta d\tau \right)^{\frac{1}{2}}.
\]

From the definition of \(g_i\), we have
\begin{equation}
(4.15)
\end{equation}

\[
\int_0^t \int_0^\infty \int_{D_n} (g_j g_j)^2 dAd\eta d\tau = \int_0^t \int_0^\infty \int_{D_n} (h_i h_i)^2 e^{-4\eta h_i} dAd\eta d\tau \\
= \int_0^t \int_0^\infty e^{-4\eta h_i} d\eta \int_{D_n} (h_i h_i)^2 e^{-4\eta h_i} dAd\eta d\tau \\
\leq \left( \frac{1}{4 \eta^2} \right)^2 \int_0^t \int_{D_n} (h_i h_i dA)^2 d\tau.
\]
For fixed $t$, we can choose $\sigma = \sigma_1$ large enough such that
\begin{equation}
\frac{p}{\mu^2} \left( \int_0^t \int_{D_0} (g_i g_i)^2 \, dA \, d\tau \right)^{\frac{1}{2}} \leq \frac{1}{8}, \tag{4.16}
\end{equation}

In a similar manner, we can also choose $\sigma = \sigma_2$ large enough such that
\begin{equation}
\frac{\text{Data}}{\mu^2} \left( \int_0^t \int_{D_0} (g_i g_i)^2 \, dA \, d\tau \right)^{\frac{1}{2}} \leq \frac{1}{8}. \tag{4.17}
\end{equation}

Thus, if we choose $\sigma = \max\{\sigma_1, \sigma_2\}$, we obtain from (4.14) that
\begin{equation}
I_2 \leq \frac{E(0, t^*)}{8} + \frac{(E(0, t^*))^2}{8}. \tag{4.18}
\end{equation}

Following the same procedure as deriving (4.18), we can get
\begin{equation}
I_3 = \frac{p}{\mu^2} \int_0^{t^*} \int_R |u_i u_j, g_i| \, dx \, d\tau \leq \frac{E(0, t^*)}{8} + \frac{(E(0, t^*))^2}{8}. \tag{4.19}
\end{equation}

Combining the above discussion, we conclude
\begin{equation}
E(0, t^*) \leq \frac{E(0, t^*)}{2} + \frac{(E(0, t^*))^2}{4} + \text{Data} \tag{4.20}
\end{equation}

that is to say
\begin{equation}
E(0, t^*) \leq \frac{(E(0, t^*))^2}{2} + \text{Data}. \tag{4.21}
\end{equation}

Using the Young’s inequality, we obtain
\begin{equation}
E(0, t^*) \leq \frac{1}{4} + \frac{\frac{1}{2} E(0, t^*)}{2} + \text{Data}. \tag{4.22}
\end{equation}

Thus, we can get
\begin{equation}
E(0, t^*) \leq \frac{8}{5} \left( \frac{1}{8} + \text{Data} \right) = \text{Data}. \tag{4.23}
\end{equation}

References
