WEAKLY DENSE IDEALS IN PRIVALOV SPACES OF HOLOMORPHIC FUNCTIONS

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Abstract. In this paper we study the structure of closed weakly dense ideals in Privalov spaces $N^p$ ($1 < p < \infty$) of holomorphic functions on the disk $D = \{z \in \mathbb{C} : |z| < 1\}$. The space $N^p$ with the topology given by Stoll’s metric [21] becomes an $F$-algebra. N. Mochizuki [16] proved that a closed ideal in $N^p$ is a principal ideal generated by an inner function. Consequently, a closed subspace $E$ of $N^p$ is invariant under multiplication by $z$ if and only if it has the form $IN^p$ for some inner function $I$. We prove that if $M$ is a closed ideal in $N^p$ that is dense in the weak topology of $N^p$, then $M$ is generated by a singular inner function. On the other hand, if $S_\mu$ is a singular inner function whose associated singular measure $\mu$ has the modulus of continuity $O(t^{(p-1)/p})$, then we prove that the ideal $S_\mu N^p$ is weakly dense in $N^p$. Consequently, for such singular inner function $S_\mu$, the quotient space $N^p/S_\mu N^p$ is an $F$-space with trivial dual, and hence $N^p$ does not have the separation property.

1. Introduction and preliminaries

Let $D$ denote the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ in the complex plane $\mathbb{C}$ and let $T$ be the boundary of $D$. For $0 < q \leq \infty$ we denote by $L^q(T)$ the Lebesgue space with respect to the normalized Lebesgue measure on $T$. Given $1 < p < \infty$, the Privalov class $N^p$ consists of all holomorphic functions $f$ on $D$ such that

$$\sup_{0 \leq r < 1} \int_0^{2\pi} (\log^+ |f(re^{i\theta})|)^p \frac{d\theta}{2\pi} < +\infty.$$ 

These classes were introduced by I. I. Privalov in [17, p. 93], where $N^p$ is denoted as $A_q$ (with $q = p > 1$). The study on the spaces $N^p$ was continued by Stoll’s work [21] in 1977 year. Further, topological and functional properties of these classes were investigated in [1], [5], [6], [12]-[14] and [16]; typically, the notation varied and Privalov was mentioned only in [12] and [14].
Recall that the Nevanlinna class $N$ consists of all holomorphic functions $f$ on $D$ such that
\[ \sup_{0 \leq r < 1} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi} < +\infty. \]
It is known that for each $f \in N$, the radial limit of $f$ defined as $f(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta})$ exists for almost every $e^{i\theta} \in T$ (see [7, p. 97]). The Smirnov class $N^+$ is the set of all functions $f \in N$ such that
\[ \lim_{r \to 1} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi} = \int_0^{2\pi} \log^+ |f(e^{i\theta})| \frac{d\theta}{2\pi}. \]
For $0 < q < \infty$ we denote by $H^q$ the classical Hardy space on $D$, and by $H^\infty$ the space of bounded holomorphic function on $D$. It is known (see [16]) that
\[ N^s \subset N^p (s > p), \quad \bigcup_{q > 0} H^q \subset \bigcap_{p > 1} N^p, \quad \text{and} \quad \bigcup_{p > 1} N^p \subset N^+ \subset N, \]
and these inclusion relations are proper.

**Theorem A** ([21, Theorem 4.2]). The Privalov space $N^p (1 < p < \infty)$ (with the notation $(\log^+ H)^p$ in [21]) with the topology given by the metric $\rho_p$ defined as
\[
(1.1) \quad \rho_p(f, g) = \left( \int_0^{2\pi} \left( \log(1 + |f(e^{i\theta}) - g(e^{i\theta})|) \right)^p \frac{d\theta}{2\pi} \right)^{1/p}, \quad f, g \in N^p,
\]
for $0 < q < \infty$ by $H^q$ the classical Hardy space on $D$, and by $H^\infty$ the space of bounded holomorphic function on $D$. It is known (see [16]) that $N^s \subset N^p (s > p), \quad \bigcup_{q > 0} H^q \subset \bigcap_{p > 1} N^p, \quad \text{and} \quad \bigcup_{p > 1} N^p \subset N^+ \subset N,$
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\]
is an $F$-algebra, i.e., a complete metrizable topological vector space in which multiplication is continuous.

It is well known [2, p. 26] that every function $f \in N^+$ admits a unique factorization of the form
\[
(1.2) \quad f(z) = B(z)S_\mu(z)F(z), \quad z \in D,
\]
where $B$ is the Blaschke product with respect to zeros $\{z_n\} \subset D$ of $f$ (the set $\{z_n\}$ may be finite), $S_\mu$ is a singular inner function, $F$ is an outer function for $N^+$, i.e.,
\[
B(z) = \prod_{n=1}^\infty \frac{|z_n| z - z_n}{1 - \bar{z}_n z}, \quad z \in D,
\]
with $\sum_{n=1}^\infty (1 - |z_n|) < \infty$, $m$ a nonnegative integer,
\[
S_\mu(z) = \exp \left( - \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \, d\mu(t) \right)
\]
with positive singular measure $d\mu$, and
\[
F(z) = \lambda \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log |f(e^{it})| \, dt \right),
\]
where $|\lambda| = 1$ and $\log |f(e^{i\theta})| \in L^1(T)$. 

Recall that a function $I$ of the form
$$I(z) = B(z)S_\mu(z), \quad z \in \mathbb{D},$$
is called an *inner function*, and $I$ is a bounded holomorphic function whose boundary values $f(e^{i\theta})$ have modulus 1 for almost every $e^{i\theta} \in \mathbb{T}$.

The inner-outer factorization theorem for the classes $N^p$ is given by Privalov [17] as follows.

**Theorem B** ([17, pp. 98–100; also see [6]). A function $f \in N^+$ factorized by (1.2) belongs to $N^p$ if and only if $\log^+ |F(e^{i\theta})| \in L^p(\mathbb{T})$.

A closed subspace $E$ of $N^p$ is called *invariant* if $zf \in E$ whenever $f \in E$. Here $z$ denotes the identity function on $\mathbb{D}$. By using a result of Mochizuki [16, Theorem 4], in Section 2 we exhibit the close relationship between inner functions and invariant subspaces of $N^p$. We prove (Theorem 2.3) that a closed subspace $E$ of $N^p$ is invariant if and only if it has the form $IN^p$ for some inner function $I$. Recall that a related result in more general form was proved by Matsugu [12, Theorem 1] using a classical result about invariant subspaces in $L^2(\mathbb{T})$ (cf. Remark 3).

For $0 < q < \infty$, in Section 3 we consider the space $F^q$ consisting of those functions $f$ holomorphic on $\mathbb{D}$ for which
$$\lim_{r \to 1}(1 - r)^{1/q} \log^+ \left( \max_{|z| \leq r} |f(z)| \right) = 0.$$Stoll [21, Theorem 3.2] proved that the space $F^q$ with the topology given by the family of seminorms $\{\| \cdot \|_{q,c} \}_{c>0}$ defined for
$$f(z) = \sum_{n=0}^{\infty} a_n z^n \in F^q$$as
$$\|f\|_{q,c} = \sum_{n=0}^{\infty} |a_n| \exp \left( -cn^{1/(q+1)} \right) < \infty,$$
is a countably normed Fréchet algebra. By a result of Eoff [5, Theorem 4.2], $F^p$ is the Fréchet envelope of $N^p$, and hence $F^p$ and $N^p$ have the same topological duals. Thus, the family of seminorms $\{\| \cdot \|_{p,c} \}_{c>0}$ induces on $N^p$ the strongest locally convex (metrizable) topology that is weaker than the metric $\rho_p$ topology.

In Section 4 we characterize the weak closure $[E]_w$ of a vector subspace $E$ of $N^p$, that is, the closure with respect to the weak topology defined on $N^p$ in the usual way. The non locally convex phenomenon of interest here is the existence of proper, closed subspaces of $N^p$ that are dense in the weak topology. We show that for any inner function $I$, the weak closure of an invariant subspace $IN^p$ of $N^p$ is again an invariant subspace. Hence, $[IN^p]_w = I_wN^p$ for some inner function $I_w$. We shall say that $I$ is a *weak outer function* in $N^p$ if $IN^p$ is weakly dense in $N^p$, that is, if $I_w \equiv 1$. We prove (Corollary 4.7) that an inner function
is weak outer in $N^p$ if and only if the space $IN^p$ is dense in $F^p$, that is, if and only if the space $IF^p$ is dense in $F^p$.

A closed subspace $E$ of $N^p$ will be said to have the separation property if each $f \in N^p$ that is not in $E$ can be separated from $E$ by a continuous linear functional on $N^p$; that is, if there exists a continuous linear functional $\varphi$ on $N^p$ such that $\varphi(E) = 0$ but $\varphi(f) \neq 0$. We will say that $E$ has the Hahn-Banach property if each continuous linear functional on $N^p$ can be extended to a continuous linear functional on the whole space $N^p$. In Section 4 (Theorem 4.10) we prove that if $I$ is any Blaschke product, then $IN^p$ has the separation property. In the case if $I$ is a finite Blaschke product, we prove that $IN^p$ has the Hahn-Banach property.

It was proved in [4, Theorem 14] that if $S$ is a nontrivial singular inner function whose associated singular measure has the modulus of continuity $O(t \log \frac{1}{t})$, then $I$ is a weak outer function in any space $H^q$ with $0 < q < 1$. In particular, it is not known whether an inner function can be weak outer for some values of $0 < p < 1$, but not for others. In our main result given in Section 5 (Theorem 5.5), for any fixed $1 < p < \infty$ we present a large class of positive singular measures $\mu$, depending on $p$, such that associated singular inner functions are weak outer functions in the Privalov space $N^p$. More precisely, Theorem 5.5 asserts that if $S$ is a non-trivial singular inner function with the associated measure $\mu$ whose modulus of continuity $\omega_\mu(t) = O(t^{(p-1)/p})$, then the ideal $S\mu N^p$ is weakly dense in $N^p$. Consequently, such a singular inner function $S\mu$ is a weak outer function in $N^p$ and the quotient space $N^p/S\mu N^p$ is an $F$-space with trivial dual (Corollaries 5.6 and 5.7).

Recall that it was shown in [19, Proposition 4] that a closed ideal in $N^+$ is weakly dense if and only if it is generated by a singular inner function $S\mu$ with $\mu$ a continuous singular measure.

Remark 1. The above notions and definitions are motivated by the famous Beurling’s theorem (see [9, Ch. 7, p. 99], [8, Lecture II]) and results from [4] related to the linear space structure of the Hardy space $H^q$ with $0 < q < 1$. Beurling’s invariant subspace theorem to $H^q$, tells that there is a one-to-one correspondence between inner functions and invariant subspaces; each invariant subspace of $H^q$ being of the form $IH^q$, where $I$ is an inner function. Beurling’s theorem can also be viewed as a result on approximation. In this formulation it states that the polynomial multiples of an $H^q$ function form a dense subset of $H^q$ if and only if that function is outer (see [2, Section 7.3]).

In [4] Duren, Romberg, and Shields added a new dimension to Beurling’s theorem by proving that when $0 < q < 1$ some inner functions (not identically 1) give rise to weakly dense invariant subspaces of $H^q$. In view of the approximation-theoretic formulation of Beurling’s theorem, Duren, Romberg, and Shields called such inner functions weakly outer.
2. Invariant subspaces of $N^p$

Denote by $\mathcal{P}$ the space of all polynomials. Let $X$ be a topological vector space of holomorphic functions on $D$ so that $H^\infty \subset X$ and convergence in $X$ implies uniform convergence on compact subsets of $D$. Suppose that $1 \in X$ and that $f \in X$ implies $Pf \in X$ for every polynomial $P \in \mathcal{P}$. If $f \in X$, then $\text{cl}(Pf)$ denotes the closure of $\mathcal{P}f := \{Pf : P \in \mathcal{P}\}$; so $\text{cl}(Pf)$ is the smallest invariant (under multiplication by $z$) closed subspace containing $f$.

Recall that an invariant subspace of $X$ is defined as a closed subspace $E$ of $X$ such that $zf \in E$ whenever $f \in E$. A function $f \in X$ is said to be cyclic in $X$ if $\text{cl}(Pf) = X$. This is equivalent to the fact that there exists a sequence $\{P_n\}_n$ of polynomials such that $P_nf \to 1$ as $n \to \infty$ in the topology of $X$. Thus, a function $f \in X$ is cyclic if it generates $X$ as an invariant subspace, that is, the smallest invariant subspace of $X$ containing the function is the whole space.

By Beurling’s theorem mentioned previously, a function in $H^q (0 < q < 1)$ is cyclic if and only if it is outer. Using the following result of Mochizuki, we can easily obtain the analogous characterization of cyclic functions in $N^p$.

**Theorem 2.1** ([16, Theorem 4]). Let $\mathcal{M}$ be a closed ideal in $N^p$ which is not identically 0. Then there is a unique (modulo constants) inner function $I$ such that $\mathcal{M} = IN^p$.

**Lemma 2.2.** A closed subspace $E$ of $N^p$ is invariant if and only if it is an ideal.

**Proof.** The proof is routine, by using the fact that $N^p$ is a topological algebra in which polynomials are dense (see [13, the proof of the assertion 2.3]), and hence may be omitted. □

As an immediate consequence of Theorem 2.1 and Lemma 2.2, we obtain the following $N^p$-analogue of Beurling’s theorem on invariant subspaces of the Hardy space.

**Theorem 2.3.** A closed subspace $E$ of $N^p$ is invariant if and only if it has the form $IN^p$ for some inner function $I$.

**Remark 2.** By [19, Theorem 2] and Lemma 2.2 with $N^+$ instead of $N^p$, it follows that Theorem 2.3 is also true for the Smirnov class $N^+$.

Theorem 2.3 shows that there is a one-to-one correspondence between inner functions and invariant subspaces of $N^p$; so each invariant subspace of $N^p$ being of the form $IN^p$, where $I$ is an inner function.

The approximative version of Theorem 2.3 is given as follows.

**Theorem 2.4** ([13, the assertion 2.3 on p. 99]). Let $f \in N^p$ with the factorization $f = \text{BSF} = IF$ given by Theorem B, and let $\mathcal{P}$ denote the space of all polynomials over $\mathbb{C}$. Then the set $\text{BSN}^p = IN^p = \{Ig : g \in N^p\}$ becomes the closure of $\mathcal{P}f := \{Pf : P \in \mathcal{P}\}$ in $N^p$ with respect to the metric topology $\rho_p$. 
Clearly, Theorem 2.4 can be formulated in terms of cyclic functions in \( N^p \) as follows.

**Theorem 2.5.** A function \( f \in N^p \) is cyclic in \( N^p \) if and only if \( f \) is an outer function.

**Remark 3.** In [12] Matsugu characterized invariant subspaces of the space \((\log^+)^p(\mathbb{T})\) \((1 \leq p < \infty)\) consisting of all measurable functions \( f \) on the circle \( \mathbb{T} \) such that

\[
\|f\| := \int_0^{2\pi} \left( \log(1 + |f(e^{i\theta})|) \right)^p \frac{d\theta}{2\pi} < +\infty.
\]

As noticed in [12], it can be easily shown that \((\log^+)^p(\mathbb{T})\) is an \( F \)-space with respect to the metric \( \tilde{\rho}_p(f, g) = \|f - g\|, \ f, g \in (\log^+)^p(\mathbb{T}) \). Furthermore, it follows by the Riesz Uniqueness Theorem that the Privalov space \( N^p \) can be identified with the space \( N^p \) of all boundary functions of \( N^p \), i.e.,

\[
N^p = \{f^* : f \in N^p\}.
\]

Then \( N^p(\equiv N^p) \) is a closed subspace of \((\log^+)^p(\mathbb{T})\) (cf. [23, Theorem 1] for \( p = 1 \) and [21, Theorem 4.2] for \( p > 1 \)). By using the classical result about invariant subspaces in \( L^2(\mathbb{T}) \) ([8, Theorems 2 and 3]; [12, Theorem A]), Matsugu [12, Theorems 1 and 2] described the set of all invariant subspaces of \((\log^+)^p(\mathbb{T})\).

These results immediately yield our Theorem 2.3 (extended for \( p = 1 \) with \( N^1 \equiv N^+ \)).

### 3. Fréchet envelope and topological dual of \( N^p \)

In connection with the spaces \( N^p (1 < p < \infty) \), Stoll in [21] also studied the spaces \( F^q (0 < q < \infty) \) with the notation \( F_{1/q} \) in [21], consisting of those functions \( f \) holomorphic on \( \mathbb{D} \) for which

\[
\lim_{r \to 1} (1 - r)^{1/q} \log^+ M_\infty(r, f) = 0,
\]

where

\[
M_\infty(r, f) = \max_{|z| \leq r} |f(z)|.
\]

Here, as always in the sequel, we will need some Stoll’s results concerning the spaces \( F^q \) only with \( 1 < q < \infty \), and hence we will assume that \( q = p > 1 \) be any fixed number.

**Theorem 3.1** ([21, Theorem 2.2]). Suppose that \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) is a holomorphic function on \( \mathbb{D} \). Then the following statements are equivalent.

(a) \( f \in F^p \).

(b) There exists a sequence \( \{c_n\}_n \) of positive real numbers with \( c_n \to 0 \) such that

\[
|a_n| \leq \exp \left( c_n n^{1/(p+1)} \right), \quad n = 0, 1, 2, \ldots.
\]
For any $c > 0$,

$$\|f\|_{p,c} := \sum_{n=0}^{\infty} |a_n| \exp \left( -cn^{1/(p+1)} \right) < \infty.$$ \hfill (3.1)

Note that in view of Theorem 3.1 ((a)$\Leftrightarrow$(c)), it is defined by (3.1) the family of seminorms $\{\|\cdot\|_{p,c}\}_{c>0}$ on $F^p$.

Moreover, Stoll defined the family of seminorms $\{\|\cdot\|_{j,p,c}\}_{c>0}$ on $F^p$ given as

$$\|f\|_{j,p,c} = \int_0^1 \exp \left( -c(1-r)^{-1/p} \right) M_p(r,f) \, dr, \quad f \in F^p,$$ \hfill (3.2)

where

$$M_p(r,f) = \left( \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p}.$$ 

By [21, Proposition 3.1], $\{\|\cdot\|_{p,c}\}_{c>0}$ and $\{\|\cdot\|_{j,p,c}\}_{c>0}$ are equivalent families of seminorms. More precisely, Stoll proved the following result.

**Theorem 3.2** ([21, Proposition 3.1]). For each $c > 0$, there is a constant $A$ depending only on $p$ and $c$, such that

$$\|f\|_{p,c} \leq \|f\|_{p,c_1} \quad \text{and} \quad \|f\|_{p,c} \leq A\|f\|_{p,c_2},$$

with $c_1 = e^{p/(p+1)}$ and $c_2 = \left( \frac{2}{1} \right)^{p/(p+1)}$.

Recall that a locally convex $F$-space is called a Fréchet space, and a Fréchet algebra is a Fréchet space that is an algebra in which multiplication is continuous.

**Theorem 3.3** ([21, Theorem 3.2]). The space $F^p$ with the topology given by the family of seminorms $\{\|\cdot\|_{p,c}\}_{c>0}$, or $\{\|\cdot\|_{p,c}\}_{c>0}$, is a countably normed Fréchet algebra. Moreover,

$$\|fg\|_{p,c} \leq \|f\|_{p,c'} \|g\|_{p,c'} \quad \text{for all} \quad f, g \in F^p,$$

where $c' = c \cdot 2^{-p/(p+1)}$. Furthermore, if $f \in F^p$, then $f_r \to f$ as $r \to 1$ in the topology of $F^p$ where $f_r(z) = f(rz)$ with $z \in \mathbb{D}$ and $0 < r < 1$.

For our purposes the most important connection between spaces $N^p$ and $F^p$ is given by the following result.

**Theorem 3.4** ([21, Theorem 4.3]). For any fixed $p > 1$ the following assertions hold.

(a) $N^p$ is a dense subspace of $F^p$.

(b) The topology on $F^p$ defined by the family of seminorms (3.1) or (3.2) is weaker than the topology on $N^p$ given by the metric $p_p$ defined by (1.1).
For each \( q > p \) there exists a function \( f \in N^p \) such that
\[
\limsup_{r \to 1} \frac{1}{q} \log M \infty(r, f_q) > 0,
\]
i.e., \( N^p \) is not contained in \( F^q \) for none \( q > p \).

**Remark 4.** Recall that the spaces \( F^p \) have also been studied independently by Zayed ([25] and [26]); many of the results in [24] parallel those of Stoll in [21], albeit in a more general setting. For \( p = 1 \), the space \( F_1 \) has been denoted by \( F^+ \) and has been studied by Yanagihara in [24] and [23]. It was shown in [24] and [23] that \( F^+ \) is actually the containing Fréchet space for \( N^+ \), i.e., \( N^+ \) with the initial topology embeds densely into \( F^+ \), under the natural inclusion, and \( F^+ \) and \( N^+ \) have the same topological duals.

Observe that the space \( F^p \) topologised by the family of seminorms \( \{ \| \cdot \|_{p, c} \}_{c > 0} \) given by (3.1) is metrizable by the metric \( d_p \) defined as
\[
(3.3) \quad d_p(f, g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\| f - g \|_{p, 1/n^{p/(p+1)}}}{1 + \| f - g \|_{p, 1/n^{p/(p+1)}}}, \quad f, g \in F^p.
\]
The following result describes the topological dual of the space \( (F^p, d_p) \).

**Theorem 3.5** ([21, Theorem 3.3]). If \( \gamma \) is a continuous linear functional on \( F^p \), then there exists a sequence \( \{ \gamma_n \}_n \) of complex numbers with \( \gamma_n = O(\exp\left(-cn^{1/(p+1)}\right)) \) for some \( c > 0 \), such that
\[
(3.4) \quad \gamma(f) = \sum_{n=0}^{\infty} a_n \gamma_n,
\]
where \( f(z) = \sum_{n=0}^{\infty} a_n z^n \in F^p \), with convergence being absolute. Conversely, if \( \{ \gamma_n \} \) is a sequence of complex numbers for which
\[
\gamma_n = O\left(\exp\left(-cn^{1/(p+1)}\right)\right),
\]
then (3.4) defines a continuous linear functional on \( F^p \).

Let us recall that if \( X = (X, \tau) \) is an \( F \)-space whose topological dual (the set of all continuous linear functionals on \( X \)) \( X^\tau \) separates the points of \( X \), then its Fréchet envelope \( X \) is defined to be the completion of the space \( (X, \tau^\tau) \), where \( \tau^\tau \) is the strongest locally convex (necessarily metrizable) topology on \( X \) that is weaker than \( \tau \). In fact, it is known that \( \tau^\tau \) is equal to the Mackey topology of the dual pair \( (X, X^\tau) \), i.e., to the unique maximal locally convex topology on \( X \) for which \( X \) still has dual space \( X^\tau \) (see [20, Theorem 1]). For each metrizable locally convex topology \( \tau \) on \( X \), \( (X, \tau) \) is a Mackey space, i.e., \( \tau \) coincides with the Mackey topology of the dual pair \( (X, X^\tau) \) (see [10, Corollary 22.3, p. 210]).

Eoff ([5, the proof of Theorem 4.2]) showed that the topology of \( F^p \), \( p > 1 \), (resp., \( F^1 = F^+ \)) is stronger than that of the Fréchet envelope of \( N^p \) (resp., \( N^+ \)). As an immediate consequence of this result, we obtain the following statements.
Theorem 3.6 ([5, Theorem 4.2, the case \( p > 1 \)]). For each \( p > 1 \), \( F^p \) is the Fréchet envelope of \( N^p \).

Theorem 3.7 ([14, Theorem 2]). The spaces \( N^p \) and \( F^p \) have the same dual spaces in the sense that every continuous linear functional on \( F^p \) (given by (3.4)) restricts to one on \( N^p \), and every continuous linear functional on \( N^p \) extends continuously to one on \( F^p \).

Remark 5. Theorem 3.7 is proved in [14] directly, by using the characterization of multipliers from \( N^q \) into \( H^\infty \) ([14, Theorem 1]). The dual space of the Smirnov class \( N^+ \) is completely described by Yanagihara in [23] and McCarthy in [15].

The following result establishes the fact that the dual of \( N^p \) contains many elements.

Theorem 3.8. For any fixed \( \xi \in \mathbb{D} \) and all \( k = 0, 1, 2 \ldots \), the functional \( \delta^{(k)}_\xi \) defined as \( \delta^{(k)}_\xi(f) = f^{(k)}(\xi) \), \( f \in N^p \), is a continuous linear functional on \( N^p \). In particular, for \( k = 0 \) every point evaluation \( \delta_\xi \) defined as \( \delta_\xi(f) = f(\xi) \), \( f \in N^p \), is a multiplicative continuous linear functional on \( N^p \).

Proof. It is easy to verify that for a fixed \( \xi \in \mathbb{D} \), the sequence \( \{\gamma_n\}_n : = \{\xi^n\}_n \) satisfies the condition from Theorem 3.5 for any \( p > 1 \), and hence it generates by (3.4) the continuous linear functional \( \gamma \) such that \( \gamma(f) = f(\xi) = \delta_\xi(f) \) for \( f \in F^p \). Obviously, \( \delta_\xi \) is multiplicative.

Similarly, for fixed \( \xi \in \mathbb{D} \) and \( k \in \mathbb{N} \), the sequence \( \{(\xi^n)^{(k)}\}_n \) with terms

\[
(\xi^n)^{(k)} = \begin{cases} 
  n(n-1) \cdots (n-k+1)\xi^{n-k} & \text{for } n \geq k \\
  0 & \text{for } n < k
\end{cases}
\]

also satisfies the growth estimate from Theorem 3.5 for any \( p > 1 \), and thus by Theorem 3.7, every functional \( \delta^{(k)}_\xi = f^{(k)}(\xi) \) is a continuous linear functional on \( N^p \). \( \square \)

4. Invariant subspaces and linear functionals on \( N^p \)

Recall that all the notions, statements and their proofs in this section are analogous to those for the spaces \( H^q \) with \( 0 < q < 1 \) given in [4, Section 6].

By Theorem 2.3, each invariant subspace of \( N^p \) being of the form \( IN^p \), where \( I \) is an inner function. Thus there is a one-to-one correspondence between inner functions and invariant subspaces of \( N^p \). Since \( N^p \) is not locally convex, it is possible that some closed ideals are weakly dense.

Recall that we may introduce the weak topology on \( N^p \) in the usual way. The basic weak neighborhoods of zero are defined by

\[
V(\varphi_1, \ldots, \varphi_n; \varepsilon) = \{ f \in N^p : |\varphi_i(f)| < \varepsilon, i = 1, \ldots, n \},
\]

where \( \varepsilon > 0 \) and \( n \in \mathbb{N} \) are arbitrary, and \( \varphi_1, \ldots, \varphi_n \) are arbitrary continuous linear functionals on \( N^p \).
For a subset $E$ of $N^p$, denote by $[E]$ the closure of $E$ in $N^p$, by $[E]_w$ the closure of $E$ in the weak topology of $N^p$, and by $[E]_{F^p}$ the closure of $E$ in $F^p$ with respect to the topology of $F^p$ given by the family of seminorms (3.1) or (3.2). Denote by $(N^p)^*$ the topological dual of $N^p$, i.e., the set of all continuous linear functional on $N^p$ with respect to the metric topology $p$. It is easy to check that a weakly closed subset of $N^p$ is also closed in $N^p$ with respect to the metric topology $p$; that is, the weak topology is weaker than the initial $N^p$ topology. The weak topology is always locally convex. Clearly, by Theorem 3.8, $N^p$ has the point separation property. This means that for every function $f \in N^p$ which is not identically equal to zero, there is a non-trivial continuous linear functional $\varphi$ on $N^p$ such that $\varphi(f) \neq 0$, and hence the weak topology is also Hausdorff.

For vector subspaces of $N^p$ there is another description of the weak closure that we shall use (see [4, p. 54]). The weak topology of $N^p$ is locally convex, and hence by [10, p. 154, Corollary 17.3], a linear functional on $N^p$ is weakly continuous if and only if it is continuous with respect to the metric topology $p$ given by (3.3). Thus $[E]_w$ consists of all those functions $f \in N^p$ which cannot be separated from $E$ by a linear functional in $(N^p)^*$. Namely, by [10, p. 154, 17.1], in any locally convex topological vector space a convex set $E$ is closed if and only if it is weakly closed, or equivalently, if and only if each point not in it can be separated from it by a linear functional. In the case when $E$ is a vector subspace, this becomes: $f \in [E]_w$ if and only if $\phi(f) = 0$ for every functional $\phi \in E^\perp$, where $E^\perp$ is the set of all $\phi \in (N^p)^*$ which vanish on $E$. Thus we have the following result.

**Lemma 4.1.** A closed subspace of $N^p$ has the separation property if and only if it is weakly closed.

**Lemma 4.2.** The weak closure of any invariant subspace $IN^p$ of $N^p$ is again an invariant subspace.

**Proof.** We follow the proof of Lemma 7 in [4, pp. 54–55]. Suppose that $f \in [IN^p]_w$, where $I$ is an inner function. We must prove that a function $zf(z)$ is in the weak closure of $IN^p$. According to the argument preceding Lemma 4.1, this is equivalent to the fact that $\phi(zf) = 0$ for each $\phi \in (IN^p)^\perp$. Now define the linear functional $\phi_1$ on $N^p$ as

$$\phi_1(f) = \phi(zf), \quad f \in N^p.$$  

Since $\phi_1(Ig) = \phi(Izg) = 0$ for each $g \in N^p$, we see that $\phi_1 \in (IN^p)^\perp$. From this and the assumption $f \in [IN^p]_w$ it follows that $\phi_1(f) = 0$, and hence $\phi(zf) = 0$. This concludes the proof. $\square$

**Theorem 4.3.** For any inner function $I$ there is a unique (modulo constants) inner function $I_w$ such that

$$[IN^p]_w = I_w N^p.$$  

Furthermore, $J = I/I_w$ is again an inner function.
Proof. By Lemma 4.2, \([IN^p]_w\) is an invariant subspace of \(IN^p\) for any inner function \(I\). Thus by Theorem 2.3, there is an inner function \(I_w\) for which
\[
[IN^p]_w = I_w N^p.
\]
The uniqueness (modulo constants) of a function \(I_w\) follows immediately from the uniqueness of the factorization of \(N^p\) functions (Theorem B). Finally, since the weak topology is weaker than the metric topology \(\rho_p\) on \(N^p\), we conclude that \(IN^p = [IN^p] \subseteq [IN^p]_w\). Therefore, it follows by (4.1) that \(IN^p \subseteq I_w N^p\), and hence \(J = I/I_w\) is an inner function. This completes the proof. \(\Box\)

Lemma 4.4. If \(E\) is a vector subspace of \(N^p\), then
\[
[E]_w = [E]_{F^p} \cap N^p.
\]
Furthermore, \([E] \subseteq [E]_w\) and \([E]_w\) is closed in \(N^p\).

Proof. As noticed above, \([E]_w\) consists of all those functions \(f \in N^p\) that cannot be separated from \(E\) by a linear functional on the dual space \((N^p)^*\) of \(N^p\). On the other hand, by Theorem 3.7, the spaces \(N^p\) and \(F^p\) have the same dual spaces, and hence \([E]_{F^p} \cap N^p\) consists of all those \(f \in N^p\) that cannot be separated from it by a linear functional. Therefore, \([E]_w = [E]_{F^p} \cap N^p\). Further, \([E] \subseteq [E]_w\) is immediate from the fact that the weak topology is weaker than the metric topology \(\rho_p\) on \(N^p\). Finally, it remains to show that \([E]_w\) is closed in \(N^p\). Suppose \(\{f_n\}\) is a sequence in \([E]_w\) such that \(f_n \to f\) in \(N^p\) as \(n \to \infty\) for some \(f \in N^p\). We must show that \(f \in [E]_w\). As \(\varphi(f_n) \to \varphi(f)\) as \(n \to \infty\) for each \(\varphi \in (N^p)^*\), this means that \(f_n \to f\) weakly in \(N^p\) as \(n \to \infty\). Hence, \(f \in [E]_w = [E]_w\), as desired. \(\Box\)

Corollary 4.5. If \(M\) is an ideal in \(N^p\), then both \([M]_w\) and \([M]_w\) are also ideals in \(N^p\).

Proof. Clearly, since the multiplication is continuous in \(N^p\), if \(M\) is an ideal in \(N^p\), so is \([M]_w\). Moreover, since the multiplication is also continuous in \(F^p\) and by Theorem 3.4(a), \(N^p\) is a dense subspace of \(F^p\), it follows that \([M]_{F^p}\) is an ideal in \(F^p\). This fact together with (4.2) yields that \([M]_w\) is an ideal in \(N^p\). \(\Box\)

Although the weak topology of \(N^p\) is not metrizable, the following corollary shows that, for vector subspaces of \(N^p\) the weak closure can be formed by adjoining limits of sequences.

Corollary 4.6. If \(E\) is a vector subspace of \(N^p\), then each function in the weak closure of \(E\) is the weak limit of a sequence of elements of \(E\).

Proof. Assume \(f \in [E]_w\). By Lemma 4.4, \(f \in [E]_{F^p}\) and so there is a sequence \(\{f_n\}_n\) in \(E\) such that \(f_n \to f\) in the topology of \(F^p\). Hence \(f_n \to f\) weakly in \(F^p\), which is the same thing as \(f_n \to f\) weakly in \(N^p\). \(\Box\)

Corollary 4.7. Let \(I\) be an inner function. The following three statements are equivalent.
(i) \([INP]_w = NP\).
(ii) \(INP\) is dense in \(FP\).
(iii) \(IFP\) is dense in \(FP\).

Proof. (i)\(\Leftrightarrow\)(ii) follows immediately from (4.2) of Lemma 4.4 and the fact that \(NP\) is a dense subspace of \(FP\).

(ii)\(\Rightarrow\)(iii) is obvious, in view of the fact that \(INP \subset IFP\).

(iii)\(\Rightarrow\)(ii). If \(f \in FP\) is arbitrary, then by (iii) there is a sequence \(\{f_n\}_n\) in \(FP\) such that \(If_n \to f\) in \(FP\). Since \(NP\) is a dense subspace of \(FP\), there exists a sequence \(\{g_n\}_n\) in \(NP\) such that \(f_n - g_n \to 0\) in \(FP\). From this fact and the continuity of multiplication in \(FP\), it follows that \(If_n - Ig_n \to 0\) in \(FP\), and hence \(Ig_n \to f\) in \(FP\). Thus \(f\) belongs to \([INP]_{FP}\), and so (ii) is satisfied. \(\square\)

If \(T\) is a continuous linear operator on \(NP\), then the adjoint operator \(T^*\) on \((NP)^*\) is defined by the usual formula

\[ (T^* \varphi)(f) = \varphi(Tf), \quad \varphi \in (NP)^*, \ f \in NP. \]

Lemma 4.8. Every continuous linear operator \(T\) on \(NP\) is weakly continuous.

Proof. Assume \(f_n \to f\) weakly, that is, \(\psi(f_n) \to \psi(f)\) for all \(\psi \in (NP)^*\). Then we have

\[ \psi(Tf_n) = (T^* \psi)(f_n) \to (T^* \psi)(f) = \psi(Tf), \]

whence it follows that \(Tf_n \to Tf\) weakly, as desired. \(\square\)

Lemma 4.9. Let \(I\) and \(J\) be arbitrary inner functions. Then

\[ [I_w JNP] \subseteq [IJNP]_w, \]

where \(I_w\) is the inner function such that \([INP]_w = I_w NP\).

Proof. For a fixed function \(g \in NP\), define the linear operator \(T_g\) on \(NP\) as

\[ T_g(f) = Jgf, \quad f \in NP. \]

Since the multiplication in \(NP\) is continuous, it follows that \(T_g\) is a continuous linear operator, and hence by Lemma 4.8, \(T_g\) is weakly continuous. Therefore, if we choose a sequence \(\{f_n\}_n\) in \(NP\) such that \(If_n \to I_w\) weakly, then \(T_g(If_n) \to T_g(I_w)\) weakly, i.e., \(IJ(gf_n) \to JI_wg\) weakly. Hence, \(JI_wg \in [IJNP]_w\), which implies \(I_w JNP \subseteq [IJNP]_w\). Therefore, in view of the fact that by Lemma 4.4, \([IJNP]_w = [IJNP]_w\), we obtain

\[ [I_w JNP] \subseteq [IJNP]_w = [IJNP]_w. \]

This yields a desired inclusion. \(\square\)

Recall that a closed subspace \(E\) of \(NP\) will be said to have the separation property if each function \(f \in NP\) that is not in \(E\) can be separated from \(E\) by a continuous linear functional on \(NP\); that is, if there exists a continuous linear functional \(\varphi\) on \(NP\) such that \(\varphi(E) = 0\) but \(\varphi(f) \neq 0\). We will say that \(E\) has the Hahn-Banach property if each continuous linear functional on \(E\) can be extended to a continuous linear functional on the whole space \(NP\) (we do not
require that the norm of the functional be preserved in the extension). We are now ready to state the following result.

**Theorem 4.10.** Let \( I \) be an inner function. Then the following assertions hold.

(i) If \( I \) is a finite Blaschke product, then \( IN^p \) has both the separation property and the Hahn-Banach property.

(ii) If \( I \) is any Blaschke product, then \( IN^p \) has the separation property.

(iii) If the space \( IN^p \) does not have the separation property and if \( J \) is any inner function, then also \( IJN^p \) does not have the separation property.

**Proof.** (i) If \( I \) is a finite Blaschke product, then the space \( IN^p \) has a finite co-dimension. Hence, the assertion (i) follows immediately from the fact that in any topological vector space a closed subspace of finite co-dimension has both the separation and the Hahn-Banach properties.

(ii) If \( I \) is a Blaschke product and if \( h \not\in IN^p \), then either \( h \) does not vanish at one of the zeros of \( I \) or \( h \) has a zero of lower multiplicity. In the first case, let \( \xi \in \mathbb{D} \) be a zero of \( I \) such that \( h(\xi) \neq 0 \). For such a \( \xi \) consider the point evaluation \( \delta_\xi \) on \( N^p \) defined as \( \delta_\xi(f) = f(\xi), \; f \in N^p \). Then by Theorem 3.8, \( \delta_\xi \) is a continuous linear functional on \( N^p \). Obviously, \( \delta_\xi(g) = 0 \) for all \( g \in IN^p \) but \( \delta_\xi(h) = h(\xi) \neq 0 \), and hence \( \delta_\xi \) is a desired separating linear functional. In the second case, suppose that \( \xi \in \mathbb{D} \) is a zero of \( I \) of the multiplicity \( k \geq 1 \), and hence of the multiplicity less than \( k \) for \( h \), that is, \( h^{(k)}(\xi) \neq 0 \). Now consider a linear functional on \( N^p \) by \( \delta_\xi^{(k)}(f) = f^{(k)}(\xi), \; f \in N^p \). By Theorem 3.8, \( \delta_\xi^{(k)} \) is also a continuous linear functional on \( N^p \). Clearly, \( \delta_\xi^{(k)}(g) = 0 \) for all \( g \in IN^p \) while \( \delta_\xi^{(k)}(h) = h^{(k)}(\xi) \neq 0 \), and thus \( \delta_\xi^{(k)} \) is a desired separating linear functional.

(iii) Suppose that \( IJN^p \) has the separation property for some inner function \( J \), and so by Lemma 4.1,

\[
[IJN^p]_w = IJN^p.
\]

By the assumption, the space \( IN^p \) does not have the separation property, and hence by Lemma 4.1, it is not weakly closed. This means that \( I \neq I_w \) (modulo constants), where \( I_w \) is an inner function associated to \( I \) (see Theorem 4.3) such that

\[
[IN^p]_w = I_w N^p.
\]

Since \( IN^p \) is a weakly dense subspace of \( I_w N^p \), by (4.3) and Lemma 4.9, we have

\[
IJN^p = [IJN^p]_w \supseteq [I_w JN^p]_w \supseteq I_w JN^p \supseteq IJN^p.
\]

From the above inclusions we see that must be \( I_w JN^p = IJN^p \). Thus, \( I_w J \) is in \( IJN^p \), and hence there is an inner function \( I_1 \) such that \( I_w J = IJI_1 \). It follows that \( I_w = I_{II_1} \), and hence \( I_w / I = I_1 \) is an inner function. On the other hand, by Theorem 4.3, \( I / I_w \) is also an inner function. Therefore, we infer that
I = I_w (modulo constants). This contradiction yields that \( IJN^p \) does not have the separation property. This completes the proof of the theorem. \( \square \)

5. Weakly dense ideals in \( N^p \)

Beurling’s invariant subspaces theorem holds for each Hardy space \( H^q, 0 < q < \infty \), the Smirnov class \( N^+ \) (see Remark 2), and for all Privalov spaces \( N^p \) with \( 1 < p < \infty \) (Theorem 2.3). This means that if \( E \) is a closed subspace of one of these spaces, denoting by \( X \), and if \( E \) is invariant under multiplication by \( z \), then \( E = IX \) for some \( X \)-inner function \( I \).

Since the spaces \( N^p \) are not locally convex, it is possible that some closed ideals are weakly dense (dense in the weak topology of \( N^p \)). In this section we give a construction of such ideals for any \( p > 1 \).

Recall that an inner function \( I \) is called a weak outer function in \( N^p \) if \( IN^p \) is closed in the weak topology of \( N^p \); that is, if \( [IN^p]w = N^p \). This means that \( Iw = 1 \), where \( Iw \) is an inner function as in Theorem 4.3 such that \( [IN^p]w = IwN^p \).

In other words, an inner function \( I \) is a weak outer function in \( N^p \) if and only if the principal ideal \( IN^p \) is weakly dense. The main result of this paper (Theorem 5.5) gives a large class of positive singular measures \( \mu \), depending on \( p \), such that associated singular inner functions are weak outer functions in the space \( N^p \).

By Theorem 3.2, \( \{\| \cdot \|_{p,c} \}_{c>0} \) and \( \{\| \cdot \|_{p,c} \}_{c>0} \) defined by (3.1) and (3.2), respectively, are equivalent families of seminorms. For simplicity, in this section we write \( \| \cdot \|_c \) instead of \( \| \cdot \|_{p,c} \).

For every \( c > 0 \), we define the function \( \| \cdot \|_c^\sim \) on \( F^p \) by

\[
(5.1) \quad \| f \|_c^\sim = \left( \frac{1}{\pi} \int_D |f(re^{i\theta})|^2 \exp \left( -\frac{c}{(1-r)^{1/p}} \right) rdrd\theta \right)^{1/2}, \quad f \in F^p.
\]

It follows easily by Minkowski’s inequality that \( \| \cdot \|_c^\sim \) satisfies the triangle inequality for any \( c > 0 \), and hence \( \| \cdot \|_c^\sim \) is a norm on \( F^p \).

Lemma 5.1. Let \( p > 1 \) and \( c > 0 \) be any fixed. Then there exist positive constants \( A \) and \( B \), depending only on \( p \) and \( c \), such that

\[
(5.2) \quad \| f \|_c^\sim \leq A\| f \|_{c_1}, \quad f \in F^p,
\]

and

\[
(5.3) \quad \| f \|_c \leq B\| f \|_{c_2}, \quad f \in F^p,
\]

where \( c_1 = \frac{c^{(p+1)}}{2} \) and \( c_2 = \left( \frac{c^{(p+1)}}{2^{p+1}} \right)^{1/p} \). Hence, the families \( \{\| \cdot \|_c \}_{c>0} \) and \( \{\| \cdot \|_c^\sim \}_{c>0} \) induce the same topology on \( F^p \).
Proof. Suppose that $f \in F^p$, $f \neq 0$, with Taylor expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on $D$. Given any constant $c > 0$ put $c_2 = \left( \frac{p+1}{6p(p+1)} \right)^{1/p}$. Then by [21, the first inequality on p. 146], there is $m \in \mathbb{N}$ so that for all $n \geq m$ holds

\begin{equation}
\int_0^1 r^n \exp \left( -\frac{c_2}{(1-r)^{1/p}} \right) \, dr \geq \exp \left( -6(c_2)^{p/(p+1)} n^{1/(p+1)} \right). \tag{5.4}
\end{equation}

Assuming that $L \leq 1$ is a positive constant such that

\begin{equation}
\int_0^1 r^{2n+1} \exp \left( -\frac{c_2}{(1-r)^{1/p}} \right) \, dr \geq L \exp \left( -6(c_2)^{p/(p+1)} (3n)^{1/(p+1)} \right)
\end{equation}

for all $n = 0, 1, \ldots, [m/2] - 1$, then by (5.1) and (5.4) for each $c_2 > 0$ we have

\begin{equation}
(\|f\|_{\mathcal{C}_c})^2 = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 f(\rho \cos \theta) f(\rho \sin \theta) \exp \left( -\frac{c_2}{(1-r)^{1/p}} \right) r \, dr \, d\theta
= 2 \sum_{n=0}^{\infty} |a_n|^2 \int_0^1 r^{2n+1} \exp \left( -\frac{c_2}{(1-r)^{1/p}} \right) \, dr
\geq 2 \left( L \sum_{n=0}^{[m/2]-1} |a_n|^2 \exp \left( -6(c_2)^{p/(p+1)} (3n)^{1/(p+1)} \right) \right.
\end{equation}

\begin{equation}
\left. + \sum_{n=[m/2]}^{\infty} |a_n|^2 \exp \left( -6(c_2)^{p/(p+1)} (2n + 1)^{1/(p+1)} \right) \right)
\geq 2L \left( \sum_{n=0}^{[m/2]-1} |a_n|^2 \exp \left( -6(c_2)^{p/(p+1)} (3n)^{1/(p+1)} \right) \right.
\end{equation}

\begin{equation}
\left. + \sum_{n=[m/2]}^{\infty} |a_n|^2 \exp \left( -6(c_2)^{p/(p+1)} (3n)^{1/(p+1)} \right) \right)
\geq 2L \sum_{n=0}^{\infty} |a_n|^2 \exp \left( -6(c_2)^{p/(p+1)} (3n)^{1/(p+1)} \right).
\end{equation}

By Cauchy-Schwarz inequality, for positive numbers $x_n, y_n$, with $n = 1, 2, \ldots, k$, holds

\begin{equation}
\left( \sum_{n=1}^k x_n y_n \right)^2 \leq \left( \sum_{n=1}^k x_n^2 \right) \left( \sum_{n=1}^k y_n^2 \right).
\end{equation}
The above inequality with $x_n = |a_n|$ and $y_n = \exp \left(-cn^{1/(p+1)}\right)$ as $k \to \infty$ yields
\[
(\|f\|_c)^2 = \left(\sum_{n=0}^{\infty} |a_n| \exp \left(-cn^{1/(p+1)}\right)\right)^2
\leq \left(\sum_{n=0}^{\infty} |a_n|^2 \exp \left(-cn^{1/(p+1)}\right)\right) \left(\sum_{n=0}^{\infty} \exp \left(-cn^{1/(p+1)}\right)\right)
= K \sum_{n=0}^{\infty} |a_n|^2 \exp \left(-cn^{1/(p+1)}\right)
= K \sum_{n=0}^{\infty} |a_n|^2 \exp \left(-6(c_2)^{p/(p+1)}(3n)^{1/(p+1)}\right),
\]
where $0 < K < +\infty$. The inequalities (5.5) and (5.6) immediately yield
\[
(\|f\|_{c_2})^2 \leq \frac{2L}{K} (\|f\|_c)^2.
\]
On the other hand, we have
\[
(\|f\|_{c_2})^2 = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 f(re^{i\theta}) \overline{f(re^{i\theta})} \exp \left(-\frac{c}{(1-r)^{1/p}}\right) rdrd\theta
= 2 \sum_{n=0}^{\infty} |a_n|^2 \int_0^1 r^{2n+1} \exp \left(-\frac{c}{(1-r)^{1/p}}\right) dr.
\]
Since by [21, the first inequality on p. 145 with $\beta = 1/p$], for $n = 0, 1, 2, \ldots$,
\[
r^n \exp \left(-\frac{c}{(1-r)^{1/p}}\right) \leq \exp \left(-c^{p/(p+1)}n^{1/(p+1)}\right), \quad 0 < r < 1,
\]
it follows that
\[
(\|f\|_{c_2})^2 \leq 2 \left(\sum_{n=0}^{\infty} |a_n| \exp \left(-c^{p/(p+1)}n^{1/(p+1)}\right)\right)^2
= 2 (\|f\|_{c_1})^2,
\]
with $c_1 = (c^{p/(p+1)})/2$. By setting $A = \sqrt{2}$, the above inequality yields (5.2). This concludes the proof. \hfill \Box

Recall that the \textit{modulus of continuity} $\omega_\mu$ of a finite Borel measure $\mu$ on the unit circle $T$ is defined by
\[
\omega_\mu(t) = \sup_{|L| \leq t} \mu(L) \quad (t > 0),
\]
where the supremum is taken on every subarcs $L$ of $T$ whose normalized Lebesgue measure (length) $|L| \leq t$.
Observe that from the condition $\omega_\mu(t) = O(t)$ it follows that the measure $\mu$ is \textit{absolutely continuous} with respect to the normalized Lebesgue measure $|\cdot|$.
on $\mathbb{T}$. Furthermore, it is known that there are positive singular measures with prescribed modulus of continuity (of higher order than $O(t)$); for example, with the modulus of continuity $\omega_\mu(t) = O(t \log \frac{1}{t})$. It is pointed out in [4] that such a measure can be constructed as the Lebesgue function over a Cantor set with variable ratio of dissection. Another example is based on Riesz products (see [3]) that we use here in order to obtain the following result.

**Lemma 5.2.** For any positive number $\alpha$ such that $0 < \alpha < 1$ there exists a positive singular Borel measure $\mu$ on $\mathbb{T}$ with the modulus of continuity

$$\omega_\mu(t) = O(t^\alpha) \quad (t > 0).$$

**Proof.** We will construct a continuous nondecreasing function $F$ on $[0, 2\pi]$ which generates measure $\mu$ such that

$$\mu([x, y]) = F(x) - F(y)$$

for any segment $[x, y]$ with $0 \leq y \leq x \leq 2\pi$, and $\omega_\mu(t)$ satisfies (5.7). Recall that the modulus of continuity $\omega(t; F)$ of a continuous complex-valued function $F$ defined on $[0, 2\pi]$ is given by

$$\omega(t; F) = \sup_{|x - y| \leq t} |F(x) - F(y)|.$$

Moreover, $F$ belongs to the Lipschitz class $\Lambda_\alpha$ with $0 < \alpha \leq 1$ if $\omega(t; F) = O(t^\alpha)$ as $t \to 0$.

Given $0 < \alpha < 1$, consider a constant sequence $\{a_j\}_j$ with $a_j = 1$ for all $j = 1, 2, \ldots$, and a sequence $\{n_j\}_j$ with $n_j = q^j$ for all $j = 1, 2, \ldots$, where $q > 3$ is a positive integer for which $q^{1-\alpha} \geq 2$. Now define a sequence $\{p_k(t)\}_k$ of trigonometric polynomials as a Riesz product

$$p_k(t) = \prod_{j=1}^k (1 + a_j \cos n_j t)$$

$$= 1 + \sum_{i=1}^{m_k} c_i \cos it, \quad k = 1, 2, \ldots,$$

with $m_k = n_1 + \cdots + n_k = \frac{q^{1-k}}{1-q}$ and suitable coefficients $c_i$. Since $n_{j+1}/n_j = q > 3$ for all $j = 1, 2, \ldots$, as noticed in [3, pp. 1264–1265], it follows by [18] and [27, p. 208] that

$$F(x) := \lim_{k \to \infty} \int_0^x p_k(t) \, dt$$

exists for all $x, 0 \leq x \leq 2\pi$, and $F$ is a continuous nondecreasing function on $[0, 2\pi]$. We say that the function $F$ is generated by a Riesz product. Further, we have

$$\prod_{j=1}^k (1 + |a_j|) = 2^k \leq q^{(1-\alpha)k} = (n_k)^{1-\alpha},$$
and hence sequences \(\{a_j\}_j\) and \(\{n_j\}_j\) satisfy the asymptotic condition (6) from [3, Theorem 1]. Then by the same theorem from [3], the function \(F\) belongs to the Lipschitz class \(\Lambda_n\), and so for a measure \(\mu\) defined by (5.8) we obtain

\[
\omega_{\mu}(t) = \omega(t; F) = \sup_{|x-y| \leq t} |F(x) - F(y)| = O(t^\alpha).
\]

Finally, since \(\sum_{j=1}^{\infty} a_j^2 = \infty\), by [3, Theorem A], we conclude that \(F\) is a singular function. Consequently, \(\mu\) is a singular Borel measure on \(\mathbb{T}\) whose modulus of continuity satisfies (5.7). This completes the proof. \(\square\)

We will need the following result in the proof of the main result.

**Lemma 5.3.** Let

\[
S_\mu(z) = \exp \left( - \int_0^{2\pi} H(z, e^{it}) \, d\mu(t) \right)
\]

be a singular inner function, where \(H(z, e^{it}) = (e^{it} + z)(e^{it} - z)^{-1}\), and let \(\mu\) be a positive singular Borel measure with the modulus of continuity \(\omega_{\mu}(t) = O(t^{p-1})\). Then

\[
|S_\mu(re^{i\theta})| \geq \exp \left( - \frac{C}{(1-r)^{1/p}} \right), \quad 0 \leq r < 1,
\]

for some constant \(C > 0\).

**Proof.** Obviously,

\[
-\log |S_\mu(re^{i\theta})| = \int_0^{2\pi} P(r, \theta - t) \, d\mu(t),
\]

with the Poisson kernel

\[
P(r, \theta - t) = \Re H(z, e^{it}) = \frac{1-r^2}{1 - 2r \cos(\theta - t) + r^2}, \quad z = re^{i\theta}.
\]

Since \(\sin x \geq 2x/\pi\) for each \(0 \leq x \leq \pi/2\), we have

\[
1 - 2r \cos(\theta - t) + r^2 = (1-r)^2 + 4r \sin^2 \frac{\theta - t}{2} \geq (1-r)^2 + \frac{4r}{\pi^2}(\theta - t)^2.
\]

As for \(r \geq 1/17\), \((372r/\pi^2)(\theta - t)^2 \geq 2(\theta - t)^2\), and since \(|\theta - t| \leq 2\pi\), the inequality \(91(1-r)^2 > 2(\theta - t)^2\) holds for \(r < 1/17\), in both cases it is obvious that

\[
2 \left( (1-r)^2 + (\theta - t)^2 \right) \leq 93 \left( (1-r)^2 + \frac{4r}{\pi^2}(\theta - t)^2 \right).
\]
Now from (5.11)-(5.13) we immediately obtain
\[ P(r, \theta) \leq \frac{2(1 - r)}{1 - 2\pi \cos(\theta - t) + r^2} \]
\[ \leq \frac{93(1 - r)}{(1 - r)^2 + (\theta - t)^2}, \quad z = re^{i\theta}. \]

If we put \( \delta = 2\pi/n \), then by (5.10), (5.11) and (5.14) and the assumption \( \omega_{\mu}(\delta) = O(\delta^{(p-1)/p}) \) of the lemma, we obtain
\[
\log |S_\mu(re^{i\theta})| \leq c\delta^{\frac{p-1}{p}} \sum_{k=0}^{n-1} \max_{k\delta \leq \theta \leq (k+1)\delta} \left( \frac{1 - r}{(1 - r)^2 + (\theta - t)^2} \right)
\]
\[
\leq c\delta^{\frac{p-1}{p}} \left( \frac{1}{1 - r} + \sum_{k=1}^{n-1} \frac{1 - r}{(1 - r)^2 + k^2\delta^2} \right)
\]
for a positive constant \( c \). Then assuming \( n \in \mathbb{N} \) such that \( 1/(1 - r) \leq n < 1 + 1/(1 - r) \) for such \( n \), we have \( 2\pi(1 - r)/(2 - r) < \delta \leq 2\pi(1 - r) \), and from the inequality (5.15) we obtain
\[
|S_\mu(re^{i\theta})| \geq \exp \left( -\frac{C}{(1 - r)^{1/p}} \right).
\]
This is the desired estimate (5.9). \( \square \)

**Lemma 5.4.** For any \( c > 0 \) denote by \( [E]_c \) the closure of a subspace \( E \) of \( N^p \) in the normed space \((F^p, \| \cdot \|_c)\). Then for arbitrary inner functions \( I \) and \( J \)
\[(5.16) \quad [IJN^p]_c = [I[N^p]]_c. \]
In particular, if \([JN^p]_c = N^p\), then
\[
[IJN^p]_c = [IN^p]_c.
\]
**Proof.** Since \( JN^p \subseteq [JN^p]_c \), we have \([IJN^p]_c \subseteq [I[JN^p]]_c \). Obviously, for any function \( h \in H^\infty \) with the norm \( \| h \|_\infty = \max_{z \in D} |h(z)| \) and each \( f \in N^p \) holds
\[
\| hf \|_c \leq \| h \|_\infty \| f \|_c.
\]
Therefore, for each \( f \) and \( g \) in \( N^p \) we obtain
\[
\|IJg - If\|_\infty \leq \|I\|_\infty \|Jg - f\|_\infty,
\]
and hence \([IJN^p]\) is an ideal in \( N^p \) that is weakly dense in \( N^p \). This concludes the proof. \( \square \)

According to Theorem 2.5, the principal ideal in \( N^p \) is dense in \( N^p \) if and only if it is generated by an outer function. Although we have been unable to give a complete characterization of weakly dense ideals in \( N^p \), the following theorem describes a large class of such ideals in \( N^p \).

**Theorem 5.5.** Let \( \mathcal{M} \) be a closed ideal in \( N^p \) that is weakly dense in \( N^p \). Then \( \mathcal{M} \) is a principal ideal generated by a singular inner function. Conversely, if \( S_\mu \) is a nontrivial singular inner function with the associated measure \( \mu \) whose modulus of continuity \( \omega_\mu \) satisfies
\[
\omega_\mu(t) = O\left(t^{\frac{p-1}{p}}\right), \quad t > 0,
\]
then the ideal \( S_\mu N^p \) is weakly dense in \( N^p \).

**Proof.** Let \( \mathcal{M} \) be a closed ideal with respect to the metric topology \( \rho_p \) of \( N^p \) that is weakly dense in \( N^p \). Then by Theorem 2.1, there is a unique (modulo constants) inner function \( I \) such that \( \mathcal{M} = IN^p \). Suppose that a function \( I \) vanishes in \( \mathbb{D} \), i.e., \( I(z) = B(z)S_\mu(z) \), where \( B \) is a nontrivial Blaschke factor of \( I \), and \( S_\mu \) is a singular inner factor of \( I \). Let \( \xi \) be an arbitrary zero of \( B \). Since by Theorem 3.8, the evaluation functional \( \delta_\xi \) defined as \( \delta_\xi(f) = f(\xi) \), \( f \in N^p \), is continuous, the assumption that \( \mathcal{M} \) is weakly dense implies that there is a sequence \( \{f_n\}_n \subset N^p \) such that \( \delta_\xi(BS_\mu f_n) \to \delta_\xi(1) \) as \( n \to \infty \). This means that \( B(\xi)S_\mu(\xi)f_n(\xi) \to 1 \) as \( n \to \infty \), what is impossible in view of the fact that \( B(\xi) = 0 \). This contradiction shows that \( I \) is a singular inner function.

Conversely, suppose that \( S_\mu \) is a singular inner function with the associated measure \( \mu \) whose modulus of continuity \( \omega_\mu(t) = O(t^{(p-1)/p}) \). By Theorem 3.3, \( F^p \) is a Fréchet algebra in which \( f_r \to f \) as \( r \to 1 \) for each \( f \in N^p \) with \( f_r(z) = f(rz), \ z \in \mathbb{D} \). Since each function \( f_r \) can be uniformly approximated on the closed disk \( \mathbb{D} : |z| \leq 1 \) by partial sums of its Taylor expansion, it follows that the space \( \mathcal{P} \) of polynomials is dense in \( F^p \). Hence, the density of \( S_\mu N^p \) in \( F^p \) is equivalent to the fact that the set \( \mathcal{P}S_\mu = \{PS_\mu : P \in \mathcal{P}\} \) is dense in \( F^p \). Since by Lemma 5.1, families of seminorms \( \{\|\cdot\|_c\}_{c>0} \) and \( \{\|\cdot\|_c\}_{c>0} \) given by (5.1) and (3.1), respectively, define the same topology on \( F^p \), it follows from (i)(ii) of Corollary 4.7 that the ideal \( S_\mu N^p \) is weakly dense in \( N^p \) if and only if it is dense in the normed spaces \( (F^p, \|\cdot\|_c)_{c>0} \) for each \( c > 0 \).

By (5.9) of Lemma 5.3, there is a constant \( C > 0 \) such that for the minimum modulus \( m(r) = \min_{|z|=r} |S_\mu(z)| \) of \( S_\mu \) holds
\[
m(r) \geq \exp\left(-\frac{C}{(1-r)^{1/p}}\right) \quad \text{for} \quad 0 \leq r < 1.
\]
uniformly on compact subsets of \( \mathbb{D} \), and hence \( p_n(z)S_\mu(z) \to 1 \) for each \( z \in \mathbb{D} \). By [11, Kap.1, Satz 1, p. 22] there holds
\[
\max_{|z|=r} |p_n(z)| \leq \max_{|z|=r} \frac{1}{|S_\mu(z)|},
\]
whence by (5.17) immediately follows
\[
(5.18) \quad \max_{|z|=r} |p_n(z)| \leq \frac{1}{m(r)} \leq \exp \left( \frac{C}{(1-|z|)^{1/p}} \right) \quad \text{for } 0 \leq |z| = r < 1.
\]
Since \( |S_\mu(z)| < 1 \) for each \( z \in \mathbb{D} \), combining (5.18) and the inequality \( |1+v|^2 \leq 2(1+|v|^2) \), for any positive constant \( c \) and \( 0 \leq |z| < 1 \) we obtain
\[
|1-p_n(z)S_\mu(z)|^2 \exp \left( -\frac{c}{(1-|z|)^{1/p}} \right)
\leq 2|p_n(z)|^2 \exp \left( -\frac{c}{(1-|z|)^{1/p}} \right) + 2 \exp \left( -\frac{c}{(1-|z|)^{1/p}} \right)
\leq 2 \exp \left( -\frac{c - C}{2(1-|z|)^{1/p}} \right) + 2 \exp \left( -\frac{c}{(1-|z|)^{1/p}} \right).
\]
It follows from the above inequality that for a constant \( c \) such that \( c \geq C \), a sequence \( \{f_n\}_n \) of functions defined as
\[
f_n(z) = |1-p_n(z)S_\mu(z)|^2 \exp \left( -\frac{c}{(1-|z|)^{1/p}} \right), \quad z \in \mathbb{D}, \; n \in \mathbb{N},
\]
is bounded by modulus by a function which is integrable on \( \mathbb{D} \) with respect to the area normalized Lebesgue measure \( r sr dv \). Since \( p_n(z)S_\mu(z) \to 1 \) as \( n \to \infty \) for each \( z \in \mathbb{D} \), it follows by the Dominated Convergence Theorem that for any fixed \( c \geq C \) a sequence \( \{\|p_nS_\mu - 1\|_c^-\}^2 \) defined as
\[
\left(\|p_nS_\mu - 1\|_c^-\right)^2_n = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 |1-p_n(re^{i\theta})S_\mu(re^{i\theta})|^2 \exp \left( -\frac{c}{(1-|re^{i\theta}|)^{1/p}} \right) r sr dv, \quad n \in \mathbb{N},
\]
converges to zero as \( n \to \infty \). Hence, the space \( S_\mu \mathcal{P} \) is dense in \((F^p, \| \cdot \|_c^-)\) for each \( c \geq C \).

Now suppose that \( 0 < c < C \). Let \( \alpha \) be a positive integer such that \( 1/\alpha < c/C \). Then applying the above argument to the singular inner function \( (S_\mu)^{1/n} \), we conclude that the space \( (S_\mu)^{1/\alpha} \mathcal{P} \) is dense in \((F^p, \| \cdot \|_c^-)\), i.e., according to the notation from Lemma 5.4, we have \([(S_\mu)^{1/\alpha}N^P]_c = F^p \). Therefore, by (5.16) of Lemma 5.4, we have
\[
[(S_\mu)^{2/n}N^P]_c = [(S_\mu)^{1/n}[(S_\mu)^{1/n}N^P]_c]_c
= [(S_\mu)^{1/n}F^p]_c
= F^p.
\]
Hence, repeating the application of (5.19) \( n - 1 \) times, we obtain \([S_{\mu}N^p]_c = F^p\). Therefore, \( S_{\mu}N^p \) is dense in the space \((F^p, \{ \| \cdot \|_c \})\) for each \( c > 0 \). This completes the proof. \( \square \)

Recall that an inner function \( I \) is said to be a weak outer function in \( N^p \) if the set \( IN^p := \{ If : f \in N^p \} \) is dense in the weak topology of \( N^p \). Then the second assertion of Theorem 5.5 can be formulated as follows.

**Corollary 5.6.** Every non-trivial singular inner function \( S_\mu \) with the associated measure \( \mu \) whose modulus of continuity \( \omega_\mu(t) = O(t^{(p-1)/p}) \) is a weak outer function in \( N^p \).

For a closed subspace \( E \) of \( N^p \), and for any \( f \in N^p \), let \( \bar{f} \) denote the coset of \( f + N^p \) in the quotient space \( N^p/E \). Define

\[ \|f\| = \inf \{ \|g\| : g \in \bar{f} \}, \quad f \in N^p, \]

where \( \|g\| = \rho_p(g, 0) \). Then the function \( \bar{\rho}_p \) defined as

\[ \bar{\rho}_p(\bar{f}, \bar{g}) = \|\bar{f} - \bar{g}\|, \quad \bar{f}, \bar{g} \in N^p/E, \]

is an additively invariant metric that induces the quotient topology on \( N^p/E \). It is well known that \( N^p/E \) is an \( F \)-space with respect to the metric \( \bar{\rho}_p \) (cf. [22, Theorem 12.3.5, p. 264]).

**Corollary 5.7.** Let \( S_\mu \) be a singular inner function described in Theorem 5.5. Then the following statements about the closed ideal \( S_\mu N^p \) in \( N^p \) hold.

(i) If \( \phi \) is a continuous linear functional on \( N^p \) which annihilates \( S_\mu N^p \), then \( \phi \) is the zero functional.

(ii) The quotient space \((N^p/S_\mu N^p, \bar{\rho}_p)\) is an \( F \)-space with the trivial dual.

(iii) The space \( S_\mu N^p \) does not have the separation property, and hence does not have the Hahn-Banach property.

**Proof.** Properties (i) and (ii) immediately follow from the fact that by Theorem 5.5, the ideal \( S_\mu N^p \) is weakly dense in \( N^p \) (cf. [4, Theorem 16, p. 59], where the analogous assertions are given for any topological vector space with enough continuous linear functionals to separate points). Since \([S_\mu N^p]_w = N^p \neq S_\mu N^p\), we see that \( S_\mu N^p \) is not weakly closed, and by Lemma 4.1, \( S_\mu N^p \) does not have the separation property. \( \square \)

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