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REPRESENTATIONS OF CERTAIN MEDIAL ALGEBRAS

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1. Introduction

Let (A, Ω) be an (abstract) algebra. We say an m-ary operation f and an n-ary operation g in Ω *commute* if

$$f(g(x_{11}, x_{12}, \cdots, x_{1n}), g(x_{21}, x_{22}, \cdots, x_{2n}), \cdots, g(x_{m1}, x_{m2}, \cdots, x_{mn}))$$
$$=g(f(x_{11}, x_{21}, \cdots, x_{m1}), f(x_{12}, x_{22}, \cdots, x_{m2}), \cdots, f(x_{1n}, x_{2n}, \cdots, x_{mn}))$$

for all x_{ij} in A, $i=1, 2, \cdots, m$, $j=1, 2, \cdots, n$. An algebra is called *medial* if every pair of operations (not necessarily distinct) commute.

Let (A, f, g) be a medial algebra with an m-ary operation f and an n-ary operation g. Since any unary operation of a medial algebra is nothing more than a homomorphism of the algebra, we may assume $2\leq m\leq n$. For any element e of A, let $\sigma_1, \cdots, \sigma_m$ and $\tau_1, \cdots, \tau_n$ be mappings of A into A defined by

$$\sigma_i : x \rightarrow f(e, \cdots, e, x, e, \cdots, e) \text{ and } \tau_i : x \rightarrow g(e, \cdots, e, x, e, \cdots, e) \quad (1)$$

with x at the i-th place. We call σ_i *the i-th translation by e with respect to f*. An element e is called *i-regular* (resp. an *i-identity*) with respect to f if σ_i is a bijection (resp. the identity mapping). The similar definitions go with g. An element c is called *regular* (resp. *an identity*) if it is *i-regular* (resp. an i-identity) with respect to both f and g for all i. Finally, an element e is called *idempotent* if $f(e, e, \cdots, e)=g\ (e, e, \cdots, e)=e$.

It is known that any medial algebra in certain varieties (varieties in which every algebra has a modular lattice of congruences) can be represented as a module over a commutative ring ([2], [4], [5]). However, the condition is very strong and we want to replace the condition with weaker condition and still obtain a representation of a medial algebra as a familiar algebra. In this paper, the condition is

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replaced with the condition that the algebra has an idempotent regular element. Quite a lot work has been done in this way for medial groupoids ([1], [3], [6]), and there are some results for medial algebras with one operation ([3]). We will see that any medial algebra with an idempotent regular element can be reconstructed from a monoid. We can extend this result slightly to medial groupoids without regular elements, but with elements which are idempotent and i-, j-regular for two different i and j.

2. Medial algebras with idempotent regular elements

LEMMA 1. *Let (A, f, g) be a medial algebra with an identity element e, then*

$$f(x_1, x_2, \cdots, x_m) = g(x_{\pi 1}, x_{\pi 2}, \cdots, x_{\pi m}, e, \cdots, e)$$

for any permutation π on $\{1, 2, \cdots, m\}$ and for all $x_1, x_2, \cdots, x_m$ in A.

Proof. For any permutation π on $\{1, 2, \cdots, m\}$ and $x_1, x_2, \cdots, x_m$ in A,

$$\begin{aligned} & f(x_1, x_2, \cdots, x_m) \\ &= f(g(e, \cdots, e, \underset{\pi^{-1}1\text{-th}}{x_1}, e, \cdots, e), \cdots, g(e, \cdots e, \underset{\pi^{-1}m\text{-th}}{x_m}, e, \cdots, e)) \\ &= g(f(e, \cdots, e, \underset{\pi 1\text{-th}}{x_{\pi 1}}, e, \cdots, e), \cdots, f(e, \cdots, e, \underset{\pi m\text{-th}}{x_{\pi m}}, e, \cdots, e), \cdots, g(e, \cdots, e)) \\ &= g(x_{\pi 1}, \cdots, x_{\pi m}, e, \cdots, e). \end{aligned}$$

COROLLARY. *If (A, f) is a medial n-groupoid with an identity element, then*

$$f(x_1, x_2, \cdots, x_n) = f(x_{\pi 1} x_{\pi 2}, \cdots, x_{\pi n})$$

for any permutation π on $\{1, 2, \cdots, n\}$ and for all $x_1, x_2, \cdots, x_n$ in A.

LEMMA 2. ([1]). *Every medial groupoid with an identity element is a commutative semigroup.*

THEOREM 3. *If (A, f, g) is a medial algebra with an identity element e, then there is a commutative semigroup $(A, +)$ with e as the identity element such that*

$f(x_1, x_2, \cdots, x_m) = x_1 + x_2 + \cdots + x_m$ and $g(x_1, x_2, \cdots, x_n) = x_1 + x_2 + \cdots + x_n$ *for all $x_1, \cdots, x_m, \cdots, x_n$ in A.*

Proof. Define a binary operation '+' on A by

$$x+y=f(x, y, e, \cdots, e) \tag{2}$$

for all x, y in A. We note that $x+y=g(x, y, e, \cdots, e)$ by Lemma 1. For $x, y, z, w \in A$,

$$\begin{aligned}
&(x+y)+(z+w)\\
&=f(f(x, y, e, \cdots, e), f(z, w, e, \cdots, e), e, \cdots, e)\\
&=f(f(x, y, e, \cdots, e), f(z, w, e, \cdots, e), f(e, \cdots, e), \cdots, f(e, \cdots, e))\\
&=f(f(x, z, e, \cdots, e), f(y, w, e, \cdots, e), f(e, \cdots, e), \cdots, f(e, \cdots, e))\\
&=f(f(x, z, e, \cdots, e), f(y, w, e, \cdots, e), e, \cdots, e)\\
&=(x+z)+(y+w).
\end{aligned}$$

Thus, $(A, +)$ is medial. Trivially, e is the identity element of $(A, +)$, and so $(A, +)$ is a commutative semigroup by Lemma 2. Suppose $f(x_1, \cdots, x_i, e, \cdots, e)=x_1+\cdots+x_i$, then

$$\begin{aligned}
&f(x_1, \cdots, x_i, x_{i+1}, e, \cdots, e)\\
&=f(f(x_1, e, \cdots, e), \cdots, f(x_i, e, \cdots, e), f(e, x_{i+1}, e, \cdots, e), (f(e, \cdots, e),\\
&\quad \cdots, f(e, \cdots, e))\\
&=f(f(x_1, \cdots, x_i, e, \cdots, e), f(e, \cdots, e, x_{i+1}, e, \cdots, e), f(e, \cdots, e), \cdots, f(e,\\
&\quad \cdots, e))\\
&=f((x_1+\cdots+x_i), x_{i+1}, e, \cdots, e)\\
&=x_1+\cdots+x_i+x_{i+1}.
\end{aligned}$$

Thus, by induction, $f(x_1, x_2, \cdots, x_m)=x_1+x_2+\cdots+x_m$. Similarly $g(x_1, x_2, \cdots, x_n)=x_1+x_2+\cdots+x_n$.

Let e be an idempotent element of (A, f, g) and let σ_i be the i-th translation defined in (1). For $x_1, x_2, \cdots, x_n$ in A,

$$\begin{aligned}
\sigma_i g(x_1, x_2, \cdots, x_n)&=f(e, \cdots, e, g(x_1, x_2, \cdots, x_n), e, \cdots, e)\\
&=f(g(e, \cdots, e), \cdots, g(x_1, x_2, \cdots, x_n), \cdots, g(e, \cdots, e))\\
&=g(f(e, \cdots, x_1, \cdots, e), f(e, \cdots, x_2, \cdots, e), \cdots, f(e, \cdots, x_n,\\
&\quad \cdots, e))\\
&=g(\sigma_i x_1, \sigma_i x_2, \cdots, \sigma_i x_n).
\end{aligned}$$

Similarly, we have $\sigma_i f(x_1, x_2, \cdots, x_m)=f(\sigma_i x_1, \sigma_i x_2, \cdots, \sigma_i x_m)$. Thus, σ_i is a homomorphism of (A, f, g). By the same arguments, we can show each τ_i is a homomorphism. Clearly, σ_i and τ_j are automorphisms if e is regular. Now, for any x in A,

$$\begin{aligned}
\sigma_i \tau_j x&=f(e, \cdots, e, g(e, \cdots, x, \cdots, e), e, \cdots, e)\\
&=f(g(e, \cdots, e), \cdots, g(e, \cdots, x, \cdots, e), \cdots, g(e, \cdots, e))\\
&=g(f(e, \cdots, e), \cdots, f(e, \cdots, x, \cdots, e), \cdots, f(e, \cdots, e))
\end{aligned}$$

$=\tau_j\sigma_i x$

and hence $\sigma_i\tau_j=\tau_j\sigma_i$. Similarly, $\sigma_i\sigma_j=\sigma_j\sigma_i$ and $\tau_i\tau_j=\tau_j\tau_i$ for every i and j. By this we have proved:

LEMMA 4. *Let (A,f,g) be a medial algebra and e an idempotent element. Then the translations defined in* (1) *are endomorphisms and they commute pairwise. If, furthermore, e is regular then they are automorphisms.*

LEMMA 5. *Let (A,f,g) be a medial algebra and $\alpha_1,\cdots,\alpha_m,\beta_1,\cdots,\beta_n$ be pairwise commuting endomorphisms of (A,f,g). Let f^* and g^* be operations on A defined by*

$f^*(x_1,\cdots,x_m)=f(\alpha_1x_1,\cdots,\alpha_mx_m)$ *and* $g^*(x_1,\cdots,x_n)=f(\beta_1x_1,\cdots,\beta_nx_n)$.

Then (A,f^*,g^*) is also a medial algebra. Furthermore, $\alpha_1,\cdots,\alpha_m,\beta_1,\cdots,\beta_n$ are endomorphisms of (A,f^*,g^*).

Proof. For $x_{ij}\in A$, $i=1,2,\cdots,m$, $j=1,2,\cdots,n$,

$$\begin{aligned}&f^*(g^*(x_{11},\cdots,x_{1n}),\cdots,g^*(x_{m1},\cdots,x_{mn}))\\&=f(g(\alpha_1\beta_1x_{11},\cdots,\alpha_1\beta_nx_{1n}),\cdots,g(\alpha_m\beta_1x_{m1},\cdots,\alpha_m\beta_nx_{mn}))\\&=f(g(\beta_1\alpha_1x_{11},\cdots,\beta_n\alpha_1x_{1n}),\cdots,g(\beta_1\alpha_mx_{m1},\cdots,\beta_n\alpha_mx_{mn}))\\&=g(f(\beta_1\alpha_1x_{11},\cdots,\beta_1\alpha_mx_{m1}),\cdots,g(\beta_n\alpha_1x_{1n},\cdots,\beta_n\alpha_mx_{mn}))\\&=g^*(f^*(x_{11},\cdots,x_{m1}),\cdots,f^*(x_{1n},\cdots,x_{mn})).\end{aligned}$$

Thus f^* and g^* commute. Similarly, f^* and g^* commute with themselves. Now,

$$\begin{aligned}\alpha_ig^*(x_1,x_2,\cdots,x_n)&=\alpha_ig(\beta_1x_1,\beta_2x_2,\cdots,\beta_nx_n)=g(\alpha_i\beta_1x_1,\alpha_i\beta_2x_2,\cdots,\alpha_i\beta_nx_n)\\&=g(\beta_1\alpha_ix_1,\beta_2\alpha_ix_2,\cdots,\beta_n\alpha_ix_n)=g^*(\alpha_ix_1,\alpha_ix_2,\cdots,\alpha_ix_n).\end{aligned}$$

Similarly, $\alpha_if^*(x_1,x_2,\cdots,x_m)=f^*(\alpha_ix_1,\alpha_ix_2,\cdots,\alpha_ix_m)$. Hence, α_i is an endomorphism of (A,f^*,g^*). By the same way, each β_j is an endomorphism of (A,f^*,g^*).

LEMMA 6. *Let (A,f,g) be a medial algebra with a regular idempotent elemente. Define new operations f^* and g^* on A by*

$$f^*(x_1,\cdots,x_m)=f(\sigma_1^{-1}x_1,\cdots,\sigma_m^{-1}x_m)\text{ and }g^*(x_1,\cdots,x_n)=f(\tau_1^{-1}x_1,\cdots,\tau_n^{-1}x_n)\qquad(3)$$

where $\sigma_1,\cdots,\sigma_m$, $\tau_1,\cdots,\tau_n$ are translations defined in (1). *Then (A,f^*,g^*) is a medial algebra with e as an identity element.*

Proof. By Lemma 4, $\sigma_1,\cdots,\sigma_m,\tau_1,\cdots,\tau_n$ are commuting automorphisms, and hence so are $\sigma_1^{-1}, \cdots, \sigma_m^{-1}, \tau_1^{-1}, \cdots, \tau_n^{-1}$. By Lemma 5, (A, f^*, g^*) is a medial algebra. Now

$$f^*(e, \cdots, x, \cdots, e)=f(\sigma_1^{-1}e, \cdots, \sigma_i^{-1}x, \cdots, \sigma_m^{-1})=f(e, \cdots, \sigma_i^{-1}, \cdots, e)$$
$$=x.$$

Similarly, $g^*(e, \cdots, x, \cdots, e)=x$. Thus e is an identity element of (A, f^*, g^*).

THEOREM 7. *Let (A, f, g) be a medial algebra with a regula idempotent element e. Then there is a commutative semigroup $(A, +)$ with e as the identity element and pairwise commuting automorphisms $\sigma_1, \cdots, \sigma_m, \tau_1, \cdots, \tau_n$ of $(A, +)$ such that*

$$f(x_1, \cdots, x_m)=\sigma_1 x_1+\cdots+\sigma_m x_m \text{ and } g(x_1, \cdots, x_n)=\tau_1 x_1+\cdots+\tau_n x_n$$

for all $x_1, \cdots, x_m, \cdots, x_n$ in A. (4)

Proof. Define operations f^* and g^* as are in (3). (A, f^*, g^*) is a medial algebra with an identity element e, by Lemma 6. Thus, by Theorem 3, there is a commutative semigroup $(A, +)$ with e as the identity element such that

$$f^*(x_1, \cdots, x_m)=f(\sigma_1^{-1}x_1, \cdots, \sigma_m^{-1}x_m)=x_1+\cdots+x_m$$

and

$$g^*(x_1, x_2, \cdots, x_n)=g(\tau_1^{-1}x_1, \tau_2^{-1}x_2, \cdots, \tau_n^{-1}x_n)=x_1+x_2+\cdots+x_n.$$

Thus, (4) holds. By, Lemmas 4 and 5, $\sigma_1, \cdots, \sigma_m, \tau_1, \cdots, \tau_n$ are pairwise commuting automorphisms of (A, f^*, g^*). As is in (2), '+' is defined by $x+y=f^*(x, y, e, \cdots, e)=g^*(x, y, e, \cdots, e)$. Thus,

$$\sigma_i(x+y)=\sigma_i f^*(x, y, e, \cdots, e)=f(\sigma_i x, \sigma_i y, \sigma_i e, \cdots, \sigma_i e)$$
$$=f^*(\sigma_i x, \sigma_i y, e, \cdots, e)=\sigma_i x+\sigma_i y.$$

That is, σ_i is an automorphism of $(A, +)$ for each i. Similarly, τ_j is an automorphism of $(A, +)$ for each j.

3. Medial algebras without regular elements

LEMMA 8. *Let (A, f, g) be a medial algebra, π a permutation on $\{1, 2, \cdots, m\}$ and ρ a permutation on $\{1, 2, \cdots, n\}$. Let f^* and g^* be operations on A defined by*

$$f^*(x_1, \cdots, x_m)=f(x_{\pi 1}, \cdots, x_{\pi m}) \text{ and } g^*(x_1, \cdots, x_n)=g(x_{\rho 1}, \cdots, x_{\rho n}).$$

Then, (A, f^*, g^*) is also a medial algebra.

Proof. We only show that f^* and g^* commute, and others can be proved similarly. For $x_{ij}\in A$,

$$\begin{aligned}&f^*(g^*(x_{11},\cdots,x_{1n}),\cdots,g^*(x_{m1},\cdots,x_{mn}))\\&=f(g(x_{\pi1\rho1},\cdots,x_{\pi1\rho n}),\cdots,g(x_{\pi m\rho1},\cdots,x_{\pi m\rho n}))\\&=g(f(x_{\pi1\rho1},\cdots,x_{\pi m\rho1}),\cdots,f(x_{\pi1\rho n},\cdots,x_{\pi m\rho n}))\\&=g^*(f^*(x_{11},\cdots,x_{m1}),\cdots,f^*(x_{1n},\cdots,x_{mn})),\end{aligned}$$

as is wanted.

THEOREM 9. *Let (A,f,g) be a medial algebra with an idempotent element e which is i– and j–regular with respect to both f and g for fixed i and j $(i\neq j)$, Then there is a commutative semigroup $(A,+)$ with e as the identity element and pairwise commuting endomorphisms $\sigma_1,\cdots,\sigma_m,\tau_1,\cdots,\tau_n$ of $(A,+)$ such that*

$$f(x_1,\cdots,x_m)=\sigma_1x_1+\cdots+\sigma_mx_m \text{ and } g(x_1,\cdots,x_n)=r_1x_1+\cdots+r_nx_n$$

for all $x_1,\cdots,x_m,\cdots,x_n$ in A. Furthermore, $\sigma_i,\sigma_j,\tau_i,\tau_j$ are automorphisms.

Proof. Due to the preceding lemma, we may assume that e is 1– and 2–regular. Let $\sigma_1,\cdots,\sigma_m,\tau_1,\cdots,\tau_n$ be the translations defined in (1). Then, they are pairwise commuting endomorphisms of (A,f,g) by Lemma 4 and $\sigma_1,\sigma_2,\tau_1,\tau_2$ are automorphisms. With the definition (1) of these mappings in mind, we have

$$\begin{aligned}&f(\sigma_1^{-1}x,\sigma_2^{-1}y,e,\cdots,e)\\&=f(g(\tau_1^{-1}\sigma_1^{-1}x,e,\cdots,e),g(e,\tau_2^{-1}\sigma_2^{-1}y,e,\cdots,e),g(e,\cdots,e),\cdots,g(e,\\&\quad\cdots,e))\\&=g(f(\tau_1^{-1}\sigma_1^{-1}x,e,\cdots,e),f(e,\tau_2^{-1}\sigma_2^{-1}y,e,\cdots,e),f(e,\cdots,e),\cdots,f(e,\\&\quad\cdots,e))\\&=g(f(\sigma_1^{-1}\tau^{-1}x,e,\cdots,e),f(e,\sigma_2^{-1}\tau_2^{-1}y,e,\cdots,e),e,\cdots,e)\\&=g(\tau_1^{-1}x,\tau_2^{-1}y,e,\cdots,e)\end{aligned}$$

for all $x,y\in A$. With this, we define a binary operation '+' on A by

$$x+y=f(\sigma_1^{-1}x,\sigma_2^{-1}y,e,\cdots,e)=g(\tau_1^{-1}x,\tau_2^{-1}y,e,\cdots,e). \qquad (5)$$

One can easily verify that $(A,+)$ is a medial groupoid with e as the identity element. Thus, $(A,+)$ is a commutative semigroup by Lemma 2. Furthermore, it can be seen that $\sigma_1,\cdots,\sigma_m,\tau_1,\cdots,\tau_n$ are endomorp hisms of $(A,+)$. From (5), $f(x_1,x_2,e,\cdots,e)=\sigma_1x_1+\sigma_2x_2$. Suppose $f(x_1,\cdots,x_i,e,\cdots,e)=\sigma_1x_1+\cdots+\sigma_ix_i$, then

$$\begin{aligned}
&f(x_1, \cdots, x_i, x_{i+1}, e, \cdots, e) \\
&=f(f(\sigma_1^{-1}x_1, e, \cdots, e), \cdots, f(\sigma_1^{-1}x_i, e, \cdots, e), f(e, \sigma_2^{-1}x_{i+1}, e, \cdots, e), \cdots \\
&\quad , f(e, \cdots, e)) \\
&=f(f(\sigma_1^{-1}x_1, \cdots, \sigma_1^{-1}x_i, e, \cdots, e), f(e, \cdots, e, \sigma_2^{-1}x_{i+1}, e, \cdots, e), \cdots, f \\
&\quad (e, \cdots, e)) \\
&=f(\sigma_1^{-1}f(x_1, \cdots, x_i, e, \cdots, e), \sigma_2^{-1}f(e, \cdots, e, x_{i+1}, e, \cdots, e), \cdots, e) \\
&=f(x_1, \cdots, x_i, e, \cdots, e)+f(e, \cdots, e, x_{i+1}, e, \cdots, e) \\
&=\sigma_1 x_1+\cdots+\sigma_i x_i+\sigma_{i+1}x_{i+1}.
\end{aligned}$$

Thus, by induction, $f(x_1, x_2, \cdots, x_m)=\sigma_1 x_1+\cdots+\sigma_m x_m$. Simlarly $g(x_1, x_2, \cdots, x_n)=\tau_1 x_1+\cdots+\tau_n x_n$.

4. Closing

For a medial algebra (A, Ω) with many operations, if we assume the relevant properties of an element with respect to every operation in Ω, then we can get the similar result as before.

For groupoids, the existence of a regular elements (not being idempotent) is sufficient for the operation of a medial groupoid to be defined on a commutative monoid by translating the operation obtained in Theorem 7 ([3], [6]). That is, for any medial groupoid $(G, \cdot)$ with a regular element, there is a commutative monoid $(G, +)$, two automorphisms α and β of $G(, +)$, and an element d of G such that $x \cdot y = ax+\beta y+d$ for all x, y in G. Our question is how we do this kind of work for medial algebras with a regular element which is not idempotent. Is it always possible to represent a medial algebra with a regular element in this way?

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