RESTRICTED FLEXIBLE ALGEBRAS

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1. Introduction

An algebra $A$ with multiplication denoted by $xy$ over a field $F$ is called flexible if it satisfies the flexible law $(xy)x = x(yx)$ for all $x, y \in A$. For an element $x$ of $A$, let $L_x$ and $R_x$ denote the left and right multiplications in $A$ by $x$. Following Schafer's work [13] on restricted noncommutative Jordan algebras of characteristic $p > 2$, we call a flexible algebra of characteristic $p > 2$ restricted if $A$ is strictly power-associative and satisfies

$$(1) \quad L_x^p = L_x \quad \text{or} \quad R_x^p = R_x$$

for all $x \in A$. All algebras considered in this note are assumed to be finite-dimensional. Recall that a flexible algebra is called a noncommutative Jordan algebra if it satisfies the Jordan identity $(x^2 y)x = x^2 (yx)$. Well known noncommutative Jordan algebras are the commutative Jordan and alternative algebras which are shown to be restricted for characteristic $> 0$ [4]. There exist simple flexible power-associative algebras of characteristic $> 2$ which are not noncommutative Jordan ([6] and [8]). An example of a simple flexible algebra of characteristic $p > 0$ which is not restricted has been given by Schafer [13].

It is the purpose of this note to extend the results of Schafer [13] for restricted noncommutative Jordan algebras to restricted flexible algebras. Since flexible power-associative algebras of characteristic 0 have a satisfactory structure theory and enjoy those properties constrained by the restricted identity (1), we may regard those algebras of characteristic 0 as restricted algebras.

2. Nodal algebras

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A power-associative algebra $A$ over $F$ with identity element 1 is said to be nodal in case every element of $A$ is of the form $\alpha 1 + z$, where $\alpha \in F$ and $z$ is nilpotent, and $A$ is not of the form $A = F1 + N$ for a nilsubalgebra $N$ of $A$. It is well known that there are no nodal algebras which are alternative of arbitrary characteristic, commutative Jordan of characteristic $\neq 2$, or noncommutative Jordan of characteristic 0 (Jacobson [3] and McCrimmon [9]). However, nodal noncommutative Jordan algebras of characteristic $p > 0$ do exist, and Kokoris [7] gave the first construction of such algebras, called Kokoris algebras, which have also been shown to be useful for the study of simple Lie algebras of prime characteristic (see Schafer [14] and Strade [16], for example). Scribner [15] has constructed a nodal noncommutative Jordan algebra of infinite dimension. Schafer [13] has proven that Kokoris algebras cannot be restricted by showing that there are no nodal, restricted noncommutative Jordan algebras of characteristic $p > 2$. We here extend this result to restricted flexible algebras. Attached to an algebra $A$ over a field $F$ of characteristic $\neq 2$ is the commutative algebra denoted by $A^+$ with multiplication $x \cdot y = \frac{1}{2}(xy + yx)$ defined on the vector space $A$. We begin with

**Lemma 1.** Let $A$ be a strictly power-associative algebra with identity element 1 over a field $F$ of characteristic $\neq 2$ such that every element of $A$ is of the form $\alpha 1 + z$ for $\alpha \in F$ and a nilpotent element $z$ in $A$. Then $A$ is the vector space direct sum $A = F1 + N$ where $N$ is a subspace of nilpotent elements in $A$ and $N^+$ is the maximal nil ideal of $A^+$.

**Proof.** Let $N$ denote the set of nilpotent elements in $A$ and let $M$ be a maximal ideal of $A^+$. Then, $M \subseteq N$, since if $M \subseteq N$, then there is an element $\alpha 1 + z$ in $M$ for $\alpha \neq 0$ and a nilpotent element $z$, and hence $\alpha 1 + z$ is invertible by power-associativity. Thus $A^+/M$ is a simple commutative, strictly power-associative algebra of degree one. If the characteristic of $F$ is greater than two, then Oehmke [11] has shown that such an algebra must be a field, and if the characteristic is zero, then the same holds by a result of Albert [1]. Therefore, $A^+/M$ is a field, and it must be that $M = N$, which is a subspace of $A$.

Extending the known result for commutative Jordan algebras, as an immediate consequence of Lemma 1, we have

**Corollary 2.** There is no nodal commutative strictly power-associative
algebra of characteristic ≠ 2.

**Theorem 3.** There is no nodal restricted flexible algebra over a field of characteristic ≠ 2.

**Proof.** Suppose that such a nodal algebra \( A \) exists. It is readily seen that any homomorphic image of a restricted algebra is also restricted. Note also that any nonzero homomorphic image of a nodal algebra is also nodal (Schafer [12, p.116]). By Lemma 1, we can write \( A = F_1 + N \) where \( N \) is a subspace of \( A \) and is the set of all nilpotent elements in \( A \). If \( M \) denotes a maximal ideal of \( A \), then as in the proof of Lemma 1 we have \( M \subseteq N \). Then, the quotient algebra \( \overline{A} = A/M \) is a simple restricted flexible algebra of degree one, and by a result of Kleinfeld and Kokoris [5] \( \overline{A}^+ \) must be an associative algebra. Hence, \( \overline{A} \) is a nodal restricted noncommutative Jordan algebra, which contradicts the result of Schafer [13] that such algebras do not exist.

3. **Semisimple algebras**

A power-associative algebra \( A \) is called *semisimple* if the maximal nil ideal of \( A \) is zero. Oehmke [10] has proven that any semisimple flexible strictly power-associative algebra \( A \) over \( F \) of characteristic ≠ 2 has an identity element and is the direct sum of simple ideals, and that any simple such algebra of characteristic ≠ 2, 3 is one of the algebras: a commutative Jordan algebra (for degree ≥ 3); a quasi-associative algebra; a flexible algebra of degree 2; and an algebra of degree one. We make use of this result to prove the following structure theorem.

**Theorem 4.** Any semisimple restricted flexible algebra \( A \) over a field \( F \) of characteristic ≠ 2, 3 is the direct sum \( A = A_1 \oplus \cdots \oplus A_n \) of simple ideals \( A_i \) of \( A \). If \( A \) is simple, then \( A \) is one of the following:

(a) a simple commutative Jordan algebra of characteristic ≠ 2 (for degree ≥ 3),

(b) a simple restricted flexible algebra of degree two,

(c) an algebra \( B(\lambda) \) with multiplication \( x*y = \lambda xy + (1-\lambda)yx \) defined on a simple associative algebra \( B \) over \( F \) for a fixed \( \lambda \neq 1/2 \) in the prime field of \( F \), where \( xy \) denotes the product in \( B \),

(d) a field (for degree one).

**Proof.** The first part and the algebras described in (a) and (b) fol-
low from the result of Oehmke noted above. If $A$ is a simple quasi-associative algebra over $F$, then it is shown in [13] that the restricted condition (1) gives the algebra described by (c). Assume then that $A$ is a simple algebra of degree one. Since $A$ cannot be nodal by Theorem 3, by a result of Block [2] we conclude that $A^+$ is a simple algebra of degree one which must be a field. Here we have used the fact that any scalar extension of a restricted algebra is also restricted (see [13, p.143]). Therefore, by Lemma 1, $A$ is a field also in this case. This completes the proof.

By the known classification, all algebras but case (b) in Theorem 4 are completely described. It is not known whether there exists a simple restricted flexible algebra (of degree 2) which is not noncommutative Jordan. We note that if $A$ is noncommutative Jordan, then one of the two restricted conditions in (1) implies the other. This is due to the fact that a commutative Jordan algebra is restricted and flexibility is equivalent to the identity $L_x R_x = R_x L_x$. In fact, let $T_x = \frac{1}{2} (L_x + R_x)$ denote the right multiplication in the Jordan algebra $A^+$ by $x$. Since any Jordan algebra of characteristic $p > 0$ is restricted [4], $\frac{1}{2} (L_x^+ + R_x^+)$ = $T_x^+ = T_x^+ = \frac{1}{2} (L_x^+ + R_x^+)$, since $L_x$ commutes with $R_x$. Hence one of the conditions (1) implies the other.

References

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