MAPPING THEOREMS FOR LOCALLY EXPANSIVE OPERATORS

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1. Introduction

It is well-known fact [2, p.62] that if a local homeomorphism of a Banach space \( X \) into a Banach space \( Y \) is a local expansion, in the sense that for a continuous nonincreasing function \( c : [0, \infty) \to (0, \infty) \) with \( \int_0^\infty c(t)dt = \infty \), each point \( x \) of \( X \) has a neighborhood \( U_x \) such that

\[
c(\max \{ \|u\|, \|v\| \}) \|u-v\| \leq \|Tu-Tv\|
\]

for each \( u, v \) in \( U_x \), then \( T(X) = Y \).

Kirk and Schöneberg [3], and Ray and Walker [5] proved that a similar result can be obtained within the class of mappings whose graphs are closed subsets of \( X \times Y \). Also Torrejon [6] obtained the same result without assuming that \( c \) is nonincreasing. Moreover, Bae and Yie [1] proved a more stronger result by giving the precise range of the operator \( T \), that is, they proved that under the same situation \( T(B(0; K)) \) contains \( B(T(0); \int_0^K c(t)dt) \).

This note is a continuation of the above program; by developing Torrejon's method, we replace the domain of \( c \) by \( X \) instead of \( [0, \infty) \) and give more general results which contain all the above mentioned results of [1, 2, 3, 5, 6].

First we give some notations and definitions.

If \( D \) is a subset of \( X \), then \( \overline{D} \) and \( \partial D \) denote, respectively, the closure and boundary of \( D \) in \( X \). Recall that a mapping \( T : D \to Y \) is said to have closed graph if for each sequence \( \{x_n\} \subseteq D \) with \( x_n \to x \in D \) and \( T x_n \to y \) as \( n \to \infty \), it follows that \( Tx = y \). We denote by \( B(x, r) \) the set \( \{y ; \|y-x\| < r \} \), and conveniently we set \( B(x; \infty) = X \) if \( x \in X \).

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A continuous curve \( h : [0, s] \longrightarrow X, 0 \leq s < \infty \), is rectifiable if there exists a constant \( M > 0 \) such that for any subdivision of \([0, s]\) of the form
\[
0 = t_0 < t_1 < \cdots < t_n = s
\]
we have
\[
\sum_{i=1}^{n} \| h(t_i) - h(t_{i-1}) \| \leq M.
\]
The least such constant \( M \) is called the length of the curve. A continuous curve \( h : [0, s] \longrightarrow X \) is said to be parametrized by arc length if for any \( t, \ 0 \leq t \leq s \), the length of the curve \( h([0, t]) \) is exactly \( t \).

**Remark 1.** Note that if \( h : [0, s] \longrightarrow X \) is a parametrized curve by arc length with \( s < \infty \), then \( \lim_{t \to s^-} h(t) \) always exists and \( h \) can be extended to \([0, s]\).

A nonlinear operator \( T \) mapping a subset \( D \) of \( X \) into a metric space \( Y \) is said to be locally \( m \)-expansive, where \( m : D \longrightarrow (0, \infty) \) is a continuous function, if each point \( x \) in \( D \) has a neighborhood \( U_x \) such that
\[
(\ast) \quad \min \{ m(u), m(v) \} \| u - v \| \leq d(Tu, Tv)
\]
for each \( u, v \) in \( U_x \).

Following Menger [4], a metric space \( Y \) is said to be metrically convex if for each \( x, y \) in \( Y \) with \( x \neq y \) there exists \( z \) in \( Y \), distinct from \( x \) and \( y \), such that \( d(x, y) = d(x, z) + d(z, y) \).

**2. Main results**

Now we state our first theorem.

**Theorem 1.** Let \( X \) be a Banach space, \( D \) an open subset of \( X \), and let \( Y \) be a complete metric space with metric convexity. Let \( m : D \longrightarrow (0, \infty) \) be a continuous function such that
\[
(\ast\ast) \quad \int_0^\infty m(h(t)) \, dt = \infty \quad \text{for any continuous curve } h : [0, \infty) \longrightarrow D
\]
parametrized by arc length.

Let \( T : D \longrightarrow Y \) be a locally \( m \)-expansive mapping on \( D \) having closed graph. If \( T \) maps open subsets of \( D \) onto open subsets of \( Y \), then for each \( y \in Y \) the followings are equivalent.
There exists $x_0 \in D$ such that $d(Tx_0, y) \leq d(Tx, y)$ for all $x \in \partial D$.

**Proof.** We only need to prove that (2) $\implies$ (1). We let $g : [0, d(Tx_0, y)] \to Y$ be an isometry such that $g(0) = Tx_0$ and $g(d(Tx_0, y)) = y$. The existence of $g$ is assured by Menger's result [4]. Let $M$ denote the set of all $\tau$ in $[0, d(Tx_0, y)]$ for which there exists a unique continuous curve $h : [0, \tau] \to D$ such that $h(0) = x_0$ and $Th(t) = g(t)$ for each $t$, $0 \leq t \leq \tau$. Let $\tau_0 = \sup \{\tau ; \tau \in M\}$. Then $\tau_0 > 0$ since $T$ is assumed to be an open and locally m-expansive mapping on $D$. Now we claim that $\tau_0 \in M$, so that we conclude that $\tau_0 = d(Tx_0, y)$ since $M$ is an open subset of $[0, d(Tx_0, y)]$, and hence $Th(\tau_0) = y \in T(D)$. Since $\tau_0 = \sup \{\tau ; \tau \in M\}$, there is a unique continuous curve $h : [0, \tau_0] \to D$ with $h(0) = x_0$ and $Th(t) = g(t)$ for each $t$, $0 \leq t < \tau_0$.

**Lemma 1.** For each $\tau \in [0, \tau_0)$, the curve $h|_{[0, \tau]}$ is rectifiable.

**Proof.** Since $[0, \tau]$ is compact, $\inf \{m(h(t)) ; 0 \leq t \leq \tau\} = m > 0$. Then it is easily seen that for any subdivision

$$0 = t_0 < t_1 < \cdots < t_n = \tau$$

of $[0, \tau]$ we have

$$\sum_{i=1}^{n} \|h(t_i) - h(t_{i-1})\| \leq \frac{\tau}{m},$$

so that $h|_{[0, \tau]}$ is rectifiable.

**Continuation of the proof of Theorem 1.** By Lemma 1, for each $t$, $0 \leq t < \tau_0$, the length of the curve $h|_{[0, 0]}$ exists, and we denote it by $s(t)$. Then note that $s : [0, \tau_0) \to [0, \infty)$ is a continuous strictly increasing function. To complete our proof, we need another lemma.

**Lemma 2.** For each fixed $t \in [0, \tau_0)$, we have

$$m(h(t))D^+s(t) \leq 1,$$

where $D^+v$ is the right-upper Dini derivative of the function $v$.

**Proof.** For any given $\varepsilon > 0$, we have a neighborhood $U_{h(t)}$ of $h(t)$ in $D$ such that $m(x) \geq (1 - \varepsilon)m(h(t))$ for all $x \in U_{h(t)}$ and (*) holds. Now choose $r > 0$ such that for all $t'$ with $t \leq t' \leq t + r < \tau_0$, $h(t') \in U_{h(t)}$. Also we can choose a subdivision
Jong Sook Bae

$$t = t_0 < t_1 < \cdots < t_n = t + r$$
of $[t, t+r]$ such that

$$s(t+r) - s(t) \leq (1+\varepsilon) \sum_{i=1}^{n} ||h(t_i) - h(t_{i-1})||.$$

Therefore we have

$$(1-\varepsilon)m(h(t)) (s(t+r) - s(t)) \leq (1+\varepsilon) \sum_{i=1}^{n} d(Th(t_i), Th(t_{i-1})) = (1+\varepsilon) r.$$\[183x351]150x341\[150x341]F(t) = \int_{0}^{s(t)} m(h(t^*(s))) ds,$$\[65x391]s \text{ continuous and strictly increasing, its inverse exists, say } t^* : [0, s_0) \rightarrow [0, \tau_0).$$

Then note that $h(t^*(s))$ is a parametrized curve by arc length with parameter $s$. Now set

$$F(t) = \int_{0}^{s(t)} m(h(t^*(s))) ds,$$\[0x391]0 \leq t < \tau_0.$$\[65x391]By Lemma 2, we have $D^+ F(t) = m(h(t)) D^+ s(t) \leq 1,$ thus we obtain, for all $t$, $0 \leq t < \tau_0$,\[184x279]fSCt ) \int_{0}^{s(t)} m(h(t^*(s))) ds \leq t.$$\[65x262]Therefore the condition (***) gives that $s_0 < \infty,$ which also yields that \lim_{t \rightarrow \tau_0} h(t) = x \in D$ exists by Remark 1. Now since $T$ has closed graph,\[77x139]As a direct consequence of Theorem 1, we have the following

**Corollary 1.** Let $X$ be a Banach space and $Y$ a complete metric space with metric convexity. Let $T : X \rightarrow Y$ be a locally m-expansive mapping having closed graph, where $m : X \rightarrow (0, \infty)$ is a continuous function such
that \( \int_0^\infty m(h(t))dt = \infty \) for every continuous curve \( h : [0, \infty) \to X \) parametrized by arc length. If \( T \) maps open subsets of \( X \) onto open subsets of \( Y \), then \( T(X) = Y \).

**Remark 2.** Note that if \( c : [0, \infty) \to (0, \infty) \) is a continuous function for which \( \int_0^\infty c(t)dt = \infty \), then the function \( m(x) = c(\|x\|) \) satisfies the condition (**). Therefore Theorem 1 contains the results of [3] and [6].

More generally we can give the precise range of the operator \( T \) as in [1].

**Theorem 2.** Let \( X \) be a Banach space, \( D \) an open subset of \( X \), and let \( Y \) be a complete metric space with metric convexity. Let \( x_0 \) be in \( D \), and \( m : D \to (0, \infty) \) a continuous function such that there is an \( N > 0 \) such that

\[
(***) \quad \text{if } h : [0, s) \to D, \ 0 \leq s \leq \infty, \text{ is a continuous curve parametrized by arc length for which } h(0) = x_0 \text{ and } \int_0^s m(h(t))dt < N, \text{ then it follows that } s < \infty \text{ and } \lim_{t \to s} h(t) \in D.
\]

If \( T : D \to Y \) is an open and locally \( m \)-expansive mapping having closed graph, then \( T(D) \) contains the ball \( B(Tx_0; N) \).

**Proof.** Let \( y \in B(Tx_0; N) \), that is, \( d(y, Tx_0) < N \). As in the proof of Theorem 1, let \( g : [0, d(Tx_0, y)] \to Y \) be an isometry with \( g(0) = Tx_0 \) and \( g(d(Tx_0, y)) = y \), and let \( M \) denote the set of all \( \tau \) in \([0, d(Tx_0, y)]\) for which there exists a unique continuous curve \( h : [0, \tau] \to D \) such that \( h(0) = x_0 \) and \( Th(t) = g(t) \) for each \( t, 0 \leq t \leq \tau \). Let \( \tau_0 = \sup \{ \tau ; \tau \in M \} \). As in the proof of Theorem 1, we get \( \tau_0 > 0 \) and we claim that \( \tau_0 \in M \), so that we conclude that \( \tau_0 = d(Tx_0, y) \) and \( Th(\tau_0) = y \in T(D) \). Also let \( h : [0, \tau_0) \to D \) be a unique continuous curve with \( h(0) = x_0 \) and \( Th(t) = g(t) \) for each \( t, 0 \leq t < \tau_0 \). Also by Lemma 1, we let \( s(t) \) be the length of the curve \( h|_{[0, t]} \) for each \( t \in [0, \tau_0) \), and let \( s_0 = \sup \{ s(t) ; 0 \leq t < \tau_0 \} \). Now we know that the inverse \( t^* : [0, s_0) \to [0, \tau_0) \) of \( s \) exists. By applying Lemma 2, we get for each \( t, 0 \leq t < \tau_0 \),

\[
\int_0^{s(t)} m(h(t^*(s)))ds \leq t,
\]
so that
\[ \int_0^{t_0} m(h(t^*(s))) \, ds < \tau_0 \leq d(Tx_0, y) < N. \]

By the condition (***)", we have \( s_0 < \infty \) and \( \lim_{t \to -s_0} h(t^*(s)) \in D \). Therefore \( \lim_{t \to -s_0} h(t) = \lim_{t \to s_0} h(t^*(s)) = x \in D \) exists. Thus by defining \( h(t^0) = x \), we have \( \tau_0 \in M \), which completes our proof.

Now if we put \( D = B(x_0; K) \), we have the following.

**Corollary 2.** Let \( X \) be a Banach space and \( Y \) a complete metric space with metric convexity. Let \( x_0 \in X, \ K \geq 0 \), and let \( m : B(x_0; K) \to (0, \infty) \) be a continuous function such that there is an \( N > 0 \) such that

\[ (***)' \text{ if } h : [0, s) \to B(x_0; K), \ 0 < s < \infty, \ \text{is a continuous curve parametrized by arc length for which } h(0) = x_0 \text{ and } \int_0^s m(h(t)) \, dt < N, \text{ then it follows that } s < \infty \text{ and } \lim_{t \to s} \|h(t) - x_0\| < K. \]

If \( T : B(x_0; K) \to Y \) is an open and locally \( m \)-expansive mapping having closed graph, then \( T(B(x_0; K)) \) contains \( B(Tx_0; N) \).

**Remark 3.** Note that if \( c : [0, K) \to (0, \infty) \) is continuous and if \( \int_0^K c(t) \, dt = N \), then the function \( m(x) = c(\|x - x_0\|) \) satisfies the condition (***)'. Therefore for locally expansive mappings, Theorem 2 is a generalized version of results of [1, 2, 3, 5, 6]. It is interesting to point out that Corollary 1 is also an immediate consequence of Corollary 2. Also note that all results in this paper can be applied to the class of locally strongly \( \phi \)-accretive operators as in [1].

**References**


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