FIXED POINT THEOREMS OF GENERALIZED NONEXPANSIVE MAPS

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1. Introduction

Let \((C, d)\) be a metric space. A map \(T: C \to C\) is said to be generalized nonexpansive if, for any \(x, y \in C\),

\[
d(Tx, Ty) \leq a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) \\
+ a_4 d(x, Ty) + a_5 d(y, Tx),
\]

where \(a_1 \geq 0\) and \(\sum_{i=1}^{5} a_i \leq 1\).

In [10], Goebel, Kirk and Shimi proved that if \(T\) is a continuous generalized nonexpansive selfmap of a nonempty closed convex bounded subset \(C\) of a uniformly convex Banach space, then \(T\) has a fixed point. Bogin [4] generalized this result for a weakly compact convex subset \(C\) having normal structure in a Banach space without assuming the continuity of \(T\). In this paper, we study various types of maps which are particular to (1) by classifying \(a_i\)'s, and obtain several new fixed point theorems. Especially, in section 2, we classify \(a_i\)'s, and study various cases of (1) without assuming normal structure. In section 3, we generalize the result of [4] for the case that \(C\) has asymptotic normal structure, and obtain new fixed point theorems for some variations of (1), and common fixed point theorems for a commuting family of generalized nonexpansive maps.

2. Generalized nonexpansive maps

By interchanging \(x\) and \(y\), (1) is equivalent to the condition

\[
d(Tx, Ty) \leq a d(x, y) + b d(x, Tx) + d(y, Ty) \\
+ c d(x, Ty) + d(y, Tx),
\]

for all \(x, y \in C\), where \(a, b, c \geq 0\) and \(a + 2b + 2c \leq 1\), by putting \(a = a_1\).
If \( a+2b+2c<1 \) and \( C \) is complete, a number of authors showed that \( T \) has a unique fixed point, and any iteration \( \{ T^n x \} \) converges to the fixed point of \( T \) for each \( x \in C \). Therefore, we may assume that \( a+2b+2c=1 \). Then the following cases can be occurred:

- **Case I.** \( a=1, \ b=c=0 \).
- **Case II.** \( a=c=0, \ b=\frac{1}{2} \).
- **Case III.** \( b>0, \ c>0 \).
- **Case IV.** \( a>0, \ b>0, \ c=0 \).
- **Case V.** \( b=0, \ c>0 \) (this case contains the case \( a=b=0 \)).

**Case I.** \( a=1, \ b=c=0 \). In this case, the map \( T \) is said to be nonexpansive. In 1965, Browder [6] showed that any nonexpansive selfmap of a closed convex bounded subset \( C \) of a Hilbert space has a fixed point by using monotone operator theory. Also, in the same year, Kirk [16] obtained the same result for a closed convex bounded subset \( C \) of a reflexive Banach space provided that \( C \) has normal structure.

The concept of normal structure was introduced by Brodskii and Mil'man [5]. A nonempty closed convex bounded subset \( C \) of a Banach space is said to have normal structure if, for any closed convex subset \( C_0 \) of \( C \) which has more than one point, there exists a point \( x \in C_0 \) satisfying

\[
\sup \{ \| x-y \| ; \ y \in C_0 \} < \delta(C_0),
\]

where \( \delta(C_0) \) denotes the diameter of the set \( C_0 \).

Moreover, if \( C \) is a weakly compact convex subset of a Banach space and \( C \) has asymptotic normal structure, then \( T \) has a fixed point by [2]. But, Alspach [1] gave an example of a weakly compact convex subset of \( L_1[0,1] \) that fails to have the fixed point property for nonexpansive maps.

If \( T \) is affine, that is, \( T(\lambda x+(1-\lambda)y)=\lambda Tx+(1-\lambda) Ty, \ 0 \leq \lambda \leq 1 \), then \( T \) is weakly continuous. Therefore, if \( C \) is weakly compact and convex, and if \( T \) is an affine nonexpansive selfmap of \( C \), then \( T \) has a fixed point by the Tychonoff fixed point theorem. Otherwise, if we impose a condition similar to that \( T \) is affine, then we have the following
THEOREM 2.1. Let $C$ be a nonempty weakly compact convex subset of a Banach space, and $T : C \to C$ be nonexpansive. Suppose that there is a strictly increasing continuous function $\gamma : [0, \infty) \to [0, \infty)$ with $\gamma(0) = 0$ such that, for all $x, y \in C$ and $0 \leq \lambda \leq 1$,

$$\|T(\lambda x + (1-\lambda)y) - \lambda x - (1-\lambda)y\| \leq \gamma(\|x - Tx\| + \|y - Ty\|).$$

Then $T$ has a fixed point.

Proof. Choose a decreasing sequence $\{\varepsilon_n\}$ of positive reals such that $\varepsilon_n \to 0$ as $n \to \infty$, and $\gamma(2\varepsilon_{n+1}) \leq \varepsilon_n$ for $n \geq 1$. By the Banach contraction principle, we can choose $x_n \in C$ such that $\|Tx_n - x_n\| \leq \varepsilon_n$. Since $C$ is weakly compact, we may assume that $x_n$ converges weakly to a point $x \in C$.

Call a sequence $\{y_n\}$ a $c$-subsequence (see [12]) of $\{x_n\}$ provided that there is a sequence of integers $1 = p_1 \leq q_1 < p_2 \leq q_2 < \ldots$ and coefficients $\alpha_i \geq 0$ such that

$$\sum_{i=p_n}^{q_n} \alpha_i = 1, \quad y_n = \sum_{i=p_n}^{q_n} \alpha_i x_i.$$

Since every closed convex subset of a Banach space is weakly closed, we may choose a $c$-subsequence $\{y_n\}$ of $\{x_n\}$ such that $y_n$ converges strongly to $x$. Then, by (3), and by using induction, we get

$$\|Ty_n - y_n\| = \|T\left(\sum_{i=p_n}^{q_n} \alpha_i x_i\right) - \sum_{i=p_n}^{q_n} \alpha_i x_i\|$$

$$\leq \gamma(2\varepsilon_{p_n}) \leq \varepsilon_{p_n-1}.$$  

By setting $n \to \infty$, we have $\lim \|Ty_n - y_n\| = 0$, so that $x$ is a fixed point of $T$.

REMARK 2.1. Under the same hypothesis of Theorem 2.1, we know that every weak cluster point of $\{S^n x\}$ is a fixed point of $T$ for any $x \in C$, where $S_\lambda = \lambda I + (1-\lambda)T$, $0 < \lambda < 1$, since, by Ishikawa [14], $S_\lambda$ is asymptotically regular. Note that every affine map satisfies (3). Also note that every nonexpansive selfmap of a closed convex bounded subset of a uniformly convex Banach space satisfies (3) by Bruck[7]. Actually, he showed that if $T$ is a nonexpansive selfmap of a closed convex bounded subset $C$ of a uniformly convex Banach space, then there is a strictly increasing continuous function $\alpha : [0, \infty) \to [0, \infty)$ with $\alpha(0) = 0$ such that, for all $x, y \in C$, and $0 \leq \lambda \leq 1$,

$$\|\lambda x + (1-\lambda)y\| \leq \|x - y\| - \|Tx - Ty\|.$$
Since \( C \) is bounded, we may assume that \( \alpha(t) \to \infty \) as \( t \to \infty \), so that \( \alpha^{-1} \) exists. Therefore by putting \( r(t) = \alpha^{-1}(t) + t \), \( T \) satisfies (3). Moreover, note that we can easily construct a nonexpansive map which satisfies (3), but not (3).

By putting \( \phi(x) = \|x - Tx\| \), \( x \in C \), in Theorem 2.1, the existence of a fixed point of \( T \) is equivalent to the fact that \( \phi \) attains its minimum. It is well-known that if \( \phi \) is convex, then \( \phi \) attains its minimum. Note that the condition (3) is a weakened form of the convexity of \( \phi \).

Case II. \( a = c = 0, \ b = \frac{1}{2} \). In this case, \( T \) is called a Kannan–type map. In 1973, Kannan [15] showed that every nonempty closed convex bounded subset of a reflexive Banach space having normal structure has the fixed point property for Kannan type maps. Note that every nonexpansive map is continuous, while a Kannan–type map need not be continuous. The existence of fixed points of Kannan-type maps related to close-to-normal structure. A closed convex bounded subset \( C \) of a Banach space is said to have close-to-normal structure if, for each closed convex subset \( C_0 \) of \( C \) having more than one point, there exists \( x \in C_0 \) such that \( \|x - y\| < \delta(C_0) \) for any \( y \in C_0 \). In [24], Wong showed that any nonempty weakly compact convex subset \( C \) of a Banach space has the fixed point property for Kannan-type maps if and only if \( C \) has close-to-normal structure. Furthermore, he posed a question whether every closed convex bounded subset of a reflexive Banach space has close-to-normal structure. But Tan [22] showed that the answer is negative by giving an example of a reflexive Banach space which has asymptotic normal structure, but does not have close-to-normal structure.

Now we have the following

**Theorem 2.2.** Any reflexive Banach space \( X \) admits an equivalent norm \( \| \|_1 \) such that any selfmap \( T \) of a nonempty closed convex bounded subset \( C \) of \( X \) satisfying, for \( x, y \in C \),

\[
\|Tx - Ty\|_1 \leq \frac{1}{2} \{\|x - Tx\|_1 + \|y - Ty\|_1\}
\]

has a unique fixed point.

**Proof.** By Troyanski [23], \( X \) admits an equivalent norm \( \| \|_1 \) so
that \((X, \| \cdot \|_1)\) is locally uniformly convex. Therefore with the new norm \(\| \cdot \|_1\), \(C\) has close-to-normal structure, and by [24] \(T\) has a unique fixed point.

REMARK 2.2. Note that Dulst [9] showed that every separable Banach space \(X\) admits an equivalent norm such that every nonexpansive selfmap (with the new norm) of a weakly compact convex subset of \(X\) has a fixed point. Since this new norm satisfies the Opial condition [20], the result of [9] can be applied to Kannan-type maps. Also note that every separable Banach space has an equivalent norm which is strictly convex, so that every closed convex bounded subset has close-to-normal structure.

Case III. \(b>0, c>0\). In this case, Bogin [4] showed that if \(C\) is a complete metric space, then any iteration \(\{T^n x\}\) converges to the unique fixed point of \(T\).

Case IV. \(a>0, b>0, c=0\). In this case, Gregus [11] showed that if \(C\) is a closed convex subset of a Banach space, then \(T\) has a unique fixed point. Actually, he proved that any iteration \(\{U^n x\}\) converges to the unique fixed point of \(T\), where \(Ux=(T^2x+T^3x)/2\).

Case V. \(b=0, c>0\). In this case, we have the following lemma.

**Lemma 2.1.** Let \((C, d)\) be a bounded metric space, and let \(T: C \rightarrow C\) be a map satisfying

\[
d(Tx, Ty) \leq ad(x, y) + cd(x, Ty) + cd(y, Tx)
\]

for all \(x, y \in C\), where \(a \geq 0\), \(c > 0\) and \(a+2c=1\). Then \(T\) is asymptotically regular, i.e., for any \(x \in C\),

\[
\lim d(T^{n+1}x, T^n x) = 0.
\]

**Proof.** For \(x_0 \in C\), let \(x_n = T^n x_0\). Then, for \(n \geq 1\), by (4), we get

\[
d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq ad(x_n, x_{n-1}) + cd(x_n, Tx_n) + cd(x_{n-1}, Tx_{n-1}),
\]

so that we have

\[
d(x_{n+1}, x_n) \leq \frac{a+c}{1-c} d(x_n, x_{n-1}) = d(x_n, x_{n-1}).
\]

Therefore, the sequence \(\{d(x_{n+1}, x_n)\}\) is nonincreasing, so that \(\lim d(x_{n+1}, x_n) = r\) exists. We must show that \(r = 0\). Suppose \(r > 0\). Then there exists a positive integer \(s\) such that the diameter of \(C = d_1 < (s+\)

1) \( r/2 \). Since \( c > 0 \), there exists \( \varepsilon > 0 \) such that \( \{1 - (s+1)c^1\} (r + \varepsilon) + (s+1)rc^2/2 < r \) (this is possible for \( 0 < \varepsilon \leq (s+1)rc^2/2 \)). Then there exists a positive integer \( N \) such that \( n \geq N \) implies \( r \leq d(x_{n+1}, x_n) < r + \varepsilon \).

Now we claim that, for \( n \geq 0 \) and \( k \geq 1 \)

\[ d(x_{n+k+1}, x_{n+k}) \leq \{1 - (k+1)c^1\} d(x_{n+1}, x_n) + c^k d(x_{n+k+1}, x_n). \]

To prove (5), let \( k = 1 \). Then, we get, by (4),

\[ d(x_{n+k}, x_{n+1}) \leq d(x_{n+1}, x_n) + cd(x_{n+k+1}, x_n), \]

which asserts (5), since \( a = 1 - 2c \). In order to use induction for \( k \), assume that (5) is true for \( k \geq 1 \). Then we have

\[
\begin{align*}
d(x_{n+k+2}, x_{n+k+1}) & \leq \{1 - (k+1)c^1\} d(x_{n+2}, x_{n+1}) + c^k d(x_{n+k+2}, x_{n+1}) \\
& \leq \{1 - (k+1)c^1\} d(x_{n+1}, x_n) + c^k \{d(x_{n+k+1}, x_n) + cd(x_{n+k+2}, x_n) + c^k d(x_{n+k+1}, x_{n+1})\} \\
& \leq \{1 - (k+1)c^1\} d(x_{n+1}, x_n) + c^k \{a(k+1) + c^k\} d(x_{n+1}, x_n) + c^k d(x_{n+k+2}, x_n) \\
& \leq \{1 - (k+2)c^k\} d(x_{n+1}, x_n) + c^k + 1 d(x_{n+k+2}, x_n),
\end{align*}
\]

by using \( d(x_{n+k+1}, x_n) \leq (k+1)d(x_{n+1}, x_n) \) and \( d(x_{n+k+1}, x_{n+1}) \leq kd(x_{n+1}, x_n) \), which proves (5).

Then, for \( n \geq N \) and \( k = s \), by (5), we have

\[
\begin{align*}
d(x_{n+s+1}, x_{n+s}) & \leq \{1 - (s+1)c^1\} (r + \varepsilon) + c^s d_1 \\
& \leq \{1 - (s+1)c^1\} (r + \varepsilon) + \frac{c^k(s+1)r}{2} \\
& < r,
\end{align*}
\]

which leads a contradiction. Therefore, we have \( r = 0 \).

In Lemma 2.1, if \( a = 0 \), then we have stronger conclusion that \( T \) is uniformly asymptotically regular, that is, \( d(T^{n+1}x, T^nx) \to 0 \) uniformly as \( n \to \infty \) for all \( x \in C \).

**Lemma 2.2.** Let \((C, d)\) be a bounded metric space, and let \( T : C \to C \) be a map satisfying

\[
(6) \quad d(Tx, Ty) \leq \frac{1}{2} \{d(x, Ty) + d(y, Tx)\}
\]

for all \( x, y \in C \). Then \( T \) is uniformly asymptotically regular. Moreover we have

\[
(7) \quad d(T^{n+1}x, T^nx) \leq \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} \delta(C),
\]

where \( \delta(C) \) is the diameter of the set \( C \).
Proof. Since
\[\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \sin^{2n} x \, dx\]
converges to 0 as \( n \to \infty \), we need only to prove (7).

First we claim that, for \( n \geq 2 \),
\[
d(T^n x, T x) \leq \frac{1}{2} d(T^n x, x) + \frac{1}{2^2} d(T^{n-1} x, x) + \ldots + \frac{1}{2^n-1} d(T^2 x, x).
\]
To prove (8), for \( n = 2 \), by (6), we have
\[
d(T^2 x, T x) \leq \frac{1}{2} d(T x, T x) + \frac{1}{2} d(T^2 x, x) = \frac{1}{2} d(T^2 x, x).
\]
Assume that (8) is true for \( n - 1 \geq 2 \). Then by the assumption and (6), we obtain
\[
d(T^n x, T x) \leq \frac{1}{2} d(T^n x, x) + \frac{1}{2} d(T^{n-1} x, T x)
\leq \frac{1}{2} d(T^n x, x)
+ \frac{1}{2^n} \left( \frac{1}{2} d(T^{n-1} x, x) + \ldots + \frac{1}{2^{n-2}} d(T^2 x, x) \right)
= \frac{1}{2} d(T^n x, x) + \frac{1}{2^n} d(T^{n-1} x, x) + \ldots + \frac{1}{2^n-1} d(T^2 x, x),
\]
which asserts (8) by induction.

Next we shall prove that, for \( n \geq 1 \),
\[
d(T^{n+1} x, T^n x) \leq \frac{a_{n+1,1}}{2^n} d(T^{n+1} x, x) + \frac{a_{n+1,2}}{2^{n+1}} d(T^n x, x)
+ \ldots + \frac{a_{n+1,n}}{2^{2n-1}} d(T^2 x, x),
\]
where \( a_{n+1,k}'s \) \((1 \leq k \leq n)\) are inductively given by the rule;
\[a_{2,1} = 1\]
and, for \( n \geq 2 \),
\[a_{n+1,1} = a_{n,1},\]
\[a_{n+1,2} = a_{n,1} + a_{n,2},\]
\[\ldots\]
\[a_{n+1,n-1} = a_{n,1} + a_{n,2} + \ldots + a_{n,n-1},\]
\[a_{n+1,n} = a_{n,1} + a_{n,2} + \ldots + a_{n,n-1}.\]
To prove (9), assume that (9) is true for \(n-1\geq1\). Then, by (8), we have

\[
d(T^{n+1}x, \; T^nx) \leq \frac{a_{n-1}}{2^{n-1}}d(T^{n+1}x, \; Tx) + \frac{a_{n-2}}{2^{n-2}}d(T^nx, \; Tx) + \cdots + \frac{a_1}{2}d(T^2x, \; x) + d(Tx, \; x)
\]

\[
\leq \frac{a_{n-1}}{2^{n-1}} \left[ \frac{1}{2} d(T^{n+1}x, \; x) + \frac{1}{2} d(T^nx, \; x) \right] + \cdots + \frac{a_1}{2} \left[ \frac{1}{2} d(T^3x, \; x) + \frac{1}{2} d(T^2x, \; x) \right] + \frac{a_0}{2} d(T^2x, \; x)
\]

\[
= \frac{a_{n+1}}{2^n} d(T^{n+1}x, \; x) + \cdots + \frac{a_{n+1}}{2^{n-1}} d(T^2x, \; x)
\]

which proves (9) by induction.

Now, by elementary calculus, it is easy to see that

\[
\frac{a_{n+1}}{2^n} + \frac{a_{n+2}}{2^{n+1}} + \cdots + \frac{a_{n+1}}{2^{n-1}} = \frac{1}{2} \cdot \frac{3}{4} \cdots 2n - 1.
\]

Therefore, by (9), we complete the proof of (7).

Using the above lemmas, we have the following

**Theorem 2.3.** Let \((C, d)\) be a compact metric space, and let \(T : C \to C\) be a map satisfying (4). Then \(T\) has a fixed point, and any iteration \(\{T^nx\}\) converges to a fixed point of \(T\) for each \(x \in C\).

**Proof.** For each \(x \in C\), there exists a subsequence \(\{T^{ni}x\}\) of \(\{T^nx\}\) which converges to some \(p\) in \(C\), by the compactness of \(C\). Then, by (4), we get

\[
d(Tp, \; T^{ni}x) \leq ad(p, \; T^{ni-1}x) + cd(p, \; T^{ni}x) + cd(Tp, \; T^{ni-1}x)
\]

\[
\leq ad(p, \; T^{ni}x) + cd(T^{ni}x, \; T^{ni-1}x) + cd(p, \; T^{ni}x) + cd(Tp, \; T^{ni}x) + cd(T^{ni}x, \; T^{ni-1}x),
\]

so that we obtain

\[
d(Tp, \; T^{ni}x) \leq \frac{a+c}{1-c} \{d(p, \; T^{ni}x) + d(T^{ni}x, \; T^{ni-1}x)\}
\]

\[
= d(p, \; T^{ni}x) + d(T^{ni}x, \; T^{ni-1}x).
\]

Therefore we have

\[
d(Tp, \; p) = \lim_{i \to \infty} d(Tp, \; T^{ni}x)
\]

\[
\leq \lim_{i \to \infty} \{d(p, \; T^{ni}x) + d(T^{ni}x, \; T^{ni-1}x)\}.
\]

Since the left hand side of the above inequality tends to 0 as \(i \to\)
by Lemma 2.1, we have $T\rho = \rho$.
Moreover, by (4), we have
$$d(T^{n+1}x, \rho) \leq ad(T^nx, \rho) + cd(T^nx, \rho) + cd(T^{n+1}x, \rho),$$
so that we obtain
$$d(T^{n+1}x, \rho) \leq \frac{a+c}{1-c}d(T^nx, \rho) = d(T^nx, \rho).$$
Therefore, the sequence \{d(T^n x, \rho)\} is nonincreasing, and the whole sequence \{T^nx\} converges to $\rho$.

**Corollary.** Let $(C, d)$ be a compact metric space, and let $T : C \to C$ be a map satisfying (6). Then $T$ has a fixed point, and any iteration \{T^n x\} converges to a fixed point of $T$, for each $x \in C$.

**Remark 2.3.** If $(C, d)$ is not bounded, then Lemma 2.1 and 2.2. are not valid. For example, consider a map $T : R \to R$ such that $Tx = x + a$ ($a \neq 0$) with the usual metric on $R$. Then $T$ satisfies (4) and (6), but $d(T^{n+1}x, T^nx) = |a|$ for all $x \in R$ and all $n \geq 1$.

Also note if $C$ is a weakly compact convex subset of a Banach space and $C$ has normal structure, and if $T$ is a selfmap of $C$ satisfying (4), then $T$ has a fixed point by [4]. In section 2, we shall extend this result for the case that $C$ has asymptotic normal structure. But the following example shows that the conditions on $C$ are indispensable.

**Example.** Let $C[0, 1]$ be a Banach space of all continuous real valued functions on $[0, 1]$ with the uniform norm. Let $C = \{ f \in C[0, 1] ; 0 \leq f \leq 1, f(0) = 0, f(1) = 1 \}$. Then $C$ is a closed convex bounded subset of $C[0, 1]$. Define a map $T : C \to C$ by $Tf(x) = xf(x)$. Then $T$ is nonexpansive. Also we can prove that $T$ satisfies (6), and so that $T$ satisfies (4) for all $a, c \geq 0$ with $a + 2c = 1$. But $T$ has no fixed point.

3. **Asymptotic Normal Structure and Fixed Points**

Recall that a closed convex bounded subset $C$ of a Banach space has asymptotic normal structure (see [2]) if, for each closed convex subset $C_0$ of $C$ consisting of more than one point and each sequence $\{x_n\}$ in $C_0$ satisfying $x_{n+1} - x_n \to 0$ as $n \to \infty$, there is a point $x \in C_0$ such that
$$\liminf_n \|x_n - x\| < \delta(C_0).$$
Note that if $C$ has normal structure, then $C$ has asymptotic normal structure. But the converse is not true.
Now, we state our main result in this section.

**THEOREM 3.1.** Let $C$ be a nonempty weakly compact convex subset of a Banach space. Suppose that $C$ has asymptotic normal structure. Let $T : C \to C$ be a map satisfying

$$
\| Tx - Ty \| \leq a \| x - y \| + b \left( \| x - Tx \| + \| y - Ty \| \right) + c \left( \| x - Ty \| + \| y - Tx \| \right)
$$

for all $x, y \in C$, with $a, c \geq 0$, $0 \leq b < \frac{1}{2}$ and $a + 2b + 2c = 1$. Then $T$ has a fixed point.

**Proof.** If $b = c = 0$, then the theorem is true by Baillon and Schöneberg [2]. If $b > 0$, $c > 0$, then the theorem is valid by Bogin [4]. If $b > 0$, $c = 0$, then also the theorem remains true by Gregus [11]. Therefore we need only to prove the theorem for the case $b = 0$ and $c > 0$, so that we may assume that $T$ satisfies (4).

By the standard Zorn's Lemma argument using weak compactness of $C$, there exists a nonempty weakly compact convex subset $C_0$ of $C$ which is minimal in the sense that it contains no proper closed convex subset which is invariant under $T$.

Now we claim that $C_0$ is a singleton, whose element is a fixed point of $T$. Suppose that $C_0$ has more than one point.

Let $x_0$ be any fixed element in $C_0$, and let $x_n = T^n x_0$. Then by Lemma 2.1, $x_{n+1} - x_n \to 0$ as $n \to \infty$. Next we claim that, for each $x \in C_0$,

$$
\lim \| x_n - x \| = \delta(C_0).
$$

Therefore, we have a contradiction to the asymptotic normal structure of $C$. To prove (10), let $y \in C_0$, and let $s = \lim \sup \| x_n - y \|$. Let $D = \{ x \in C_0; \lim \sup \| x_n - x \| \leq s \}$, which is nonempty closed and convex. Then, by (4), we have

$$
\| Tx - x_n \| = \| Tx - Tx_{n-1} \|
\leq a \| x - x_{n-1} \| + c \| x - Tx_{n-1} \| + c \| x_{n-1} - Tx \|
\leq (a + c) \| x - x_n \| + c \| x_n - Tx \| + (a + c) \| x_n - x_{n-1} \|,
$$

so that we obtain

$$
\| Tx - x_n \| \leq \frac{a + c}{1 - c} \| x - x_n \| + \frac{a + c}{1 - c} \| x_n - x_{n-1} \|
= \| x - x_n \| + \| x_n - x_{n-1} \|,
$$
which shows that $D$ is invariant under $T$. By the minimality of $C_0$, $D=C_0$. Choose a subsequence $\{x_{n_i}\}$ so that $\lim \|x_{n_i} - y\| = s'$ exists. Suppose that there exists $z$ in $C_0$ and a subsequence $\{x_{n_j}\}$ of $\{x_{n_i}\}$ such that $\lim \|x_{n_j} - z\| = t$. Let $E = \{x \in C_0 : \limsup \|x_{n_j} - x\| \leq \min \{t, s'\}\}$. Repeating the above argument, we find $E = C_0$. Therefore $y, z \in C_0 = E$, and so $t = s'$. Thus, for each $x \in C_0$, $\lim \|x_{n_j} - x\|$ exists and equals $s'$.

We complete the proof by showing that $s' = r = \delta(C_0)$. From this it follows that $\|x_{n_i} - y\| \to r$ whenever $\|x_{n_i} - y\|$ converges. Therefore by the boundedness of $\{x_{n_i}\}$, $\|x_n - y\| \to r$ for the entire sequence.

For this purpose, consider $F = \{u \in C_0 : \sup \{|u - x| ; x \in C_0\} \leq s'\}$. Then $F$ is nonempty because we can choose a weakly convergent subsequence, again denoted by $\{x_{n_i}\}$ with the limit $z$. Since $\|x_{n_i} - x\| \to s'$ for each $x \in C_0$, it follows that $\|x - z\| \leq s'$, so that $z \in F$. Now if $s' < r$, then $F$ is a proper closed convex subset of $C_0$. However, this contradicts the minimality of $C_0$ because $F$ is invariant under $T$. To see the latter, let $w$ be in $F$, and let $\sup \{|Tw - x| ; x \in C_0\} = s_1$. We must prove that $s_1 \leq s'$. Suppose $s_1 > s'$. Choose $\varepsilon$ with $0 < \varepsilon < (a + c) (s_1 - s') / 2$. Then there exists $u \in C_0$ such that $s_1 < |Tw - u| + \varepsilon$. By the minimality of $C_0$, we can see that the closed convex hull of $TC_0$ is actually $C_0$. Therefore we can choose $v = \sum \lambda_i Tv_i$ with $v_i \in C_0$, $\lambda_i > 0$, $\sum \lambda_i = 1$, and $\|u - v\| \leq \varepsilon$. Then we have

\[
s_1 \leq |Tw - u| + \varepsilon \\
\leq |Tw - v| + |v - u| + \varepsilon \\
\leq \sum \lambda_i |Tw - Tv_i| + 2\varepsilon \\
\leq \sum \lambda_i (a|w - v_i| + c|w - Tv_i| + c|v_i - Tw|) + 2\varepsilon \\
\leq \sum \lambda_i (as' + cs' + cs_i) + 2\varepsilon \\
= (a + c) s' + cs_1 + 2\varepsilon < s_1,
\]

which is a contradiction. Therefore, we have $s_1 \leq s'$. This completes the proof.

**Remark 3.1.** Theorem 3.1 is a generalization of [4] and [10] except for the case $b=1/2$. However, the conclusion of Theorem 3.1 does not hold for the case $b=1/2$ by [22].

In Theorem 3.1, instead of asymptotic normal structure of $C$, let
us consider close-to-normal structure (cf. [13]).

**Theorem 3.2.** Let \( C \) be a nonempty weakly compact convex subset of a Banach space. Suppose that \( C \) has close-to-normal structure, and that \( T : C \to C \) is a generalized nonexpansive map with \( b > 0 \) in (2). Then \( T \) has a unique fixed point.

**Proof.** From [4], [11] and [24], the proof is clear.

Next we consider a family of generalized nonexpansive maps and their common fixed points. For this end, we need the following lemma.

**Lemma 3.1.** ([19]). Let \( C \) be a closed convex subset of a strictly convex Banach space, and let \( T : C \to C \) be a generalized nonexpansive map with \( a > 0 \) in (2). Then the set \( F(T) \) of all fixed points of \( T \) is closed and convex.

**Theorem 3.3.** Let \( C \) be a nonempty weakly compact convex subset of a strictly convex Banach space. Suppose that \( C \) has asymptotic normal structure, and that \( \mathcal{I} \) is an arbitrary commuting family of selfmaps of \( C \) such that each member of \( \mathcal{I} \) satisfies (2) with \( a > 0 \). Then \( \mathcal{I} \) has a common fixed point, i.e., there is a point \( p \in C \) such that \( Tp = p \) for every \( T \in \mathcal{I} \).

**Proof.** By Theorem 3.1, each member \( T \) of \( \mathcal{I} \) has a nonempty fixed point set \( F(T) \). Moreover, by Lemma 3.1, each \( F(T) \) is a closed convex subset of \( C \), so that it is weakly compact. Let \( \mathcal{A} = \{ F(T); T \in \mathcal{I} \} \). Now we claim that \( \mathcal{A} \) satisfies the finite intersection condition, so that \( \bigcap \{ F(T); T \in \mathcal{I} \} \) is nonempty, and it is the set of common fixed points of \( \mathcal{I} \).

Suppose \( T_1, T_2, \ldots, T_n \in \mathcal{I} \). Since \( T_1T_2 = T_2T_1 \), \( F(T_1) \) is invariant under \( T_2 \). Therefore \( T_2 \) has a fixed point in \( F(T_1) \), so that \( F(T_1) \cap F(T_2) \) is nonempty, closed and convex. Since \( T_3 \) commutes \( T_1 \) and \( T_2 \), \( F(T_1) \cap F(T_2) \) is invariant under \( T_3 \), so that \( T_3 \) has a fixed point in \( F(T_1) \cap F(T_2) \). Therefore \( F(T_1) \cap F(T_2) \cap F(T_3) \) is nonempty, closed and convex. By the same argument and by induction, we have \( \bigcap_{i=1}^{n} F(T_i) \neq \emptyset \), which shows that \( \mathcal{A} \) satisfies the finite intersection condition.

By the same line as in the proof of Theorem 3.3, we have the following
THEOREM 3.4. Let $C$ be a nonempty weakly compact convex subset of a Banach space. Suppose that $C$ has close-to-normal structure, and that $\mathcal{F}$ is a commuting family of selfmaps of $C$ such that each member of $\mathcal{F}$ satisfies (2) with $b > 0$. Then $\mathcal{F}$ has a unique common fixed point. In particular, if one of members in $\mathcal{F}$ satisfies (2) with $b > 0$, then $\mathcal{F}$ has a unique common fixed point.

REMARK 3.2. For a family $\mathcal{F}$ of nonexpansive maps, a number of authors ([3], [8], [17], [18] and [21]) investigated common fixed points of $\mathcal{F}$. Note that every nonexpansive map is continuous, but a generalized nonexpansive map need not be continuous in general. Therefore, in a strictly convex Banach space, Theorem 3.3 is a generalization of [3], [17], [18] and [21].

Finally we shall prove two fixed point theorems which are concerned with variations of (2). At first, we have the following

THEOREM 3.5. Let $C$ be a nonempty weakly compact convex subset of a Banach space, and let $T$ be a selfmap of $C$ satisfying

$$
\|Tx - Ty\| \leq \max \left\{ \|x - y\|, \frac{1}{2}\|x - Ty\| + \frac{1}{2}\|y - Tx\| \right\},
$$

for all $x, y \in C$. If $C$ has normal structure, then $T$ has a fixed point.

Proof. By Zorn’s lemma, we have a nonempty minimal closed convex subset $C_0$ of $C$ in the sense that it contains no proper closed convex subset which is invariant under $T$. Suppose that $C_0$ has more than one point. For an arbitrary $x_0 \in C_0$, put $x_n = T^n x_0$. Then, by Bogin [4], there exists $r > 0$ such that $C_1 = \{x \in C_0; \lim \sup \|x_n - x\| \leq r\}$ is a nonempty proper closed convex subset of $C_0$. Let $x \in C_1$ and $\lim \sup \|x_n - Tx\| = r_0$. Since

$$
\|x_n - Tx\| = \|Tx_{n-1} - Tx\|
$$

$$
\leq \max \left\{ \|x_{n-1} - x\|, \frac{1}{2}\|Tx_{n-1} - x\| + \frac{1}{2}\|x_{n-1} - Tx\| \right\},
$$

we have

$$
r_0 = \lim \sup \|Tx - x_n\|
$$

$$
\leq \lim \sup \left\{ \max \left\{ \|x_{n-1} - x\|, \frac{1}{2}\|x_n - x\| + \frac{1}{2}\|x_{n-1} - Tx\| \right\} \right\}
$$

$$
\leq \max \left\{ r, \frac{1}{2}r + \frac{1}{2}r_0 \right\}.
$$
Therefore, we have \( r_0 \leq r \), and so that \( Tx \in C_1 \), which shows that \( C_1 \) is invariant under \( T \). This is a contradiction. Therefore \( C_0 \) is a singleton.

Let \( (C, d) \) be a complete metric space. We define the Kuratowski measure of noncompactness \( \alpha \) as a nonnegative real valued function on the set of all bounded subsets of \( C \) such that \( \alpha (D) = \inf \{ r > 0; D \text{ is covered by finitely many sets with diameter less than } r \} \). It is well-known that \( \alpha (D) = 0 \) if and only if the closure \( \bar{D} \) of \( D \) is compact.

A map \( T : C \to C \) is said to be condensing if, for each bounded subset \( D \) of \( C \), \( TD \) is bounded and

\[
\alpha (TD) < \alpha (D) \quad \text{for all } \alpha (D) \neq 0.
\]

Note that we do not assume that \( T \) is continuous.

**Theorem 3.6.** Let \( (C, d) \) be a bounded complete metric space, and let \( T : C \to C \) be a generalized nonexpansive and condensing (not necessarily continuous) map with \( c > 0 \) in (2). Then \( T \) has a fixed point, and any iteration \( \{ T^nx \} \) converges to a fixed point of \( T \) for each \( x \in C \).

**Proof.** By Lemma 2.1 and [4], \( T \) is asymptotically regular. Let \( x_0 \) be an arbitrary point in \( C \), and let \( x_n = T^nx_0 \). Then we claim that \( \alpha (\{ x_n \}_{n=0}^{\infty}) = 0 \). Suppose \( \alpha (\{ x_n \}_{n=0}^{\infty}) > 0 \). Since \( T \) is condensing, we have

\[
\alpha (\{ x_n \}_{n=0}^{\infty}) = \alpha (T \{ x_n \}_{n=1}^{\infty}) < \alpha (\{ x_n \}_{n=0}^{\infty}).
\]

But \( \alpha (\{ x_n \}_{n=0}^{\infty}) = \max \{ \alpha (\{ x_0 \}), \alpha (\{ x_n \}_{n=1}^{\infty}) \} \) gives that the above inequality is a contradiction. Therefore, \( \{ x_n \} \) is relatively compact, so that we can choose a convergent subsequence \( \{ x_{n_i} \} \) with the limit \( p \). Then, by (2), we have

\[
d(Tp, x_{n_i}) \leq ad(p, x_{n_i-1}) + b \{ d(p, Tp) + d(x_{n_i-1}, x_{n_i}) \}
+ c \{ d(p, x_{n_i}) + d(x_{n_i-1}, Tp) \},
\]

so that we obtain

\[
d(Tp, x_{n_i}) \leq \frac{a+c}{1-c} d(p, x_{n_i}) + \frac{a+b+c}{1-c} d(x_{n_i-1}, x_{n_i}) + bd(p, Tp).
\]

By letting \( i \to \infty \), we have \( d(Tp, p) \leq bd(p, Tp) \), so that \( Tp = p \). By the same way in the proof of Theorem 2.3 we can prove that \( \{ T^nx_0 \} \) converges to \( p \).

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References
9. D. V. Dulst, Equivalent norms and the fixed point property for nonexpansive mappings, J. London Math. Soc. 25 (1982), 139-144.


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