CERTAIN SUBGROUPS OF HOMOTOPIY GROUPS

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D.H. Gottlieb ([1], [2]) has defined and studied the evaluation subgroup \( G_n(X) \) of the homotopy group \( \pi_n(X) \). On the other hand, many authors have studied the homotopy groups of function spaces. In particular, S.S. Koh ([6]) proved some theorems concerning function spaces. In this paper we will define a subgroup of \( \pi_n(X) \) which contains \( G_n(X) \). Using the properties of the group defined here, we will generalize the results of S.S. Koh.

1. Introduction

The paper is divided into 5 sections. In section 2 we define a subgroup \( G_n(X, A, *) \) of \( \pi_n(X, *) \). The relationship between this group and the evaluation map from function space \( X^A \) to \( X \) is examined and it is shown that \( G_n(X, A, *) \) contains \( G_n(X, *) \). Moreover we give an example for which \( G_n(X, *) \subset G_n(X, A, *) \subset \pi_n(X, *) \).

In section 3, we will prove that \( G_n(X, A, *) \) is an invariant of homotopy type in the category of spaces homotopically equivalent to \( CW \)-pairs.

In section 4, we study some conditions concerning the cell in \( CW \)-pair, and some other conditions.

In section 5, we investigate the relationship between \( G \)-spaces and \( W \)-spaces. In final section 6, we devote ourself to study function spaces and their homotopy groups.

2. Group \( G_n(X, A, *) \)

Let \( (X, *) \) and \( (A, *) \) be any two pointed topological spaces and \( f : (A, *) \rightarrow (X, *) \) be a fixed map. Consider the class of continuous functions \( F : A \times S^1 \rightarrow X \)

such that \( F(a, *) = f(a) \).
Then the map \( h : (S^n, *) \rightarrow (X, *) \) defined by \( h(s) = F(*, s) \) represents an element \([h] \in \pi_n(X, *)\).

**DEFINITION 2.1.** The set of all elements \([h] \in \pi_n(X, *)\) obtained in the above manner from some \( F \) will be denoted by \( G_n^f(X, A, *) \).

Thus for every \([h] \in G_n^f(X, A, *)\), there is at least one map \( F : A \times S^n \rightarrow X \) which satisfies the above conditions. We say that \( F \) is an affiliated map to \([h]\) with respect to \( A \). Note that \([h]\) may have many affiliated maps to \([h]\) with respect to \( A \) which are not homotopic. We will abbreviate an affiliated map to \([h]\) with respect to \( A \) to an affiliated map to \([h]\) if no confusion arise. It is easy to see that \( G_n^f(X, A, *) \) form a subgroup of \( \pi_n(X, *) \).

Let \( A \) be locally compact and regular, and \( X^A \) be the space of mappings from \( A \) to \( X \) with compact-open-topology. The map \( p : X^A \rightarrow X \) given by \( p(g) = g(*) \) is continuous. We call \( p \) an evaluation map. Thus \( p \) induces homomorphisms

\[
p_\ast : \pi_n(X^A, f) \rightarrow \pi_n(X, *)
\]

for all \( n \). Then we have

**THEOREM 2.1.** \( p_\ast(\pi_n(X^A, f)) = G_n^f(X, A, *) \).

**Proof.** Since \( A \) is locally compact, any continuous map

\[
H : (S^n, *) \rightarrow (X^A, f)
\]

gives rise to a continuous associated map

\[
\phi(H) : A \times S^n \rightarrow X.
\]

Since \( \phi(H)(*, s) = (H(s))(*) = (pH)(s) \) and \( \phi(H)(a, *) = (H(*(a))) (a) = f(a) \), we have \([pH] = p_\ast[H] \in G_n^f(X, A, *)\).

Conversely, if \( F \) is an affiliated map to \([F(*, *)]\), define \( H \) by \( H = pF^{-1}(F) \). Then \([H] \in p_\ast(\pi_n(X^A, f)) \) and \( H(s) = F(*, s) \). This completes the theorem.

**REMARK.** Note that \( G_n(X, *, *) = \pi_n(X, *) \) and \( G_n^1(X, X, *) = G_n(X, *) \), where \( G_n(X, *) \) is the evaluation subgroup defined by Gottlieb [2].

D.H. Gottlieb proved the following result for \( CW \)-complexes \( A \) and \( X \) [2].

**THEOREM 2.2.** For any topological spaces \( A, X \) and any \( f : (A, *) \rightarrow (X, *) \), we have \( G_n(X, *) \leq G_n^f(X, A, *) \).
THEOREM 2.3. If $A$ is a subspace of $X$ and $i : (A, *) \longrightarrow (X, *)$ is inclusion, then $G^n_i(X, A, *) \subseteq G^n_f(X, A, *)$ for any map $f : (A, *) \longrightarrow (X, *)$ such that $f(A) \subseteq A$.

Proof. If $[h] \in G^n_i(X, A, *)$, there is an affiliated map

$F : A \times S^1 \longrightarrow X$

to $[h]$. Define a map $H : A \times S^1 \longrightarrow X$ by $H(a, s) = F(f(a), s)$.

DEFINITION 2.2. $G^n_i(X, A, *)$ will be denoted by $G^n(X, A, *)$.

DEFINITION 2.3. A space $X$ is an $H$-space iff there a point $* \in X$ and a continuous map $u : X \times X \longrightarrow X$ such that $u(x, *) = u(*, x) = x$ for all $x \in X$. We will write $u(x, y)$ by $x \cdot y$.

THEOREM 2.4. Suppose that $X$ is an $H$-space, then

$G_n(X, *) = G_n(X, A, *) = \pi_n(X, *)$.

Proof. By Gottlieb ([1], [2]).

The fact that $G_n(X, *) \leq G_n(X, A, *) \leq \pi_n(X, *)$ leads naturally to the questions: Is there topological pair $(X, A)$ for which $G_n(X, *) < G_n(X, A, *) < \pi_n(X, *)$? For this we give an example.

EXAMPLE. Let $X = \{ z \in C \mid |z| = 1, |z - 2| = 1 \}$, $A = S^1 = \{ z \in C \mid |z| = 1 \}$. Take $*$ by the point $(1, 0)$. Define $F : A \times S^1 \longrightarrow X$ by $F(z, w) = \pi w$. Then $F$ is well defined and continuous. Moreover $[F(*, *])$ is one of the generators of the free group $\pi_1(X, *)$ on two generators and $F(*, *)$ is inclusion $i : (A, *) \longrightarrow (X, *)$. Thus we have $G_1(X, A, *) \cong \mathbb{Z}$. On the other hand, $G_1(X, *) = \{ 0 \}$ (cf. Gottlieb [1]).

3. Some fundamental theorems

In this section we wish to study the category $\mathcal{V}^2$ whose objects are the pairs $(X, A)$ of spaces and morphisms are maps of pairs.

Let $\sigma : I \longrightarrow X$ be a path such that $\sigma(0) = x_0$ and $\sigma(1) = x_1$. Then $\sigma$ induces an isomorphism $\sigma_* : G_n(X, x_1) \cong G_n(X, x_0)$. Similarly we will show that, in the usual sense, $G_n(X, A, *)$, viewed as a subgroup of $\pi_n(X, *)$, is independent of the base point.

THEOREM 3.1. Let $(X, A)$ be a topological pair and $\sigma : I \longrightarrow A \subseteq X$ be a path such that $\sigma(0) = x_0$, $\sigma(1) = x_1$. Then $\sigma$ induces an isomorphism $\sigma_* : G_n(X, A, x_1) \cong G_n(X, A, x_0)$. 

Proof. If \( \alpha \in G_n(X, A, x_1) \), there exists an affiliated map

\[ F : A \times S^n \to X. \]

to \( \alpha \) such that \([F(x_1, \cdot)] = \alpha, F(\cdot, *) = i : A \to X\). Now define \( h : S^n \times I \to X \) by

\[ h(t, s) = F(\sigma(1-s), t). \]

It is clear that \([h(\cdot, 0)] = \alpha \) and \( h(\cdot, 1) \) represents \( \sigma_*(\alpha) \in \pi_n(X, x_0) \). Moreover \( F \) is an affiliated map to \([h(\cdot, 1)]\). Thus we have \( \sigma_*(G_n(X, A, x_1)) \subseteq G_n(X, X, x_0) \).

On the other hand, the reverse path \( \sigma^{-1} : I \to A \subseteq X \) induces the inverse isomorphism, \( (\sigma^{-1})_* : \pi_n(X, x_0) \to \pi_n(X, x_1) \), to \( \sigma_* \). Hence we complete the theorem.

It is not true that \( f : (X, A) \to (Y, B) \) induces a homomorphism from \( G_n(X, A, *) \) to \( G_n(Y, B, f(*)) \) \cite{1}. But, for some maps, it is true that \( f_* \) maps \( G_n(X, A, *) \) to \( G_n(Y, B, f(*)) \). Suppose \( r : (Y, B) \to (X, A) \) be map. We say that \( r \) has a right homotopy inverse if there is a map \( j : (X, A) \to (Y, B) \) such that \( lj \) is homotopic to \( 1_{(X, A)} \) (with homotopy of pair). Similarly we can define a left homotopy inverse.

**Theorem 3.2.** Let \((X, A)\) and \((Y, B)\) be in \( \mathcal{D}_2 \). Suppose \((X, A)\) is a CW-pair and \( B \) is path-connected. If \( r : (Y, B) \to (X, A) \) has a right homotopy inverse, then \( r_* : \pi_n(Y, *) \to \pi_n(X, r(*)) \) carries \( G_n(Y, B, *) \) into \( G_n(X, A, r(*)) \).

**Proof.** First we need a lemma.

**Lemma.** Under the same assumption of Theorem 3.2, there is a right homotopy inverse

\[ j' : (X, A) \to (Y, B) \]

such that \( j'(r(*)) = * \).

**Proof of Lemma.** Let \( j : (X, A) \to (Y, B) \) be a right homotopy inverse of \( r \) and \( \alpha : I \to B \subseteq Y \) be a path such that \( \alpha(0) = jr(*) \), \( \alpha(1) = * \). By the homotopy extension property, in the diagram

\[
\begin{array}{ccc}
  r(* \times I \cup A \times 0) & \to & B \subseteq Y \\
  \downarrow & & \\
  A \times I & \to & \\
  \alpha \cup j | A
\end{array}
\]

we have an extension \( K : A \times I \to B \subseteq Y \).

Again in the diagram...
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\[ A \times I \cup X \times 0 \xrightarrow{K \cup j} Y \]
\[ \xrightarrow{X \times I} \]

we have an extension \( K' : X \times I \longrightarrow Y \). Let \( j' = K'(,1) \), then \( j' = K'(,0) = j \) and \( rj' \sim rj \sim 1_{(X,A)} \) (homotopy of pair). Moreover \( j'(r(*)) = K'(r(*),1) = \alpha(1) = * \).

Now we continue the proof of Theorem 3.2.

If \( \alpha \in G_n(Y,B,*), \) there is an affiliated map

\[ F : B \times S^n \longrightarrow Y \]

to \( \alpha \). Define \( F' : A \times S^n \longrightarrow X \) by

\[ F'(a,s) = \tau (F(j'(a),s)). \]

Then \( F'(,*) = rj' \). Since \( rj' \) is homotopic to \( 1_{(X,A)} \), we can find a homotopy \( H : (X \times I, A \times I) \longrightarrow (X, A) \) such that

\[ H|_{A \times 0} = F'|_{A \times *, H|_{A \times 1} = 1_A}. \]

Define a map \( G : (A \times * \times I) \cup (A \times S^n \times 0) \longrightarrow X \) by

\[ G(a,*t) = H(a,t), \ G(a,s,0) = F'(a,s). \]

Then \( G \) is well defined and continuous. By the homotopy extension property, we have a homotopy

\[ (A \times * \times I) \cup (A \times S^n \times 0) \longrightarrow X \]
\[ \xrightarrow{G} \]
\[ A \times S^n \times I \]

\( H' : A \times S^n \times I \longrightarrow X \) connecting \( H'(,0) = F' \) to \( H'(,1) \).

Note that \( F'|_{r(*) \times S^n} : S^n \longrightarrow X \) represents \( r_*(a) \). Now let \( \alpha : I \longrightarrow X \) be given by

\[ \sigma(t) = H'(r(*),*,t) \subseteq A. \]

Thus by Theorem 3.1, \( \sigma \) induces an isomorphism

\[ \sigma_* : G_n(X,A,r(*)) \cong G_n(X,A,r(*)). \]

Let \( h : S^n \longrightarrow X \) be given by \( h = H'(r(*),1) \). Then \( \sigma_*[h] = r_*(a) \).

Moreover \([h] \in G_n(X,A,r(*)) \). Consequently \( r_*(a) = \sigma[h] \in G_n(X,A,r(*)) \).
COROLLARY 3.3. If \( r : (Y, B) \to (X, A) \) is a retract, \((X, A)\) is a CW-pair and \( B \) is path-connected, then \( r_* : \pi_n(Y, *) \to \pi_n(X, r(*)) \) carries \( G_n(Y, B, *) \) into \( G_n(X, A, r(*)) \).

THEOREM 3.4. Let \((X, A)\) and \((Y, B)\) be in \( \mathbb{H}^2 \). Let \((X, A)\) be a CW-pair and \( B \) be path-connected. If \( j : (Y, B) \to (X, A) \) has a lift homotopy inverse, then \( j_* (\alpha) \in G_n(X, A, x_0) \) implies \( \alpha \in G_n(Y, B, y_0) \) where \( j(y_0) = x_0 \).

Proof. Since \( j : (Y, B) \to (X, A) \) has a left homotopy inverse and \((X, A)\) is a CW-pair, we can find \( r : (X, A) \to (Y, B) \) such that \( r(x_0) = y_0 \) and \( rj \sim 1_{(Y, B)} \) (homotopy of pair) by the homotopy extension property. Let \( h_t : Y \to Y \) be the homotopy from \( rj \) to \( 1_{(Y, B)} \). Let \( \sigma : I \to Y \) be a closed path given by \( \sigma(t) = h_t(y_0) \). Then \( r_*j_* = \sigma_* : \pi_n(Y, y_0) \to \pi_n(Y, y_0) \).

If \( j_* (\alpha) \in G_n(X, A, x_0) \), then \( r_*j_* (\alpha) \in G_n(Y, B, r(x_0)) \) by Theorem 3.2. Hence \( \alpha = \sigma_*^{-1}r_*j_* (\alpha) \in G_n(Y, B, y_0) \) by Theorem 3.1.

Now we can prove that \( G_n(X, A) \) is a homotopy type invariant by using Theorems 3.2 and 3.4.

THEOREM 3.5. Suppose that \((X, A)\) and \((Y, B)\) are both the same homotopy type of a path-connected CW-pair. If \( f : (X, A) \to (Y, B) \) is a homotopy equivalence, then \( f_* \) carries \( G_n(X, A, *) \) isomorphically onto \( G_n(Y, B, f(*)) \).

Proof. First we assume that \((Y, B)\) is a CW-pair. By Theorem 3.2, we have \( f_*^{-1} (G_n(Y, B, f(*)) \subseteq G_n(X, A, *)) \). Similarly by Theorem 3.4, \( f_* (G_n(X, A, *)) \subseteq G_n(Y, B, f(*)) \). Thus \( G_n(Y, B) = f_* f_*^{-1} (G_n(Y, B)) \subseteq f_* (G_n(X, A)) \). Hence \( f_* (G_n(X, A)) = G_n(Y, B) \). Since \( f_* \) is an isomorphism, the theorem is true for the special case that \((Y, B)\) is a CW-pair.

Now in general, \((Y, B)\) is homotopy equivalent to a CW-pair \((Z, C)\). Let \( g : (Y, B) \to (Z, C) \) be a homotopy equivalence. Then \( gf \) is a homotopy equivalence. Thus \( g_*f_* \) carries \( G_n(X, A) \) isomorphically onto \( G_n(Z, C) \) and \( g_* \) carries \( G_n(Y, B) \) isomorphically onto \( G_n(Z, C) \). Hence \( f_* \) must carry \( G_n(X, A) \) isomorphically onto \( G_n(Y, B) \).

THEOREM 3.6. If \((X, A)\) and \((Y, B)\) are homotopy type of path-connected CW-pairs, then \( G_n(X \times Y, A \times B, (x_0, y_0)) \equiv G_n(X, A, x_0) \oplus G_n(Y, B, y_0) \).
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Proof. Since there exists an isomorphism \( h : \pi_n(\times Y, (x_0, y_0)) \to \pi_n(X, x_0) \oplus \pi_n(Y, y_0) \) such that \( h([\alpha]) = p_*([\alpha]) \oplus q_*([\alpha]) \), where \( p_* \) and \( q_* \) are induced homomorphisms from the projection \( X \times Y \) onto \( X \) and \( Y \) respectively. Now \( h(G_n(X \times Y, A \times B, (x_0, y_0)) \subseteq G_n(X, A, x_0) \oplus G_n(Y, B, y_0) \) as may readily be seen by noting that \( p \) and \( q \) are retractions and applying Corollary 3.3.

On the other hand, let \( [\alpha] \) and \( [\beta] \) be elements of \( G_n(X, A, x_0) \) and \( G_n(Y, B, y_0) \) respectively. Now \( h^{-1}([\alpha] \oplus [\beta]) = ([j\alpha] \cdot (k\beta)) \) where \( j \) and \( k \) inject \( X \to X \times y_0 \) and \( Y \to x_0 \times Y \) respectively. Let \( H : A \times S^n \to X \) be an affiliated map to \( [\alpha] \). Define \( K : A \times B \times S^n \to X \times Y \) such that \( K(x, y, s) = (H(x, s), y) \). The existence of \( K \) show that \( [j\alpha] \in G_n(X \times Y, A \times B, (x_0, y_0)) \). Similarly \( [k\beta] \in G_n(X \times Y, A \times B, (x_0, y_0)) \). Thus the product \( [j\alpha] \cdot [k\beta] = ([j\alpha] \cdot (k\beta)) \in G_n(X \times Y, A \times B, (x_0, y_0)). \) This completes the Theorem.

4. Relations between \( G_n(X, A, *) \) and \( G_n(X, *) \)

Theorem 4.1. Suppose \( S \) is a set of integers, and \( (X, A) \) is a CW-pair such that if \( e \in X - A \) is a cell, \( \dim e \in S \). Suppose that if \( m \in S, \pi_{n+m-1}(X, *) = \{0\} \). Then \( G_n(X, A, *) = G_n(X, *) \).

Proof. Since \( G_n(X, *) \subseteq G_n(X, A, *) \) we need only to prove that \( G_n(X, A, *) \subseteq G_n(X, *) \). If \( [f] \in G_n(X, A, *) \), there is an affiliated map \( H : A \times S^n \to X \) to \( [f] \). Let \( L = (A \times S^n) \cup (X \times *) \) and define a map \( K : L \to X \) by

\[
K(a, s) = H(a, s) \\
K(x, *) = x.
\]

Since \( (X \times S^n, L) \) is a CW-pair, there exists an extension \( K' \):

\[
\begin{array}{ccc}
L & \xrightarrow{K} & X \\
\downarrow & & \searrow \\
X \times S^n & \xrightarrow{K'} & X
\end{array}
\]

such that \( K' \big|_L = K \). Thus \( K' \) is the required affiliated map to \( [f] \) with respect to \( X \). Thus we have \( [f] \in G_n(X, *) \).

Theorem 4.2. Suppose \( S \) is a set of integers and \( (X, A) \) is a CW-pair such that if \( e \in A - * \) is a cell, \( \dim e \in S \). If \( m \in S, \pi_{n+m-1}(X, *) = \{0\} \). Then \( G_n f X, A, *) = \pi_n(X, *) \) for any \( f : A \to X \).
Proof. Let \([f] \in \pi_n(X, \ast)\). Define a map \(H : (A \times \ast) \cup (\ast \times S^n) \longrightarrow X\) by \(H(a, \ast) = a, H(\ast, s) = f(s)\). By Corollary 16.3 ([3] p131), there is an extension

\[K : A \times S^n \longrightarrow X\]

such that \(K|_{(A \times \ast) \cup (\ast \times S^n)} = H\). Thus \(K\) is an affiliated map to \([f]\), so that \([f] \in G_n(X, A, \ast)\).

In particular, if we take \(A = X\) in Theorem 4.2, we have

**Corollary 4.3.** Let \(X\) be a CW-complex and \(S = \{\dim e | e \in X - \ast\}\). Suppose that if \(m \in S\), then \(\pi_{n+m-1}(X, \ast) = \{0\}\). Then \(\pi_n(X, \ast) = G_n(X, \ast) = G_n'(X, A, \ast)\) for any \(A \subseteq X\) and for any \(f : A \longrightarrow X\).

In general, \(i_\ast(\pi_n(A, \ast)) \subseteq G_n(X, A, \ast)\) for \(i : A \longrightarrow X\). But under the same assumption of Theorem 4.2, we have \(i_\ast(\pi_n(A, \ast)) \subseteq G_n(X, A, \ast)\).

**Theorem 4.4.** If \(A\) is a retract of \(X\), then

\[G_n(X, A, \ast) \cap i_\ast(\pi_n(A, \ast)) = i_\ast(G_n(A, \ast))\]

**Proof.** \(G_n(X, A, \ast) \cap i_\ast(\pi_n(A, \ast)) \supseteq i_\ast(G_n(A, \ast))\) is obvious. Conversely, if \([f] \in G_n(X, A, \ast) \cap i_\ast(\pi_n(A, \ast))\), there is a map \(g : (S^n, \ast) \longrightarrow (A, \ast)\) such that \(i_\ast[g] = [ig] = [f]\). And there is an affiliated map \(F : A \times S^n \longrightarrow X\) to \([f]\). Define \(F' : A \times S^n \longrightarrow A\) by \(F' = rF\), where \(r : X \longrightarrow A\) is a retraction. Then \([F'(*, \ast)] = [rF] = r_*[f] = [ig] = [f]\). And \(F'(\ast, \ast) = r\ast = \ast\ast\). This implies \([f] = i_\ast[g] \subseteq i_\ast(G_n(A, \ast))\).

**Corollary 4.5.** If \(A\) is a retract of \(X\), we have

\[G_1(X, A, \ast) \cap i_\ast(\pi_1(A, \ast)) \subseteq i_\ast(Z(\pi_1(A, \ast))\]

where \(Z(A)\) denotes the center of the group \(A\).

**Proof.** Gottlieb [1].

**Corollary 4.6.** Let \(A\) be a compact polyhedron such that Euler–Poincare number \(x(A) \neq 0\) and be a retract of \(X\). Then we have

\[G_1(X, A, \ast) \cap i_\ast(\pi_1(A, \ast)) = \{0\}\]

**Proof.** Gottlieb [1].

**Theorem 4.7.** Let \(X\) be a CW-complex and \(\{X_a\}\) be the set of all finite subcomplexes. Then \(G_n(X, x) = \lim G_n(X, X_a, \ast)\).
Proof. Since \( \{X_a\} \) directed by induction \( (X_a \leq X_\beta \iff X_a \subseteq X_\beta) \), we can construct an inverse system
\[
\{G_n(X, X_a, \ast), (i_{X_aX_\beta})_\ast, \{X_a\}\}
\]
where \( (i_{X_aX_\beta})_\ast : G_n(X, X_\beta, \ast) \subseteq G_n(X, X_a, \ast) \) is an inclusion homomorphism. By the definition of inverse limit of the inverse system, we obtain the required result.

5. \( G \)-spaces and \( W \)-spaces

Define a subgroup \( P_n(X, A, \ast) \) of \( \pi_n(X, \ast) \) as follows:

**Definition 5.1.** \([f] \in P_n(X, A, \ast) \) iff for every \([g] \in \pi_m(A, \ast) \) and every \( m \), there exists a map \( G : S^m \times S^n \longrightarrow X \) such that \( G(_, \ast) = f, G(\ast, \ast) = \ast \) and \( \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast
\((X, \ast)\) is continuous, \([f] \in \pi_n(X, \ast)\). Thus there is a map \(F : S^m \times S^n \longrightarrow X\) such that \(F(\ast, \ast) = g\) and \(F(\ast, \ast) = f\). The existence of \(F\) implies \(\alpha \in G_n^f(X, S^m, \ast)\).

**Corollary 5.6.** \(P_n(X, \ast) = \bigcap_{f \in G_n^f(X, S^m, \ast)} \pi_n\)

**Corollary 5.7.** \([7]. P_n(S^a, \ast) = G_n^s(S^a, \ast)\) \[
\begin{cases}
0 & \text{for } n \text{ even} \\
Z & n = 1, 3 \text{ or } 7 \\
2Z & n = \text{odd}, \ n \neq 1, 3 \text{ or } 7.
\end{cases}
\]

### 6. Function space \(X^A\)

Now suppose that \(A\) is a locally compact and ANR space, then the evaluation map \(p : X^A \longrightarrow X\) is a fibering ([5]).

Let \(F\) be the fibre \(p^{-1}(\ast)\), then we have a long exact sequence \[
\cdots \longrightarrow \pi_n(F, f) \longrightarrow \pi_n(X^A, f) \longrightarrow \pi_n(X, \ast) \longrightarrow \pi_{n-1}(F, f) \longrightarrow \cdots
\]

where \(f : (A, \ast) \longrightarrow (X, \ast)\).

By Theorem 2.1, we have the following theorem.

**Theorem 6.1.** The next three statements are equivalent:

(i) \(p_*\) is epimorphism

(ii) \(G_n^f(X, A, \ast) = \pi_n(X, \ast)\)

(iii) For any \(g : (S^n, \ast) \longrightarrow (X, \ast)\), there is a lift \(\tilde{g} : (S^n, \ast) \longrightarrow (X^A, f)\) such that \([p \tilde{g}] = [g]\).

Combining theorem 6.1 and proposition 6.2([5] p. 152) we have

**Corollary 6.2.** If the fibering \(p : X^A \longrightarrow X\) admits a cross section \(\alpha : X \longrightarrow X^A\), then

\(G_n^f(X, A, \ast) = \pi_n(X, \ast)\) \((n \geq 1)\).

**Theorem 6.3.** If \(X\) is an H-space with \(\ast\) as unit, then we have \(\pi_n(X^A, f) \cong \pi_n(F, f) \oplus \pi_n(X, \ast)\) \((n \geq 1)\).

**Proof.** Define a map \(\alpha : X \longrightarrow X^A\) by

\[(\alpha(x))(a) = x \cdot f(a)\]

Then \(\alpha\) is well defined because \(\alpha(x) : A \longrightarrow X\) is continuous. Moreover \(p\alpha = 1_X\).
Now we will show that \( \alpha \) is continuous. Since \( A \) is locally compact, 
\( \alpha : X \to X^A \) is continuous if and only if \( \alpha : X \times A \to X \) is continuous. 
But the continuity of \( \alpha \) is clear.

Moreover \( \alpha(*)=f \). This \( \alpha \) is a cross-section. Thus we have the results 
for \( n \geq 2 \) by proposition 6.2([5]p.152).

In case \( n=1 \), we have the short exact sequence, since \( X \) is an \( H \)-space,

\[
0 \to \pi_1(F,f) \to \pi_1(X^A,f) \to \pi_1(X,\ast) \to 0.
\]

Since \( \alpha : X \to X^A \) is a cross-section, it induces a homomorphism \( \alpha_* : \pi_1(X,\ast) \to \pi_1(X^A,f) \) and \( p_*\alpha_*=\alpha_*(p\alpha)=1 \). Thus need only to prove 
\( \pi_1(X^A,f) \) is abelian. But the two multiplications in \( [(S^A,\ast),(X,\ast)] \) 
are the same and they are commutative ([3] p. 65). So that \( [S,X^A] \cong \pi_1(X^A) \) is abelian. This completes the theorem.

**Corollary 6.4.** Let \( X \) be an \( H \)-space. Then

\[
\pi_n(X^S,\ast)=\pi_n(X,\ast) \oplus \pi_{n+q}(X,\ast) \quad n \geq 1
\]

**Proof.** By Whitehead theorem, we have \( \pi_n(F,f) \cong \pi_{n+q}(X,\ast) \).

**Corollary 6.5.**

\[
\begin{align*}
\pi_1(S^1\times S^1, S^0) & \cong Z \oplus Z \oplus Z \\
\pi_1(S^1\times S^1, S^0) & \cong Z \oplus Z \quad (q>0) \\
\pi_n(S^1\times S^1, S^0) & \cong \{0\} \quad (n>1, \quad q \geq 0).
\end{align*}
\]

**Corollary 6.6.** Let \( \pi \) be abelian and \( K(\pi,n) \) be an Eilenberg–Maclane space. Then

\[
\pi_m(K(\pi,n), S^0) \cong \begin{cases} 
\pi \oplus \pi & \text{if } m=n \\
\{0\} & \text{if } m \neq n
\end{cases} \quad (q=0)
\]

\[
\begin{cases} 
\pi & \text{if } m=n \\
\pi \oplus \pi & \text{if } m+q=n \quad (q>0) \\
\{0\} & \text{otherwise}
\end{cases}
\]

**Proof.** Since \( \pi \) is abelian, \( K(\pi,n) \) is an \( H \)-space.

**Remark.** (1) S.S. Koh [6] proved that if \( X \) is an \( H \)-space then 
\( \pi_p(X,S^q) / \pi_{p+q}(X) \cong \pi_p(X) \). We generalized this.

(2) We can calculate the homotopy groups for \( S^1 \times S^1 \times \cdots \times S^1, \quad S^3 \times S^3 \times \cdots \times S^3 \) and so on.
References


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