ON EXTENSIONS OF THE CARISTI-KIRK FIXED
POINT THEOREM

By Sehie Park

1. Introduction

Since the appearance of the Caristi–Kirk fixed point theorem in [4], various proofs and several applications are given by a number of authors. For the literature, see Caristi [5] and Park [13]. Among those applications are fixed point theorems for maps satisfying inwardness conditions [4], results concerning normal solvability [11], metric convexity [12], characterization of metric completeness [12], [13], and many others.

Also there have appeared generalizations of the theorem. In fact, Kasahara [10] gave an $L$-space version of a common fixed point result for a family of the Caristi–Kirk type maps. Downing and Kirk [6] obtained a generalization and some of its applications. Also, Siegel [16] gave another generalization with simple constructive proof.

In the present paper, we show that Siegel's theorem includes the results of Downing–Kirk and Kasahara in the metric version, and we provide constructive proofs of results of Downing–Kirk and Kasahara. Also, we note that Downing–Kirk's generalization is actually equivalent to the Caristi–Kirk theorem. Simultaneously, we give a number of other equivalent formulations of the Caristi–Kirk theorem and some of their applications.

2. Siegel's Theorem

Let $M$ and $N$ be complete metric spaces, $f : M \rightarrow N$ be closed, that is, for $\{ x_n \} \subset M$ the conditions $x_n \rightarrow x$ and $fx_n \rightarrow y$ imply $fx = y$, and $\phi : fM \rightarrow \mathbb{R}$ be a lower semicontinuous function.

In order to give our main result, we begin with following lemmas.

Lemma 1. Let $\{ x_n \}$ be a sequence in $M$ such that
max \{d(x_i, x_{i+1}), d(fx_i, fx_{i+1})\} \leq \phi(fx_i) - \phi(fx_{i+1})
for each i, then \(\lim_{i \to \infty} x_i = \bar{x}\) exists and
\[
\max\{d(x_i, \bar{x}), d(fx_i, f\bar{x})\} \leq \phi(fx_i) - \phi(f\bar{x})
\]
for each i.

**Proof.** Since \(\{\phi(fx_i)\}\) is decreasing to some \(r \geq 0\) and
\[
\max\{d(x_i, x_j), d(fx_i, fx_j)\} \leq \phi(fx_i) - \phi(fx_j)
\]
for \(i \leq j\), \(\{x_i\}\) and \(\{fx_i\}\) are Cauchy sequences in \(M\) and \(N\), respectively. Since \(M\) and \(N\) are complete and \(f\) is closed, there exists \(\bar{x} \in M\) such that \(x_i \to \bar{x}\) and \(fx_i \to f\bar{x}\). On the other hand, we have
\[
\max\{d(x_i, \bar{x}), d(fx_i, f\bar{x})\} = \max\{\lim_{j \to \infty} d(x_i, x_j), \lim_{j \to \infty} d(fx_i, fx_j)\}
\leq \phi(fx_i) - \lim_{j \to \infty} \phi(fx_j)
\leq \phi(fx_i) - \phi(f\bar{x})
\]
from the lower semicontinuity of \(\phi\).
Let \(h_i : M \to M\), \(1 \leq i < \infty\). The countable composition of \(\{h_i\}\) is defined by
\[
\prod_{i=1}^{\infty} h_i(x) = \lim_{i \to \infty} h_i h_{i-1} \cdots h_1(x)
\]
if the limit exists for each \(x \in M\).
Let \(\mathcal{H}^*\) denote the set of all \(h : M \to M\) satisfying the condition
\[
\max\{d(x, hx), d(fx, fhx)\} \leq \phi(fx) - \phi(fhx)
\]
for each \(x \in M\).

**Lemma 2.** \(\mathcal{H}^*\) is closed under countable composition.

**Proof.** For any \(h_1, h_2 \in \mathcal{H}^*\),
\[
\max\{d(\bar{x}, h_2 h_1 x), d(fx, fh_2 h_1 x)\}
\leq \max\{d(\bar{x}, h_1 x), d(fx, fh_1 x)\} + \max\{d(h_1 x, h_2 h_1 x), d(fh_1 x, fh_2 h_1 x)\}
\leq \{\phi(fx) - \phi(fh_1 x)\} + \{\phi(fh_1 x) - \phi(fh_2 h_1 x)\} = \phi(fx) - \phi(fh_2 h_1 x)
\]
shows that \(\mathcal{H}^*\) is closed under composition. By putting \(x_i = h_i h_{i-1} \cdots h_1(x)\) for each \(x \in X\) from Lemma 1, we have the conclusion.
For any \(A \subset M\), let \(r(A) = \text{glb}_{x \in A}\{\phi(fx)\}\). Then \(B \subset A\) implies \(r(B) \geq r(A)\). For any \(\mathcal{H} \subset \mathcal{H}^*\), let \(\mathcal{H}(x) = \{hx \mid h \in \mathcal{H}\}\).

**Lemma 3.** \(\text{diam } \mathcal{H}(x) \leq 2[\phi(fx) - r(\mathcal{H}(x))]\).

**Proof.** For any \(h_1, h_2 \in \mathcal{H}\), we have
\[
d(h_1 x, h_2 x) \leq d(x, h_2 x) + d(x, h_2 x)
\leq \phi(fx) - \phi(fh_1 x) + \phi(fx) - \phi(fh_2 x)
\leq 2[\phi(fx) - r(\mathcal{H}(x))]\].

The following is our version of Siegel's theorem.
THEOREM 1. Let \( M \) and \( N \) be complete metric spaces, \( f : M \to N \) be closed, \( \phi : fM \to \mathbb{R}_+ \) be a lower semicontinuous function, and \( \mathcal{K}^\ast \) denote the family of all \( h : M \to M \) satisfying
\[
\max \{ d(x, hx), d(fx, fhx) \} \leq \phi(fx) - \phi(fhx) \quad (\ast)
\]
for each \( x \in X \). Let \( \mathcal{K} \subset \mathcal{K}^\ast \) be closed under composition and \( x_0 \in M \).

(a) If \( \mathcal{K} \) is closed under countable composition, then there exists an \( h \in \mathcal{K} \) such that \( \bar{x} = hx_0 \) and \( h\bar{x} = \bar{x} \) for all \( h \in \mathcal{K} \).

(b) If each map in \( \mathcal{K} \) is continuous, then there exist a sequence \( \{h_i\} \) in \( \mathcal{K} \) and
\[
\bar{x} = \lim_{i \to \infty} h_i h_{i-1} \cdots h_1 (x_0)
\]
in \( M \) such that \( h\bar{x} = \bar{x} \) for each \( h \in \mathcal{K} \).

Proof. Let \( \{\epsilon_i\} \) be a positive sequence converging to 0. Choose an \( h_1 \in \mathcal{K} \) such that
\[
\phi(fh_1 x_0) - r(\mathcal{K}(x_0)) < \epsilon_1/2.
\]
Set \( x_1 = h_1 x_0 \). Since \( \mathcal{K} \) is closed under composition, \( \mathcal{K}(x_1) \subset \mathcal{K}(x_0) \) and
\[
\text{diam}(\mathcal{K}(x_1)) \leq 2 \left[ \phi(fx_1) - r(\mathcal{K}(x_1)) \right] 
\]
\[
\leq 2 \left[ \phi(fh_1 x_0) - r(\mathcal{K}(x_0)) \right] < \epsilon_1.
\]
Repeating this process, we get a sequence \( \{h_i\} \) such that
\[
x_{i+1} = h_i (x_i), \quad \mathcal{K}(x_{i+1}) \subset \mathcal{K}(x_i) \quad \text{and} \quad \text{diam}(\mathcal{K}(x_i)) < \epsilon_i.
\]

(a) Let \( \bar{h} = \prod_{i=1}^\infty h_i \) and \( \bar{x} = \bar{h}(x_0) \). Since \( \bar{x} = \prod_{i=1}^\infty h_j(x_i) \), we have \( \bar{x} \in \mathcal{K}(x_i) \) for each \( i \). Moreover, since \( \lim_{i \to \infty} \text{diam}(\mathcal{K}(x_i)) = 0 \), we have \( \bar{x} = \cap_{i=0}^\infty \mathcal{K}(x_i) \). Now it remains to check that \( h\bar{x} = \bar{x} \) for each \( h \in \mathcal{K} \). Since \( h\bar{x} = h(\prod_{i=1}^\infty h_j(x_i)) \), we have \( h\bar{x} \in \mathcal{K}(x_i) \) for each \( i \), whence \( h\bar{x} = \bar{x} \).

(b) Let \( \bar{x} = \lim_{i \to \infty} h_i h_{i-1} \cdots h_1 (x_0) = \lim_{i \to \infty} x_i \). Since \( \{x_j\}_{i<j} \subset \mathcal{K}(x_i) \) for each \( i \), we have \( \bar{x} \in \text{cl} \mathcal{K}(x_i) \), the closure of \( \mathcal{K}(x_i) \). Since \( \text{diam}(\text{cl} \mathcal{K}(x_i)) = \text{diam}(\mathcal{K}(x_i)) \), we have \( \bar{x} = \cap_{i=0}^\infty \text{cl} \mathcal{K}(x_i) \). Now observe \( h\bar{x} \in \mathcal{K}(x_i) \) for each \( i \). Since \( h \) is continuous, for any \( \epsilon > 0 \) there exists \( i_0 \) such that \( B_r(h\bar{x}) \cap \mathcal{K}(x_i) \neq \phi \), \( i > i_0 \). Therefore, for \( i > i_0 \), \( d(h\bar{x}, \bar{x}) < \epsilon + \epsilon_i \), and since \( \epsilon_i \to 0 \) we have \( d(h\bar{x}, \bar{x}) \leq \epsilon \). Since \( \epsilon \) is arbitrary, we have \( h\bar{x} = \bar{x} \).

The above proof, which is given here for the completeness, is a slight modification of that of Siegel [16]. Theorem 1 also can be deduced by the method of Bröndsted [3].

In view of Lemma 2 and Theorem 1, we have

COROLLARY. Let \( M, N, f, \) and \( \mathcal{K} \) be the same as in Theorem 1. Then the family \( \mathcal{K}^\ast \) has a common fixed point. Further, if \( h \in \mathcal{K}^\ast \) is continuous, then for any \( x_0 \in M \), \( \bar{x} = \lim_{i \to \infty} h^i x_0 \) is a fixed point of \( h \).

Kasahara [10] obtained an \( L \)-space version of the first part of Corollary.
for the case that $M = N$ and $f = 1_M$.

By putting $N = M$ and $f = 1_M$, Theorem 1 reduces to the following

**THEOREM 2 (Siegel [16]).** Let $M$ be a complete metric space, $\phi : M \to \mathbb{R}_+$ be a lower semicontinuous function, and $\mathcal{R}^*$ denote the family of all $h : M \to M$ satisfying

$$d(x, hx) \leq \phi(x) - \phi(hx)$$

for each $x \in X$. Let $\mathcal{R} \subset \mathcal{R}^*$ be closed under composition and $x_0 \in M$. Then the conclusions of Theorem 1 hold.

However, Theorems 1 and 2 are equivalent. To see this, in Theorem 2, let us introduce the metric

$$\rho(x, y) = \max \{d(x, y), d(fx, fy)\}, \quad x, y \in M,$$

on $M$. Since $f : M \to N$ is closed with $M$ and $N$ complete, $(M, \rho)$ is complete and $\phi \circ f$ is l.s.c. Hence, Theorem 2 applied to $\mathcal{R}$ on $(M, \rho)$ yields conclusions (a) and (b) relative to $(M, \rho)$ and, since $d(x, y) \leq \rho(x, y)$, $x, y \in M$, the same conclusions hold in $(M, d)$.

3. Equivalent formulations

For a single map, Corollary of Theorem 1 can be stated as follows:

**PROPOSITION 1.** Let $M, N, f, \text{ and } \phi$ be the same as in Theorem 1. If

(i) a map $h : M \to M$ satisfies the condition (*) for each $x \in M$,

then $h$ has a fixed point.

**PROPOSITION 2 (Downing–Kirk [6]).** Let $M, N, f, \text{ and } \phi$ be the same as in Theorem 1. If

(ii) $h : M \to M$ is a map and $c$ is a positive constant such that

$$\max \{d(x, hx), c d(fx, fhx)\} \leq \phi(fx) - \phi(fhx)$$

for each $x \in M$,

then $h$ has a fixed point.

**Proof.** By putting $k = \max \{1, 1/c\}$, we have

$$\max \{d(x, hx), d(fx, fhx)\} \leq k \phi(fx) - k \phi(fhx)$$

from (ii). Since $k \phi$ is also lower semicontinuous, Proposition 2 follows from Proposition 1.

In [6], the authors used Proposition 2 to prove surjectivity theorems for nonlinear closed maps $f : X \to Y$ where $X$ and $Y$ are Banach spaces.

Proposition 2 is actually an equivalent formulation of Proposition 1, for, by putting $c = 1$, (ii) implies (i).

By putting $N = M, f = 1_M$ (and $c = 1$) in Propositions 1 and 2, we have
PROPOSITION 3 (Caristi-Kirk [4]). Let $M$ be a complete metric space, $h : M \to M$ a map and $\phi : M \to \mathbb{R}_+$ a lower semicontinuous function. If
\[ d(x, hx) \leq \phi(x) - \phi(hx) \]
for each $x \in X$, then $h$ has a fixed point.

Proposition 3 is useful to locate fixed points of selfmaps $h$ such that there exist $u \in M$ and $\alpha \in [0, 1)$ satisfying
\[ d(hx, h^2 x) \leq \alpha d(x, hx) \]
for each $x$ in $\text{cl}\{h^nu\}$ and $h$ is continuous on $\text{cl}\{h^nu\}$ (Park [13]). Among such type of maps is one satisfying the condition
\[ d(hy, hy) \leq \alpha \max\{d(x, y), d(x, hx), d(y, hy), [d(x, hy) + d(y, hx)]/2\} \]
for $x, y \in M$.

As for Theorems 1 and 2, note that Propositions 1 and 3 are equivalent.

We give another equivalent form of the Downing-Kirk theorem.

PROPOSITION 4. Let $M, N, f,$ and $\phi$ be the same as in Theorem 1. If

(iii) there exists a map $g : M \to N$ and a choice function $\tilde{g}$ of $\{f^{-1}gx \mid x \in M\}$ such that $gM \subseteq fM$ and
\[ \max\{d(x, \tilde{g}x), d(fx, gx)\} \leq \phi(fx) - \phi(gx) \]
for each $x \in M$,

then there exists $\tilde{x} \in M$ such that $f\tilde{x} = g\tilde{x}$ and $\tilde{g}\tilde{x} = \tilde{x}$.

Proof. By Proposition 1, there exists $\tilde{x} \in M$ such that $g\tilde{x} = \tilde{x}$. Since $\tilde{x} = \tilde{g}\tilde{x} \in f^{-1}g\tilde{x}$, we have $f\tilde{x} = g\tilde{x}$.

In Proposition 4, since $g = f\tilde{g}$, the condition (iii) reduces to (i). Hence, Proposition 4 is equivalent to Proposition 1.

We can prove Proposition 4 by the method of the proof of the main result of [6], which is based on the idea of Brøndsted [2].

The function $\phi : M \to \mathbb{R}_+$ in the Caristi-Kirk theorem and in other results in this paper can be replaced by $\phi : M \to \mathbb{R}$ bounded from below. However, the condition “bounded from below” can not be dispensable. For example, if we put $M = N = \mathbb{R}$, $f = 1_\mathbb{R}$, $g_x = x - 1$, $\phi = 1_\mathbb{R}$ in Proposition 4, then $\tilde{g}$ has no fixed point. Also the condition $gM \subseteq fM$ in Proposition 4 can not be dispensable. For example, if we put $M = N = \mathbb{R}_+$, $f_x = x + 1$, $g = 1_{\mathbb{R}_+}$, $\phi = 1_\mathbb{R}$, in Proposition 4, then $f$ and $g$ have no coincidence.

By putting $N = M$ and $f = 1_M$. Proposition 4 reduces to the Caristi–Kirk theorem. Further, by putting $N = M$ and $g = 1_M$, Proposition 4 reduces to

COROLLARY 1. Let $M$ be a complete metric space, $f : M \to M$ a closed surjection, and $\phi : M \to \mathbb{R}_+$ a lower semicontinuous function. If for any $x \in X$
there exists $y \in f^{-1}x$ such that
\[
\max\{d(x, y), d(x, fx)\} \leq \phi(fx) - \phi(x),
\]
then $f$ has a fixed point.

From Corollary 1, we have

**COROLLARY 2.** Let $M$ be a complete metric space, $f : M \to M$ a closed surjection, and $\phi : M \to \mathbb{R}_+$ a lower semicontinuous function. If for any $y \in M$,
\[
\max\{d(y, fy), d(fy, f^2y)\} \leq \phi(f^2y) - \phi(fy)
\]
holds, then $f$ has a fixed point.

The following is also equivalent to Proposition 1.

**PROPOSITION 5.** Let $M$ and $N$ be complete metric spaces, $f : M \to N$ a closed map, and $\phi : fM \to \mathbb{R}$ a lower semicontinuous function bounded from below. Then there exists a point $p \in X$ such that
\[
\phi(fp) - \phi(fx) < \max\{d(p, x), d(fp, fx)\}
\]
for each $x \in M$ other than $p$.

This was given in [15]. From Proposition 5, we obtain

**PROPOSITION 6 (Ekeland [7], [8]).** Every lower semicontinuous function $\phi$ from a complete metric space $M$ into $\mathbb{R}_+$ has a $d$-point $p$ in $M$, that is, we have
\[
\phi(p) - \phi(x) < d(p, x)
\]
for every other point $x$ in $M$.

Proposition 6 is equivalent to the Caristi–Kirk Theorem (see [1]).

In view of Proposition 6, we give another equivalent formulations of the Caristi–Kirk theorem.

**PROPOSITION 7.** Let $X$ be a set, $M$ a complete metric space, and $f, g : X \to M$ maps such that
1. $f$ is surjective, and
2. there exists a lower semicontinuous function $\phi : M \to \mathbb{R}_+$ satisfying
\[
d(fx, gx) \leq \phi(fx) - \phi(gx)
\]
for each $x \in X$.

Then $f$ and $g$ have a coincidence.

**Proof.** By Proposition 6, $\phi$ has a $d$-point $p \in M$. Let $x \in f^{-1}p$. Suppose $fx \neq gx$. Since $p = fx$ and $gx \in M$ we have
\[
\phi(fx) - \phi(gx) < d(fx, gx),
\]
which contradicts (2).
By putting $X = M$ and $f = 1_M$, Proposition 7 reduces to the Caristi-Kirk theorem.

The condition (2) can be replaced by various contractive type conditions without affecting the conclusion. In fact, Goebel [9] used the condition

$$(2)'' \quad d(gx, gy) \leq \alpha \, d(fx, fy), \quad \alpha \in [0, 1),$$

and gave an application to the unique existence of solution of differential equation of the form $x' = f(t, x)$. Park [14] extended this fact by using the Meir-Keeler type contractive condition:

$$(2)''' \quad \text{for a given } \varepsilon > 0 \text{ there exists a } \delta(\varepsilon) > 0 \text{ such that for } x, y \in X,$$

$$\varepsilon \leq d(fx, fy) < \varepsilon + \delta \implies d(gx, gy) < \varepsilon,$$

and $fx = fy$ implies $gx = gy$.

From Proposition 7, we obtain a Downing-Kirk type result as follows:

**Proposition 8.** Let $M$ and $N$ be metric spaces and $h : M \to M$ a map. Suppose there exist a map $f : M \to N$, a lower semicontinuous function $\phi : fM \to \mathbb{R}_+$, and a constant $c > 0$, such that $fM$ is complete and for each $x \in M$,

$$\max \{d(x, hx), cd(fx, fhx)\} \leq \phi(fx) - \phi(fhx).$$

Then $h$ has a fixed point.

**Proof.** Since $(1/c)\phi$ is lower semicontinuous, putting $fh = g$ and $(1/c)\phi = \phi'$, we have

$$d(fx, gx) \leq \phi'(fx) - \phi'(gx).$$

Therefore, by Proposition 7, $f$ and $fh$ have a coincidence $\bar{x} \in X$. Since

$$d(\bar{x}, h\bar{x}) \leq \phi(f\bar{x}) - \phi(fh\bar{x}),$$

we have $\bar{x} = h\bar{x}$.

Note that, by putting $X = M$, $f = 1_M$, and $c = 1$, Proposition 8 reduces to Proposition 3.

Moreover, by putting $X = M$ and $g = 1_M$, Proposition 8 reduces to

**Proposition 9.** Let $M$ be a complete metric space and $f : M \to M$ be a surjection. If there exists a lower semicontinuous function $\phi : M \to \mathbb{R}_+$ satisfying

$$d(x, fx) \leq \phi(fx) - \phi(x)$$

for $x \in M$, then $f$ has a fixed point.

Proposition 9 may be used to show the existence of fixed point of certain maps $f$ satisfying

$$d(x, fx) \leq \alpha \, d(fx, f^2x), \quad \alpha \in [0, 1)$$

for each $x \in M$, since we can put
\[ \phi(x) = \frac{\alpha}{1-\alpha} d(x, fx). \]

Among such type are maps \( f \) satisfying
\[
d(x, y) \leq \alpha \max \{ d(fx, fy), d(x, fx), d(y, fy), [d(x, fy) + d(y, fx)]/2 \}
\]
for \( x, y \in M \).

Note also that in Proposition 9, choosing \( y \in f^{-1}x \), we have
\[
d(y, fy) \leq \phi(y) - \phi(fy)
\]
for each \( y \in M \). This shows that the Caristi-Kirk theorem follows from Proposition 9.

Finally, from Proposition 7 we have

**Corollary.** Let \( F \) be a selfmap of a Banach space \( B \), \( \alpha, \beta \) numbers, \(|\alpha| \neq |\beta| \), and \( F_{a,\beta} = \alpha 1_B + \beta F \). If \( F_{a,\beta}(B) \subset F_{\beta,a}(B) \), \( F_{\beta,a}(B) \) is closed in \( B \), and if there exists a lower semicontinuous function \( \phi : B \to \mathbb{R}^+ \) satisfying
\[
\|F_{a,\beta}(x) - F_{\beta,a}(x)\| \leq \phi(F_{\beta,a}(x)) - \phi(F_{a,\beta}(x))
\]
for any \( x \in B \), then \( F \) has exactly one fixed point.

**Proof.** From Proposition 7 \( F_{a,\beta} \) and \( F_{\beta,a} \) have a coincidence \( x \in B \). It is clear that \( x \) is the unique fixed point of \( F \).

Note that Goebel [9] showed the same result using the condition
\[
\|F_{a,\beta}(x) - F_{\beta,a}(y)\| \leq k \|F_{\beta,a}(x) - F_{\beta,a}(y)\|, \quad 0 \leq k < 1,
\]
for \( x, y \in B \), instead of the inequality in the above Corollary. In [14], this result was extended to the Meir–Keeler type condition similar to (2)'.

**References**


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Seoul National University
Seoul 151, Korea