CHARACTERIZATIONS OF $r$-SEMISTABLE PROBABILITY MEASURES ON HILBERT SPACES

BY DONG M. CHUNG

1. Introduction

A sequence $\{S_n = a_n^{-1} \sum_{j=1}^{n} X_j + b_n\}$, where $a_n > 0$, $b_n \in \mathbb{R}$, the reals, $X_j$'s are independent identically distributed real random variables, is called a sequence of normed sums. It is well known [7] that the class of stable distributions in $\mathbb{R}$ coincides with the weak limits of the sequences of distribution of normed sums. This result, on one hand, has been generalized to general topological vector spaces [1,4]. Recently, on the other hand, Kruglov [3] and Mejzler [8] considered subsequences $\{S_{k_n}\}$ of $\{S_n\}$ such that $k_n/k_{n+1} \to r$ with $0 < r \leq 1$ as $n \to \infty$, and gave a characterization (in terms of the characteristic functions) of the distributions which are weak limits of the distributions of $S_{k_n}$. These distributions are, in fact, similar to the so-called semi-stable distributions considered earlier by Levy [6].

Motivated from the work of Kruglov [3], Mejzler [8] and Levy [6], we define $r$-semistable probability (prob.) measures ($0 < r \leq 1$) on a real separable Hilbert space $H$ and obtain characterizations of these measures on $H$. We also study the relationship between stable and $r$-semistable prob. measures on $H$. The organization of this paper is as follows:

In Section 2 we present notation preliminaries which will be needed in this paper. In Section 3 we characterize non-degenerate $r$-semistable prob. measures ($0 < r \leq 1$) on a real separable Hilbert space. As a corollary of this result, we show that the class of 1-semistable prob. measures coincides with that of stable prob. measures on $H$. We also show that the class of $r$-semistable prob. measures ($0 < r < 1$) is closed with respect to the weak topology. In Section 4 we present a characterization of non-degenerate $r$-semistable prob. measures ($0 < r \leq 1$) on $H$ in terms of their characteristic functionals. Using this result, we give an example of $r$-semistable prob. measures on $\mathbb{R}$ which shows that the class of stable prob. measures is properly contained in the class of $r$-semistable prob. measures for every $r \in (0,1)$. 

Received Dec. 20, 1979
2. Preliminaries

In this section we collect necessary notation, definitions and some known results which will be used in this paper.

Let $H$ denote a real separable Hilbert space. $\mathbb{R}$ and $\mathbb{R}^+$ will denote the reals and positive reals, respectively. By a prob. measure on $H$ we will always mean that it is defined on $\mathcal{B}(H)$, the smallest $\sigma$-algebra containing all the open sets of $H$. $M(H)$ will denote the set of all prob. measures on $H$. If $\mu, \nu \in M(H)$, the convolution of $\mu$ and $\nu$ is defined by

$$\mu * \nu (B) = \int_H \mu(B - x) \nu(dx)$$

for every $B \in \mathcal{B}(H)$. The symbol $\mu^{*n}$ will denote $\mu$ convoluted $n$ times with itself. It is well known that $M(H)$ becomes an abelian topological semigroup with the topology of weak convergence of measures, and convolution as a multiplication. For $\mu \in M(H)$, the characteristic functional (ch.f.) of $\mu$, denote by $\hat{\mu}(\cdot)$, is the complex-valued function on $H$ defined by

$$\hat{\mu}(y) = \int_H e^{i\langle x, y \rangle} d\mu(x)$$

where $\langle \ , \ \rangle$ is the inner product on $H$ and $y \in H$. It is easily seen that, for every $\mu, \nu \in M(H)$, $\hat{\mu} \hat{\nu}(\cdot) = \hat{\mu}(\cdot) \hat{\nu}(\cdot)$. It is well-known that every $\mu \in M(H)$ is uniquely determined by its ch. f.. The prob. measure $\delta_x$ defined by $\delta_x(B) = 0$ if $x \in B$, and 1 if $x \in B$, is said to be degenerated at $x \in H$.

**Definition 2.1** A $\mu \in M(H)$ is said to be infinitely divisible (i. d.) if for each positive integer $n$, there exists a $\lambda_n \in M(H)$ such that $\mu = \lambda_n^{*n}$.

**Definition 2.2** A triangular array of measures $\{\lambda_{n,j}\} \subset M(H)$ ($j = 1, 2, \ldots, k_n ; n = 1, 2, \ldots$) is said to be uniformly infinitesimal if for every neighborhood of zero in $H$,

$$\lim \inf_{1 \leq j \leq k_n} \lambda_{n,j}(U) = 1$$

**Theorem 2.3** [9] A function $\phi(y)$ is the ch. f. of an i. d. prob. measure $\mu$ on $H$ if and only if it is of the form

$$\phi(y) = \exp \left\{ i\langle x_0, y \rangle - \frac{1}{2} \langle By, y \rangle + \int_H \left( e^{-i\langle x, y \rangle} - 1 - i\langle x, y \rangle \frac{1}{1 + ||x||^2} \right) dM(x) \right\}$$

where $x_0 \in H$, $S$ is an $S$-operator and $M$ is a $\sigma$-finite measure with finite outside every neighborhood of zero and

$$\int_{||x|| < 1} ||x||^2 \ dM(x) < \infty.$$ 

The above representation is unique.

If $\mu$ is an i. d. prob. measure on $H$, we denote it by $\mu = (x_0, S, M)$ where $x_0$, $S$ and $M$ are the three quantities in the representation of Theorem 2.3.
If \( \mu \) is an i.d. prob. measure on \( H \) with the representation \((x_0, S, M)\), it follows from Theorem 2.3 that for every \( t > 0 \), \((tx_0, tS, tM)\) is the ch. f. of some i.d. prob. measure on \( H \). In view of this point, we define \( \mu' \) by the i.d. prob. measure with representation \((tx_0, tS, tM)\).

3. A characterization of \( r \)-semistable probability measures on \( H \)

In this section we define \( r \)-semistable prob. measures, \( 0 < r \leq 1 \), and obtain characterizations of such measures on \( H \). We begin with a few definitions.

**Definitions 3.1** Let \( \mu \in M(H) \) and \( 0 < r \leq 1 \), then we say that \( \mu \) is \( r \)-semistable if there exist a \( \nu \in M(H) \), two sequences \( \{a_n\} \subset \mathbb{R}^+ \), \( \{x_n\} \subset H \) and an increasing sequence \( \{k_n\} \) of positive integers such that

\[
\lim_{n \to \infty} \frac{k_n}{k_{n+1}} = r \quad (3.1)
\]

and

\[
\lim_{n \to \infty} T_{a_n} \nu^k \delta_{x_n} = \mu \quad (3.2)
\]

If \( k_n = n \) for all \( n \), we say that \( \mu \) is stable.

We denote by \( S_r \) \((0 < r \leq 1)\) the set of all \( r \)-semistable prob. measures on \( H \). The following lemmas will be needed for the proof of main theorems in this section.

**Lemma 3.2** Let \( \mu \in M(H) \); suppose that there exist a \( \nu \in M(H) \), sequences \( \{a_n\}, \{x_n\} \) from \( \mathbb{R}^+ \) and \( H \), respectively, and an increasing sequence \( \{k_n\} \) of positive integers such that \((3.2)\) holds. Then \( \mu \) is infinitely divisible. Consequently, every \( r \)-semistable prob. measure on \( H \) is i.d.

**Proof.** By the same argument as in Lemma 2 of \( 2 \), we have \( \lim a_n = 0 \).

This shows that \( \{T_{a_n} \mu_j\} \) with \( \mu_j = \mu \) for \( 1 \leq j \leq k_n, \ n = 1, 2, \ldots \), is an uniformly infinitesimal triangular array of measures on \( H \). Hence it follows from \( 9, p \ 199 \) that \( \mu \) is i.d.. The second assertion is obvious.

**Lemma 3.3** Let \( \mu \in M(H) \) be non-degenerate and \( r \)-semistable \((0 < r < 1)\). Then there exist an \( a(r) \in \mathbb{R} \) and an \( x(r) \in H \) such that

\[
\mu' = T_{a(r)} \mu_k \delta_{x(r)} \quad (3.3)
\]

**Proof.** Suppose that \( \nu \in M(H) \), \( \{a_n\} \), \( \{x_n\} \) and \( \{k_n\} \) satisfy the conditions \((3.1)\) and \((3.2)\) of Definition 3.1. Then by Lemma 3.2, \( \mu \) is i.d. and hence \( \mu' \) is defined. The relation \((3.1)\) implies that, for every \( y \in H \)

\[
\lim_n T_{a_n} \nu^k(a_n y) \cdot e^{i x_n} y' = \mu(y).
\]
Hence we have
\[
(\tilde{\varphi}_{k_{n+1}}(a_{n+1}y) \cdot e^{i<z_{n+1},y>} k_{n+1}) = \tilde{\varphi}_{k_n}(a_n \cdot \frac{a_{n+1}}{a_n} y) \cdot e^{i<z_n,y>} e^{i<z_{n+1},y>},
\]
where \( z_n = \frac{k_n}{k_{n+1}} x_{n+1} \cdot \frac{a_{n+1}}{a_n} x_n \). Now putting \( b_n = \frac{a_{n+1}}{a_n} \), \( \lambda_n = T_{a_n} \mu \circ k_n \circ \delta_{z_n} \)
we have
\[
\tilde{\varphi}_{k_n}(b_n y) \cdot e^{i<z_n,y>} \longrightarrow \tilde{\mu}'(y) \quad \text{as} \quad n \to \infty. \quad (3.4)
\]

Now show that there exists a subsequence \( \{m\} \) of \( \{n\} \) such that \( b_m \to a(r) \)
\( \in \mathbb{R}^+ \) and \( z_m \to x(r) \in H \) in \( H \). Suppose \( \lim \inf b_n = 0 \) or \( \lim \sup b_n = \infty \). If \( \lim \inf b_n = 0 \), then there exists a subsequence \( \{b_m\} \) of \( \{b_n\} \) such that \( b_m \to 0 \) as \( m \to \infty \). Since \( |\tilde{\varphi}_m(b_m y)| \to |\tilde{\mu}(y)|^r \) and \( b_m \to 0 \), it follows that \( |\tilde{\mu}(y)|^r = 1 \)
for all \( y \in H \), which contradicts that \( \mu \) is non-degenerate. If \( \lim \sup b_n = 0 \),
there exists a subsequence \( \{b_m\} \) such that \( b_m \to 0 \) as \( m \to \infty \). Hence we have
\[
|\tilde{\varphi}_m(y)| = |\tilde{\varphi}_m(b_m^{-1} b_m y)| \to |\tilde{\mu}(y)| = 1
\]
for all \( y \in H \). This also contradicts that \( \mu \) is non-degenerate. Thus there exists a subsequence \( \{m\} \) of \( \{n\} \) such that \( b_m \to a(r) \in R \), which implies that \( T_{b_m} \lambda_m \to T_{a(r)} \mu \). Now show that \( z_m \to x(r) \) in \( H \). By (3.4), it follows that
\[
e^{i<z_m,y>} \longrightarrow \frac{\tilde{\mu}'(y)}{\tilde{\mu}(y)}
\]
uniformly on any bounded sphere \( \{y : \|y\| \leq M\} \) in \( H \). (note \( \mu(y) \neq 0 \) for
every \( y \in H \)) Hence one can show that \( \{z_m\} \) is a Cauchy sequence in \( H \).
Therefore, there exists an \( x(r) \) in \( H \) such that \( z_m \to x(r) \) as \( m \to \infty \).

By the above arguments, we have \( a(r) \in \mathbb{R}^+ \), \( x(r) \in H \) such that
\[
\tilde{\mu}'(y) = \tilde{\mu}(a(r) y) e^{i<x(r),y>}. \quad \text{Since} \quad \mu \in \mathbb{M}(H) \quad \text{is uniquely determined by its characteristic functional, it follows that} \quad \mu' = T_{a(r)} \mu \circ \delta_{x(r)}'. \quad \text{This completes the proof.}
\]

We now state and prove the main result.

**Theorem 3.4** Let \( \mu \in \mathbb{M}(H) \) be non-degenerate, and let \( r \in (0,1) \). Then the following statements are equivalent:

(a) \( \mu \) is \( r \)-semistable

(b) \( \mu \) is i. d. and there exist an \( a(r) \in (0,1) \) and an \( x(r) \in H \) such that
\[
\mu' = T_{a(r)} \mu \circ \delta_{x(r)}'. \quad (3.5)
\]

**Proof.** (a) \( \Rightarrow \) (b) : From Lemmas 3.2 and 3.3, we need only show that \( a(r) \in (0,1) \). We first note that the relation (3.3) implies that
\[
|\tilde{\mu}(a(r) y)| = |\tilde{\mu}(y)|^r. \quad (3.6)
\]
We also note [9, p.171] that \( \tilde{\mu}(y) \neq 0 \) for every \( y \in H \).
Suppose that \( a(r) = 1 \). Then by the relation (3.6), we have \( |\hat{\mu}(y)|^{-r} = 1 \), which contradicts that \( \mu \) is nondegenerate. Now suppose that \( a(r) > 1 \). Then since \( |\hat{\mu}(y)| \leq 1 \) and \( r \in (0, 1) \), we have \( |\hat{\mu}(a(r)y)| = |\hat{\mu}(y)|^r \geq |\hat{\mu}(y)| \) for every \( y \in H \). Hence by iterating this \( n \) times, we obtain \( |\hat{\mu}(y)| \geq |\hat{\mu}(0)| = 1 \) for every \( y \in H \). This contradicts that \( \mu \) is non-degenerate. Hence \( a(r) \in (0, 1) \).

(b) \( \Rightarrow \) (a): We first show that if \( \mu \) is a non-degenerate prob. measure on \( H \), then \( \{\mu^t : t > 0\} \) is weakly continuous. Let \( \mu = (x_0, S, M) \), and let \( \{t_n\} \) be a sequence in \( \mathbb{R}^+ \) converging to \( t \). Then we need to show that \( \mu^s = (tx_0, ts, tM) \) weakly converges to \( \mu^t = (tx_0, tS, tM) \). This is easily shown by verifying the conditions (1) - (3) of Theorem 5.5 [9, p189].

By iterating the relation (3.5) \( n \) times \( (n = 1, 2, \ldots) \), we obtain a sequence \( \{\varepsilon_n\} \) in \( H \) such that

\[
\mu^n = T_{a(r)n}^n \mu \delta_{\varepsilon_n}.
\]

Now let's choose an increasing sequence \( \{k_n\} \) of positive integers such that \( r^n \cdot k_n \rightarrow 1 \) as \( n \rightarrow \infty \) (for example, \( k_n = \text{the integral part of } r^{-n} \)). Then clearly, \( \{k_n\} \) satisfies the condition (3.1). By taking \( k_n \)-th power on both sides of the equation (3.7) and letting \( n \rightarrow \infty \), we obtain, by the weak continuity of \( \mu^t \),

\[
\lim_{n \to \infty} T_{a(r)n}^{k_n \mu^t} \delta_{y_n} = \mu,
\]

where \( y_n = k_n \varepsilon_n \). This shows that \( \mu \) is \( r \)-semistable.

**Remark 3.5.** A certain subclass of \( r \)-semistable prob. measures on \( H \) has been studied in [5], where a characterization similar to the one obtained in Theorem 3.4 is given. However, the approaches in the two papers are different.

**Theorem 3.6** Let \( \mu \in M(H) \) be non-degenerate. Then the following statements are equivalent:

(a) \( \mu \) is 1-semistable

(b) \( \mu \) is i.d. and for each \( r \in (0, 1) \), there exist an \( a(r) \in (0, 1) \) and an \( x(r) \in H \) such that \( \mu^t = T_{a(r)n} \mu \delta_{x(r)} \).

**Proof.** (a) \( \Rightarrow \) (b): Let \( \mu \) be non-degenerate and 1-semistable. Then by definition, there exist a \( \nu \in M(H) \), \( a_n \subseteq \mathbb{R}^+ \), \( x_n \subseteq H \), and an increasing sequence \( \{k_n\} \) of positive integers such that the conditions (3.1) and (3.2) hold with \( r = 1 \). By Lemma 3.2, \( \mu \) is i.d., and hence \( \mu^t \) is well-defined for every \( r > 0 \). Since \( k_n \rightarrow \infty \) and \( k_n/k_{n+1} \rightarrow 1 \), it follows that for each \( r \in (0, 1) \), we can choose a subsequence \( \{r(n)\} \) of positive integers such that \( k_n/k_{r(n)n} \rightarrow r \) as \( n \rightarrow \infty \). Now let \( r \in (0, 1) \) be fixed. Since \( k_n/k_{r(n)n} \rightarrow r \) as \( n \rightarrow \infty \), we have

\[
(\rho_{k_n}(a_{r(n)n})y) \cdot e^{i < x_{r(n)n}, y>} k_n/k_{r(n)n}.
\]
$$\hat{\mu}_n(y) \cdot e^{i \langle x_n, y \rangle} \cdot e^{i \langle a_n, y \rangle} \rightarrow \hat{\mu}^r(y)$$

where $z_n = \frac{k_n}{k_{r(n)}} \cdot x_{r(n)} - \frac{a_{r(n)}}{a_n} x_n$. Hence by using the same argument as in Lemma 3.3, it follows that there exist an $a(r) \in (0, 1)$ and an $x(r)$ such that

$$\mu^r = T_{a(r)} \cdot \mu^r \cdot \delta_{x(r)}.$$

(b) \(\Rightarrow\) (a): Let (b) hold. Then by putting $r = \frac{1}{n}$, $n = 2, 3, \ldots$, we have an sequence $\{a(n)\} \subset \mathbb{R}^+$ and $\{x(n)\} \subset H$ such that

$$\mu = T_{a(n)} \cdot \mu^r \cdot \delta_{x(n)}.$$

By taking $n$-th power on the both sides of (3.8), we obtain $\mu = T_{a(n)} \cdot \mu^r \cdot \delta_{x(n)}$ for every $n = 2, 3, \ldots$, where $z_n = nx(n)$. This shows that $\mu$ is 1-semistable.

**COROLLARY 3.7** Let $\mu$ be non-degenerate. Then the following statements are equivalent:

(a) $\mu$ is 1-semistable

(b) $\mu$ is stable

(c) is $r$-semistable for every $r \in (0, 1)$.

**PROPOSITION 3.8** For every $r \in (0, 1)$, the class $S_r$ is closed in $M(H)$; that is, if $\{\mu_n\}$ is a sequence in $S_r$ converging to some $\mu \in M(H)$, then $\mu \in S_r$.

Proof. Let $r \in (0, 1)$ be fixed, and let $\{\mu_n\}$ be a sequence in $S_r$ converging to $\mu \in M(H)$. If $\mu$ is degenerate, then, clearly, $\mu \in S_r$. Now let $\mu$ be non-degenerate. Then we have, by Theorem 3.4,

$$\mu_n = T_{a_n(r)} \cdot \mu^r \cdot \delta_{z_n(r)}$$

Since $\hat{\mu}_n(y) \rightarrow \hat{\mu}(y)$ for every $y \in H$, it follows that

$$\hat{\mu}_n(a_n(r) y) \cdot e^{i \langle x_n(r), y \rangle} \rightarrow \hat{\mu}^r(y).$$

Hence by using the same arguments, as in Lemmas 3.3 we get $\hat{\mu}^r(y) = \hat{\mu}(a(r)) \cdot e^{i \langle x(r), y \rangle}$ for every $y \in H$. Since every $\mu \in M(H)$ is uniquely determined by its ch. f., we have $\mu^r = T_{a(r)} \cdot \mu^r \cdot \delta_{x(r)}$. This shows that $\mu \in S_r$.

4. **A representation of the characteristic functional of $r$-semistable probability measures $(0 < r \leq 1)$ on $H$**

In this section we present a representation of the characteristic functional of $r$-semistable prob. measures $(0 < r \leq 1)$ on $H$. Using this result, we will give an example which shows that $S_1$ is properly contained in $S_r$ for every $r \in (0, 1)$.

The proof of the theorem in this section can be carried out by using Theorems 3.4 and 3.6 of Section 3 and the method given in [5].

**THEOREM 4.1** Let $\mu \in M(H)$ be non-degenerate, and let $r \in (0, 1)$. Then
Characterizations of $r$-semistable probability measures on Hilbert spaces

$\mu$ is $r$-semistable if and only if either

$$\hat{\mu}(y) = \exp \left\{ i\langle x_0, y \rangle - \frac{1}{2} \langle Sy, y \rangle \right\},$$

(4.1)

where $x_0 \in H$ and $S$ is an $S$-operator, or

$$\hat{\mu}(y) = \exp \left\{ i\langle x_0, y \rangle + \int_H \left( e^{i\langle x, y \rangle} - 1 - \frac{i\langle x, y \rangle}{1 + \|x\|^2} \right) dM(x) \right\},$$

(4.2)

where $x_0 \in H$ and $M$ is $\sigma$-finite measure outside every neighborhood of zero in $H$ such that

$$\int_{\|x\| < 1} \|x\|^2 dM(x) < \infty$$

(4.3)

and there exists a unique $\alpha \in (0, 2)$ such that $rM = T_{r^\alpha} M$.

In view of the above theorem, we will call $\alpha$ (0 < $\alpha$ < 2) the type of $\mu$, where $\alpha$ is the constant appearing in Theorem 4.1.

Corollary 4.2 [2, 10]. Let $\mu \in M(H)$ be non-degenerate. Then $\mu$ is 1-semistable if and only if either $\hat{\mu}$ satisfies (4.1) or $\hat{\mu}$ satisfies (4.2) and $M$ satisfies (4.3) and there exists a unique $\alpha \in (0, 2)$ such that $rM = T_{r^\alpha} M$ for all $r \in (0, 1)$.

The following example is a modification of an example of Levy [8].

Example 4.3. Let's take $H = \mathbb{R}$. For fixed $r \in (0, 1)$, we define a $\sigma$-finite Borel measure $M$ on $R$ by

$$M(\{x\}) = \begin{cases} r^{-k}, & \text{for } x = \pm r^k, \ k = 0, 1, 2, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

Then we easily note that $M(\{0\}) = 0$, and $M(U) < \infty$, for every neighborhood $U$ of 0. We also note that

$$\int_{\|x\| < 1} \|x\|^2 dM(x) = 2 \cdot \sum_{k=0}^{\infty} r^k = \frac{2}{1 - r} < \infty.$$ 

Hence, by Theorem 4.10 of [9], the function on $\mathbb{R}$ defined by

$$\phi(t) = \exp \left\{ \int_{\mathbb{R}} (\cos xt - 1) dM(x) \right\} = \exp \left\{ 2 \cdot \sum_{k=0}^{\infty} r^{-k} (\cos r^k t - 1) \right\}$$

is a ch. f. of a non-Gaussian symmetric i. d. prob. measure on $\mathbb{R}$. Further, it is easy to see that $rM = T_{r^\alpha} M$. Hence, it follows from Theorem 4.1 that $\phi$ is the ch. f. of a $r$-semistable prob. measure on $\mathbb{R}$ of the type 1.

We now assert that $\phi$ is not the ch. f. of a 1-semistable prob. measure. We choose an $r_1 \in (0, 1)$ so that $\log r_1 / \log r$ is not an integer. Then, clearly,

$$r_1 M(\{r^k\}) = T_{r_1^\alpha} M(\{r^k\}),$$

for any $\alpha \in (0, 2)$, $k = 0, \pm 1, \pm 2, \ldots$. Therefore, the assertion follows from Corollary 4.2.
References


