Polarization and Unconditional Constants of $P(2d_*(1, w)^2)$

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Abstract. We explicitly calculate the polarization and unconditional constants of $P(2d_*(1, w)^2)$.

1. Introduction

According to the Krein-Milman Theorem, every nonempty convex set in a Banach space is fully described by the set of its extreme points. We recall that if $C$ is a convex set in a Banach space, a point $e \in C$ is said to be extreme if $x, y \in C$ and $e = \lambda x + (1 - \lambda)y$ for some $0 < \lambda < 1$ implies that $x = y = e$. Let $n \in \mathbb{N}$. We write $B_E$ for the closed unit ball of a real Banach space $E$. We denote by $\text{ext}B_E$ the sets of all the extreme points of $B_E$. We denote by $L(nE)$ the Banach space of all continuous $n$-linear forms on $E$ endowed with the norm $\|T\| = \sup_{\|x]\|=1} |T(x_1, \ldots, x_n)|$. A $n$-linear form $T$ is symmetric if $T(x_1, \ldots, x_n) = T(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ for every permutation $\sigma$ on $\{1, 2, \ldots, n\}$. We denote by $L_s(nE)$ the Banach space of all continuous symmetric $n$-linear forms on $E$. A mapping $P : E \to \mathbb{R}$ is a continuous $n$-homogeneous polynomial if there exists a unique $T \in L_s(nE)$ such that $P(x) = T(x, \ldots, x)$ for every $x \in E$. In this case it is convenient to write $T = \hat{P}$. We denote by $P(nE)$ the Banach space of all continuous $n$-homogeneous polynomials from $E$ into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$. It is well-known that $\|P\| \leq \|T\| \leq \frac{n^n}{n!} \|P\| (\forall P \in P(nE))$.

For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [7].

We will denote by $T((x_1, y_1), (x_2, y_2)) = ax_1x_2 + by_1y_2 + c(x_1y_2 + x_2y_1)$ and $P(x, y) = ax^2 + by^2 + cxy$ a symmetric bilinear form and a 2-homogeneous polynomial.

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polynomial on a real Banach space of dimension 2, respectively. We denote the predual of two dimensional real Lorentz sequence space with a positive weight 0 < w < 1 by

\[ d_s(1, w)^2 := \{ (x, y) \in \mathbb{R}^2 : \| (x, y) \|_{d_s} := \max \{ |x|, \sqrt{|x|^2 + |y|^2/w}\} \}. \]

In [22] the nth polarization constant of \( E \) is defined by

\[ c_{\text{pol}}(n : E) = \inf \{ M > 0 : \| P \| \leq M \| P \| \text{ for every } P \in \mathcal{P}(n E) \}. \]

Let \( X^n \) denote the monomial \( x_1^{\alpha_1} \cdots x_m^{\alpha_m} \), where \( X = (x_1, \ldots, x_m) \in \mathbb{R}^m \) and \( \alpha = (\alpha_1, \ldots, \alpha_m) \) with \( \alpha_k \in \mathbb{N} \cup \{ 0 \} \), \( 1 \leq k \leq m \). If \( P(X) = \sum_{|\alpha| \leq n} a_\alpha X^\alpha \) is a polynomial of degree \( n \) on \( \mathbb{R}^m \), we define its modulus \( |P| \) by \( |P|(X) = \sum_{|\alpha| \leq n} |a_\alpha| X^\alpha \). We define the nth unconditional constant of \( d_s(1, w)^2 \) by

\[ c_{\text{unc}}(n : d_s(1, w)^2) = \inf \{ M > 0 : \| P \| \leq M \| P \| \text{ for every } P \in \mathcal{P}(^n d_s(1, w)^2) \}. \]

Gámez-Merino et al. [9] classify the extreme points of the unit ball of \( \mathcal{P}(^2 \square) \) and, using its extreme points, compute the polarization and unconditional constants of \( \mathcal{P}(^2 \square) \), where \( \square \) is the unit square of vertices \((0, 0), (0, 1), (1, 0), (1, 1)\). The author [14] characterized the extreme points of the unit ball of \( \mathcal{P}(^2 d_s(1, w)^2) \). Recently, the author [16] calculated the norm of symmetric bilinear form of \( \mathcal{L}_s(2 d_s(1, w)^2) \) and classified the extreme points of the unit ball of \( \mathcal{L}_s(2 d_s(1, w)^2) \). We refer to ([1–6], [8–22]) and references therein for some recent work about extremal properties of multilinear mappings and homogeneous polynomials on some classical Banach spaces. By the Krein-Milman Theorem, a convex function (like a polynomial norm, for instance) defined on a convex set (like the unit ball of a finite dimensional polynomial space) attains its maximum at one extreme point of the convex set. In this paper, using the results of [14] and [16] with the Krein-Milman Theorem, we explicitly calculate \( c_{\text{pol}}(2 : d_s(1, w)^2) \) and \( c_{\text{unc}}(2 : d_s(1, w)^2) \) as follows:

\( a \)

If \( w \leq \sqrt{2} - 1 \), then \( c_{\text{pol}}(2 : d_s(1, w)^2) = 2 \frac{(1 + w^2)}{(1 + w^2)^2} \);

\( b \)

If \( w > \sqrt{2} - 1 \), then \( c_{\text{pol}}(2 : d_s(1, w)^2) = 1 + w^2 \);

\( c \)

If \( w \leq \sqrt{2} - 1 \), then \( c_{\text{unc}}(2 : d_s(1, w)^2) = 1 + w^2 + \frac{1 + w^2}{(1 + w^2)^2} \);

\( d \)

If \( w > \sqrt{2} - 1 \), then \( c_{\text{unc}}(2 : d_s(1, w)^2) = 1 + w^2 + \frac{4 w^2}{(1 + w^2)^2} \).

2. The results

**Theorem 2.1** ([16]). Let \( T((x_1, y_1), (x_2, y_2)) := (a, b, c) \in \mathcal{L}_s(2 d_s(1, w)^2) \) with \( |b| \leq a, c \geq 0 \). Then:

**Case 1:** \( b \geq 0 \)

**Subcase 1:** \( c > a \)

If \( w \leq \frac{c - a}{c + a} \) then \( \| T \| = (a + b)w + c(1 + w^2) \).

If \( w > \frac{c - a}{c + a} \), then \( \| T \| = bw^2 + 2cw + a \).

**Subcase 2:** If \( c \leq a \), \( \| T \| = bw^2 + 2cw + a \).
Case 2: \( b < 0 \)

Subcase 1: \( c < |b| \)
If \( w \leq \frac{c}{|b|} \), then \( \|T\| = \max\{bw^2 + 2cw + a, (a - b)w + c(1 - w^2)\} \).
If \( w > \frac{c}{|b|} \), then \( \|T\| = \max\{(a - bw^2, (a - b)w + c(1 - w^2))\} \).

Subcase 2: \( c \geq |b| \)
If \( w \leq \frac{|b|}{c} \), then \( \|T\| = \max\{bw^2 + 2cw + a, (a - bw^2, (a - b)w + c(1 - w^2))\} \).
If \( w > \frac{|b|}{c} \), then \( \|T\| = \max\{bw^2 + 2cw + a, (a + b)w + c(1 + w^2)\} \).

Theorem 2.2 ([14]). Let \( P \in \mathcal{P}(2d_*(1,w)^2) \) with \( P(x,y) = ax^2 + by^2 + cxy \) for \( (x,y) \in d_*(1,w)^2 \) with \( a \geq |b| \geq 0, c \geq 0 \). Then
Case 1: \( 0 \leq c < 2|b| \)
Subcase 1: \( b < 0 \)
(a) If \( \frac{c^2}{4|b|} \leq w \), then
\[ \|P\| = a + \frac{c^2}{4|b|}. \]
(b) If \( \frac{c^2}{4|b|} > w \), then
\[ \|P\| = bw^2 + cw + a. \]
Subcase 2: \( b > 0 \), then
\[ \|P\| = bw^2 + cw + a. \]
Case 2: \( 2|b| \leq c \leq 2a \), then
\[ \|P\| = bw^2 + cw + a. \]
Case 3: \( 2a < c \)
(a) If \( \frac{a^2 - 4ab}{c - a - b} \leq w \), then
\[ \|P\| = bw^2 + cw + a. \]
(b) If \( \frac{a^2 - 4ab}{c - a - b} \geq w \), then
\[ \|P\| = \frac{(c^2 - 4ab)(1 + w)^2}{4(c - a - b)}. \]

Theorem 2.3 ([14]).
\[
\text{ext} B_{\mathcal{P}(2d_*(1,w)^2)} = \{ \pm x^2, \pm y^2, \pm \frac{1}{1 + w^2}(x^2 + y^2), \pm \frac{1}{(1 + w)^2}(x \pm y)^2 \\
\pm t(x^2 - y^2) \pm 2\sqrt{t(1 - t)xy} \left( \frac{1}{1 + w^2} \leq t \leq 1 \right), \\
\pm t(x^2 - y^2) \pm 2\sqrt{1 - t^2(1 + w)^4} \left( 0 \leq t \leq \frac{1 - w}{(1 + w)(1 + w^2)} \right) \text{.}
\]
Then \( \Phi \) and only if \( \Phi(\mathbb{B}) \in \mathbb{B}^* \).

Now, let \( \Phi : \mathbb{B} \to \mathbb{B} \) be an isometrically isomorphism. Moreover, \( \Phi \) is an isometrically isomorphism.

Lemma 2.4. Let \( w^* = \frac{1-w}{1+w} \). Then, there is an isometry \( \phi : d_*(1, w) \to d_*(1, w^*) \) such that
\[
\phi(x, y) := \left( \frac{x + y}{1+w}, \frac{x - y}{1+w} \right).
\]

Proof. By definition, the norms of \( (x, y) \in d_*(1, w) \) and \( (X, Y) \in d_*(1, w^*) \) are given by
\[
\| (x, y) \|_{d_*(1, w)} = \max \left\{ |x|, |y|, \frac{|x| + |y|}{1+w} \right\},
\]
\[
\| (X, Y) \|_{d_*(1, w^*)} = \max \left\{ |X|, |Y|, \frac{|X| + |Y|}{1+w^*} \right\}.
\]
Now, let \( (X, Y) = \phi(x, y) = \left( \frac{x + y}{1+w}, \frac{x - y}{1+w} \right) \). Then
\[
\| (X, Y) \|_{d_*(1, w^*)} = \max \left\{ \frac{|x + y|}{1+w}, \frac{|x - y|}{1+w} \right\}.
\]
\[
= \max \left\{ \frac{|x| + |y|}{1+w}, \frac{|x + y| + |x - y|}{2} \right\}.
\]
\[
= \max \left\{ \frac{|x| + |y|}{1+w}, \max\{|x|, |y|\} \right\}.
\]
\[
= \| (x, y) \|_{d_*(1, w)}.
\]

Lemma 2.5. Let \( 0 < w < 1 \), \( w^* = \frac{1-w}{1+w} \). Define \( \Phi : \mathcal{P}(d_*(1, w)^2) \to \mathcal{P}(d_*(1, w^*)^2) \) by \( \Phi(P) = P \circ \phi^{-1} \), where \( \phi \) is the isometry in Lemma 2.4. Then \( \Phi \) is an isometrically isomorphism. Moreover, \( P \in \text{ext} B_{\mathcal{P}(d_*(1, w)^2)} \) if and only if \( \Phi(P) \in \text{ext} B_{\mathcal{P}(d_*(1, w^*)^2)} \).
Remark. Note that, for \( \frac{1}{1+w} \leq t \leq 1 \),

\[
\Phi(P_t)(X,Y) = \frac{2\sqrt{t(1-t)}}{(1+w^*)^2}(X^2 - Y^2) \pm \left( \frac{2}{1+w^*} \right)^2 XY
\]

\[
= Q_s(X,Y) \text{ for a unique } 0 \leq s \leq \frac{1-w^*}{(1+w^*)(1+(w^*)^2)},
\]

where \( w^* = \frac{1-w}{1+w} \) and \( X = \frac{x}{1+w}, Y = \frac{y}{1+w} \).

**Theorem 2.6.** (a) If \( w \leq \sqrt{2} - 1 \), then \( c_{pol}(2 : d_*(1, w)^2) = \frac{2(1+w)}{(1+w^*)^2} \).

(b) If \( w > \sqrt{2} - 1 \), then \( c_{pol}(2 : d_*(1, w)^2) = 1 + w^2 \).

**Proof.** By the Krein-Milman Theorem,

\[
c_{pol}(2 : d_*(1, w)^2) = \max\{\|\tilde{P}_1\|, \|\tilde{Q}_s\|, \|\tilde{R}_k\| : \frac{1}{1+w^*} \leq t \leq 1, \quad 0 \leq s \leq \frac{1-w}{(1+w)(1+w^*)}, \quad k = 1, 2, 3, 4 \}.
\]

Note that \( \|\tilde{R}_k\| = 1 \) for \( k = 1, 2, 3, 4 \). We claim that \( \max\{\|\tilde{P}_1\| : \frac{1}{1+w^*} \leq t \leq 1\} = 1 + w^2 \). Note that \( \tilde{P}_1((x_1, y_1), (x_2, y_2)) = tx_1x_2 - ty_1y_2 + \sqrt{t(1-t)}(x_1y_1 + x_2y_2) \) for \( \frac{1}{1+w^*} \leq t \leq 1 \). Simple calculation shows that, for \( \frac{1}{1+w^*} \leq t \leq 1 \),

\[
t \leq \frac{(1+w^2)^2}{2(1+w^2)} \Leftrightarrow 2wt + \sqrt{t(1-t)}(1-w^2) \geq (1+w^2)t.
\]

Let \( g(t) = 2wt + \sqrt{t(1-t)}(1-w^2) \) for \( \frac{1}{1+w^*} \leq t \leq 1 \). Then \( g'(t) = 0 \) for \( \frac{1}{1+w^*} \leq t \leq 1 \) implies that \( t = \frac{(1+w^2)^2}{2(1+w^*)} \).

Case 1: \( w \geq \sqrt{2} - 1 \)

Obviously, \( \frac{1}{1+w^*} \leq \frac{(1+w^2)^2}{2(1+w^*)} < 1 \) and \( \frac{1}{1+w^*} \leq \frac{(1+w^2)^2}{2(1+w^*)} < \frac{3w-w^3}{1+w^*} < 1 + w^2 \). It follows that, by Theorem 2.1 (Case 2, Subcase 1),

\[
\max\{\|\tilde{P}_1\| : \frac{1}{1+w^*} \leq t \leq 1\}
\]

\[
= \max\{\max\{\|\tilde{P}_1\| : \frac{1}{1+w^*} \leq t \leq \frac{(1+w^2)^2}{2(1+w^*)}, \max\{\|\tilde{P}_1\| : \frac{1}{1+w^*} \leq t \leq 1\}\} \}
\]

\[
= \max\{\max\{1+w^2\}t, 2wt + \sqrt{t(1-t)}(1-w^2) : \frac{1}{1+w^*} \leq t \leq \frac{(1+w^2)^2}{2(1+w^*)}\}, \max\{1+w^2\}t, 2wt + \sqrt{t(1-t)}(1-w^2) : \frac{1}{1+w^*} \leq t \leq 1\}
\]

\[
= \max\{\max\{2wt + \sqrt{t(1-t)}(1-w^2) : \frac{1}{1+w^*} \leq t \leq \frac{(1+w^2)^2}{2(1+w^*)}\}, \max\{2wt + \sqrt{t(1-t)}(1-w^2) : \frac{1}{1+w^*} \leq t \leq 1\}\}
\]

\[
\max\{(1+w^2)t : \frac{1}{1+w^*} \leq t \leq 1\}
\]
\[= \max \{ \max \{ g(\frac{1}{1+w^2}), g(\frac{(1+w)^2}{2(1+w^2)}) \}, 1+w^2 \} \]
\[= \max \{ \frac{3w-w^3}{1+w^2}, \frac{(1+w)^2}{2}, 1+w^2 \} \]
\[= 1+w^2. \]

Case 2: \( w < \sqrt{2} - 1 \)

Obviously, \( \frac{(1+w)^2}{2(1+w^2)} < \frac{1}{1+w^2} \). Since \( \frac{1}{1+w^2} \leq t \leq 1 \),

\[\max\{ (1+w^2)t, 2wt + \sqrt{t(1-t)(1-w^2)} : \frac{1}{1+w^2} \leq t \leq 1 \} = (1+w^2)t.\]

It follows that by Theorem 2.1 (Case 2, Subcase 1),

\[\max\{ \| \hat{P}_t \| : \frac{1}{1+w^2} \leq t \leq 1 \} = \max\{ (1+w^2)t, 2wt + \sqrt{t(1-t)(1-w^2)} : \frac{1}{1+w^2} \leq t \leq 1 \} = (1+w^2).\]

We claim that \( \max\{ \| \hat{Q}_s \| : 0 \leq s \leq \frac{1-w}{(1+w)(1+w^2)} \} = 2(1+w^2)\).

By Lemmas 2.4–5, \( Q_s(x,y) = P_t(X,Y) \) for some \( \frac{1}{1+(w^*)^2} \leq t \leq 1 \), where \( w^* = \frac{1-w}{1+w} \) and \( X = \frac{x+y}{1+w}, Y = \frac{x-y}{1+w} \). By the above claim, it follows that

\[\max\{ \| \hat{Q}_s \| : 0 \leq s \leq \frac{1-w}{(1+w)(1+w^2)} \} = \max\{ \| \hat{P}_t((X_1,Y_1),(X_2,Y_2)) \| : \frac{1}{1+(w^*)^2} \leq t \leq 1 \} = 1+(w^*)^2 \]
\[= \frac{2(1+w^2)}{(1+w)^2}. \]

Therefore,
\[c_{pol}(2 : d_*(1,w)^2) = \max\{1+w^2, \frac{2(1+w^2)}{(1+w)^2} \}.\]

Since \( \frac{2(1+w^2)}{(1+w)^2} \geq 1+w^2 \Leftrightarrow w \leq \sqrt{2} - 1 \), we complete the proof. \( \square \)

**Theorem 2.7.** (a) If \( w \leq \sqrt{2} - 1 \), then \( c_{unc}(2 : d_*(1,w)^2) = \frac{1+w^2+\sqrt{2(1+w^4)}}{(1+w)^2} \);

(b) If \( w > \sqrt{2} - 1 \), then \( c_{unc}(2 : d_*(1,w)^2) = \frac{1+w^2+\sqrt{(1+w^2)^2+4w^2}}{2} \).
Proof. By the Krein-Milman Theorem,

\[ c_{\text{unc}}(2 : d_+(1, w)^2) = \max\{\|P_t\|, \|Q_s\|, \|R_k\| : \frac{1}{1 + w^2} \leq t \leq 1, 0 \leq s \leq \frac{1 - w}{(1 + w)(1 + w^2)}, k = 1, 2, 3, 4\}. \]

Note that \(\|R_k\| = 1\) for \(k = 1, 2, 3, 4\). We claim that \(\max\{\|P_t\| : \frac{1}{1 + w^2} \leq t \leq 1\} = \frac{1 + w^2 + \sqrt{(1 + w^2)^2 + 4w^2}}{2}\). Note that \(|P_t|((x_1, y_1), (x_2, y_2)) = tx^2 + ty^2 + \sqrt{(1 - t)xy}\) for \(\frac{1}{1 + w^2} \leq t \leq 1\). Let \(f(t) = (1 + w^2)t + 2w\sqrt{t(1 - t)}\) for \(\frac{1}{1 + w^2} \leq t \leq 1\). Then \(0 = f'(t)\) for \(\frac{1}{1 + w^2} \leq t \leq 1\) implies that \(t = \frac{1}{2} + \frac{w^2}{2(1 + w^2)^2 + 4w^2}\).

Clearly, \(\frac{1}{1 + w^2} < \frac{1}{2} + \frac{1 + w^2}{2(1 + w^2)^2 + 4w^2} < 1\). It follows that, by Theorem 2.2 (Case 1, Subcase 2),

\[ \|P_t\| = \max\{(1 + w^2)t + 2w\sqrt{t(1 - t)} : \frac{1}{1 + w^2} \leq t \leq 1\} = \max\{f\left(\frac{1}{1 + w^2}\right), f(1), f\left(\frac{1}{2} + \frac{1 + w^2}{2(1 + w^2)^2 + 4w^2}\right)\} = \max\{1 + 3w^2, 1 + w^2, 1 + w^2 + \sqrt{(1 + w^2)^2 + 4w^2}\} = \frac{1 + w^2 + \sqrt{(1 + w^2)^2 + 4w^2}}{2}. \]

By Lemmas 2.4–5, \(Q_s(x, y) = P_t(X, Y)\) for some \(\frac{1}{1 + w^2} \leq t \leq 1\), where \(w^* = \frac{1 - w}{1 + w}\) and \(X = \frac{x + w}{1 + w}, Y = \frac{x - w}{1 + w}\). By the above claim, it follows that

\[ \max\{\|Q_s\| : 0 \leq s \leq \frac{1 - w}{(1 + w)(1 + w^2)}\} = \max\{\|P_t(X, Y)\| : \frac{1}{1 + (w^*)^2} \leq t \leq 1\} = 1 + (w^*)^2 \]

\[ = \frac{1 + (w^*)^2 + \sqrt{(1 + (w^*)^2)^2 + 4(w^*)^2}}{2} = \frac{1 + w^2 + \sqrt{2(1 + w^4)}}{(1 + w)^2}. \]

Therefore,

\[ c_{\text{unc}}(2 : d_+(1, w)^2) = \max\{\frac{1 + w^2 + \sqrt{(1 + w^2)^2 + 4w^2}}{2}, \frac{1 + w^2 + \sqrt{2(1 + w^4)}}{(1 + w)^2}\}. \]

Since \(\frac{1 + w^2 + \sqrt{2(1 + w^4)}}{2(1 + w)^2} \geq \frac{1 + w^2 + \sqrt{(1 + w^2)^2 + 4w^2}}{2} \iff w \leq \sqrt{2} - 1\), we complete the proof. \(\square\)
References


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