ON PERMUTING \(n\)-DERIVATIONS IN NEAR-RINGS

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Abstract. In this paper, we introduce the notion of permuting \(n\)-derivations in near-ring \(N\) and investigate commutativity of addition and multiplication of \(N\). Further, under certain constraints on a \(n!\)-torsion free prime near-ring \(N\), it is shown that a permuting \(n\)-additive mapping \(D\) on \(N\) is zero if the trace \(d\) of \(D\) is zero. Finally, some more related results are also obtained.

1. Introduction

Throughout this paper \(N\) will denote a zero-symmetric left near ring. A near ring \(N\) is called zero symmetric if \(0x = 0\) for all \(x \in N\) (recall that in a left near ring \(x0 = 0\) for all \(x \in N\)). \(N\) is called prime if \(xNy = \{0\}\) implies \(x = 0\) or \(y = 0\). It is called semi prime if \(xNx = \{0\}\) implies \(x = 0\). Near-ring \(N\) is called \(n\)-torsion free if \(nx = 0\) implies \(x = 0\). The symbol \(Z\) will represent the multiplicative center of \(N\), that is, \(Z = \{x \in N \mid xy = yx\text{ for all } y \in N\}\). As usual, for \(x, y \in N\), \([x, y]\) will denote the commutator \(xy - yx\), while \((x, y)\) will indicate the additive group commutator \(x + y - x - y\). The symbol \(C\) will represent the set of all additive commutators of near ring \(N\). For terminologies concerning near-rings we refer to G. Pilz [10].

An additive map \(f : N \rightarrow N\) is called a derivation if \(f(xy) = f(x)y + xf(y)\) holds for all \(x, y \in N\). The concepts of symmetric bi-derivation, permuting tri-derivation and permuting \(n\)-derivation have already been introduced in rings by G. Maksa, M. A. Öztürk and K. H. Park in [4, 5, 6], and [8], respectively. These concepts of symmetric bi-derivations and permuting tri-derivations have been studied in near-rings by M. A. Öztürk and K. H. Park in [7] and [9], respectively. In the present paper, motivated by these concepts, we define permuting \(n\)-derivations in near-rings and study some properties involved there. Some relations between permuting \(n\)-derivations and \(C\), the set of all additive commutators in near-ring \(N\) have also been studied.
A map \( D : N \times N \times \cdots \times N \rightarrow N \) is said to be permuting if the equation 
\[
D(x_1, x_2, \ldots, x_n) = D(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)})
\]
holds for all \( x_1, x_2, \ldots, x_n \in N \) and for every permutation \( \pi \in S_n \), where \( S_n \) is the permutation group on \( \{1, 2, \ldots, n\} \). A map \( d : N \rightarrow N \) defined by \( d(x) = D(x, x, \ldots, x) \) for all \( x \in N \) where \( D : N \times N \times \cdots \times N \rightarrow N \) is a permuting map, is called the trace of \( D \). A permuting \( n \)-additive (i.e., additive in each argument) mapping \( D : N \times N \times \cdots \times N \rightarrow N \) is called a permuting \( n \)-derivation if 
\[
D(x^i_1, x_2, \ldots, x_n) = D(x_1, x_2, \ldots, x_n)x^i_1 + x_1D(x_1', x_2, \ldots, x_n) \]
holds for all \( x_1, x_1', \ldots, x_n \in N \). Of course, a permuting 1-derivation is a derivation and permuting 2-derivation is a symmetric bi-derivation. For an example of permuting \( n \)-derivation let \( n \geq 1 \) be a fixed positive integer, \( N \) a commutative near-ring. Then \( R = \{ ( \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} ) \mid a, b, 0 \in N \} \) is a non-commutative near-ring with regard to matrix addition and matrix multiplication. Define \( D : R \times R \times \cdots \times R \rightarrow R \) such that 
\[
D\left( \begin{pmatrix} a_1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} a_2 \\ 0 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} a_n \\ 0 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ a_1a_2 \cdots a_n \end{pmatrix}.
\]
It is easy to see that \( D \) is a permuting \( n \)-derivation of \( R \).

Now let \( D \) be a permuting \( n \)-derivation of a near-ring \( N \). Then it can be easily seen that \( D(0, x_2, \ldots, x_n) = D(0 + 0, x_2, \ldots, x_n) = D(0, x_2, \ldots, x_n) + D(0, x_2, \ldots, x_n) \). Therefore \( D(0, x_2, \ldots, x_n) = 0 \) for all \( x_2, \ldots, x_n \in N \). We also observe that \( D(-x_1, x_2, \ldots, x_n) = -D(x_1, x_2, \ldots, x_n) \) for all \( x_i \in N; i = 1, 2, \ldots, n \).

There has been a great deal of work concerning derivations, biderivations and triderivations in near-rings (see \([1, 2, 3, 4, 9]\) where further references can be found). In this paper we study the commutativity of addition and multiplication of near-rings. Many well known results for derivations, bi-derivations and tri-derivations in near-rings have been generalized for permuting \( n \)-derivation. In fact, our results generalize and complement several well known theorems for near-rings.

2. Preliminary results

We begin with the following lemmas which are essential for developing the proofs of our main results. Proofs of Lemmas 2.1 and 2.2 can be seen in \([2, \text{Lemma 3}]\) and \([3, \text{Lemma 1.2}]\), respectively.

**Lemma 2.1.** Let \( N \) be a prime near-ring.

(i) If \( z \in Z \setminus \{0\} \), then \( z \) is not a zero divisor.

(ii) If \( Z \setminus \{0\} \) contains an element \( z \) for which \( z + z \in Z \), then \( (N, +) \) is abelian.
Lemma 2.2. Let $N$ be a prime near-ring. If $z \in Z \setminus \{0\}$ and $x$ is an element of $N$ such that $xz \in Z$ or $zx \in Z$, then $x \in Z$.

Lemma 2.3. Let $N$ be a near-ring. Then $D$ is a permuting $n$-derivation of $N$ if and only if $D(x_1x'_1,x_2,\ldots,x_n) = x_1D(x'_1,x_2,\ldots,x_n) + D(x_1,x_2,\ldots,x_n)x'_1$ for all $x_1,x'_1,x_2,\ldots,x_n \in N$.

Proof. We have

$$D(x_1(x'_1 + x'_2),x_2,\ldots,x_n) = D(x_1,x_2,\ldots,x_n)(x'_1 + x'_2) + x_1D(x'_1 + x'_2,x_2,\ldots,x_n)$$

and

$$D(x_1x'_2 + x_1x'_1,x_2,\ldots,x_n) = D(x_1x'_1,x_2,\ldots,x_n) + D(x_1,x_2,\ldots,x_n)x'_2$$

Combining above two equalities we obtain that

$$D(x_1,x_2,\ldots,x_n)x'_1 + x_1D(x'_1,x_2,\ldots,x_n) = x_1D(x'_1,x_2,\ldots,x_n) + D(x_1,x_2,\ldots,x_n)x'_1.$$

Therefore, $D(x_1x'_1,x_2,\ldots,x_n) = x_1D(x'_1,x_2,\ldots,x_n) + D(x_1,x_2,\ldots,x_n)x'_1$.

Converse can be proved in a similar way. □

In a left near-ring $N$, right distributive law does not hold in general, however, we can prove the following partial distributive properties in $N$.

Lemma 2.4. Let $N$ be a near-ring. Let $D$ be a permuting $n$-derivation of $N$ and $d$ be the trace of $D$. Then for every $x_1,x'_1,\ldots,x_n,y \in N$,

(i) $\{D(x_1,x_2,\ldots,x_n)x'_1 + x_1D(x'_1,x_2,\ldots,x_n)\}y = D(x_1,x_2,\ldots,x_n)x'_1y + x_1D(x'_1,x_2,\ldots,x_n)y$.

(ii) $x_1D(x'_1,x_2,\ldots,x_n) + D(x_1,x_2,\ldots,x_n)x'_1y = x_1D(x'_1,x_2,\ldots,x_n)y + D(x_1,x_2,\ldots,x_n)x'_1y$.

(iii) $\{D(x_1,x_2,\ldots,x_n)x_1 + x_1D(x'_1,x_2,\ldots,x_n)\}y = d(x_1)y + x_1D(x'_1,x_2,\ldots,x_n)y$.

(iv) $\{D(x_1,x_2,\ldots,x_1) + d(x_1)\}y = D(x_1,x_2,\ldots,x_1)y + d(x_1)y$.

Proof. (i) For all $x_1,x'_1,\ldots,x_n \in N$

$$D((x_1x'_1)x_2,\ldots,x_n) = D(x_1,x_2,\ldots,x_n)x'_1 + (x_1x'_1)D(x'_1,x_2,\ldots,x_n).$$

$$= \{D(x_1,x_2,\ldots,x_n)x'_1 + x_1D(x'_1,x_2,\ldots,x_n)\}x'_2 + (x_1x'_1)D(x'_1,x_2,\ldots,x_n).$$
Also
\[ D(x_1(x'_1x''_1), x_2, \ldots, x_n) \]
\[ = D(x_1, x_2, \ldots, x_n)x'_1x''_1 + x_1D(x'_1x''_1, x_2, \ldots, x_n) \]
\[ = D(x_1, x_2, \ldots, x_n)x'_1x''_1 + x_1\{D(x'_1, x_2, \ldots, x_n)x''_1 + x_1D(x'_1, x_2, \ldots, x_n)\} \]
\[ = D(x_1, x_2, \ldots, x_n)x'_1x''_1 + x_1D(x'_1, x_2, \ldots, x_n)x''_1 + x_1x_1D(x'_1, x_2, \ldots, x_n). \]

Combining the above two relations, we get
\[ \{D(x_1, x_2, \ldots, x_n)x'_1 + x_1D(x'_1, x_2, \ldots, x_n)\}x''_1 \]
\[ = D(x_1, x_2, \ldots, x_n)x'_1x''_1 + x_1D(x'_1, x_2, \ldots, x_n)x''_1. \]

Putting \( y \) in the place of \( x''_1 \), we find that
\[ \{D(x_1, x_2, \ldots, x_n)x'_1 + x_1D(x'_1, x_2, \ldots, x_n)\}y \]
\[ = D(x_1, x_2, \ldots, x_n)x'_1y + x_1D(x'_1, x_2, \ldots, x_n)y. \]

(ii) It can be proved, in a similar way as above, with the help of Lemma 2.3.

(iii) In the proof (i) above putting \( x_1 = x_2 = x_3 = \cdots = x_n = x \), we get
\[ \{d(x)x'_1 + xD(x'_1, x, \ldots, x)\}y = d(x)x_1y + xD(x'_1, x, \ldots, x). \]

In particular for \( x'_1 = x_1 \) we get
\[ \{d(x)x_1 + xD(x, x, \ldots, x)\}y = d(x)x_1y + xD(x, x, \ldots, x). \]

(iv) It can be proved in a similar way as above. \( \square \)

**Lemma 2.5.** Let \( N \) be prime near-ring and \( D \) be a non zero permuting \( n \)-derivation of \( N \),

(i) If \( D(N, N, \ldots, N)x = \{0\} \) where \( x \in N \), then \( x = 0 \),

(ii) If \( xD(N, N, \ldots, N) = \{0\} \) where \( x \in N \), then \( x = 0 \).

**Proof.** (i) Given that \( D(x_1x'_1, x_2, \ldots, x_n)x = 0 \) for all \( x_1, x'_1, \ldots, x_n \in N \). This yields that \( \{D(x_1, x_2, \ldots, x_n)x'_1 + x_1D(x'_1, x_2, \ldots, x_n)\}x = 0 \). By hypothesis and Lemma 2.4(i) we have \( D(x_1, x_2, \ldots, x_n)Nx = \{0\} \). But since \( N \) is a prime near ring and \( D \neq 0 \), we have \( x = 0 \).

(ii) It can be proved in a similar way. \( \square \)

**Lemma 2.6.** Let \( D \) be a nonzero permuting \( n \)-derivation of a prime near ring \( N \). Then \( D(C, C, \ldots, C) \neq \{0\} \) where \( C \neq \{0\} \).

**Proof.** If possible assume \( D(C, C, \ldots, C) = \{0\} \), then \( D(c_1, c_2, \ldots, c_n) = 0 \) for all \( c_1, c_2, \ldots, c_n \in C \). For all \( r_1 \in N \) and \( c_1 \in C \) we get \( r_1c_1 \in C \). Also \( D(r_1c_1, c_2, \ldots, c_n) = 0 \) implies \( r_1D(c_1, c_2, \ldots, c_n) + D(r_1, c_2, \ldots, c_n)c_1 = 0 \).

Thus we get
\[ D(r_1, c_2, \ldots, c_n)c_1 = 0. \]
Replacing $c_1$ by $xc_1$ in equation (2.1) where $x \in N$ we find that
$$D(r_1,c_2,\ldots,c_n)Nc_1 = \{0\}.$$ Primeness of $N$ yields,
\begin{equation}
D(r_1,c_2,\ldots,c_n) = 0.
\end{equation}
Now putting $r_2c_2 \in C$ in place of $c_2$ where $r_2 \in N$ in the equation (2.2) and proceeding as above we have $D(r_1,r_2,c_3,\ldots,c_n) = 0$. Proceeding inductively we conclude that $D(r_1,r_2,\ldots,r_n) = 0$ for all $r_1,r_2,\ldots,r_n \in N$ leading to a contradiction.

**Lemma 2.7.** Let $N$ be a $m!$-torsion free near-ring, where $(N,+)$ is an abelian group. Suppose $y_1,y_2,\ldots,y_m \in N$ satisfy $\alpha y_1 + \alpha^2 y_2 + \cdots + \alpha^m y_m = 0$ for $\alpha = 1,2,\ldots,m$. Then $y_i = 0$ for all $i$.

**Proof.** Let $A = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
2 & 2^2 & \cdots & 2^m \\
\vdots & \vdots & \ddots & \vdots \\
m & m^2 & \cdots & m^m
\end{pmatrix}$ be any $m \times m$ matrix. Then by our assumption
$$A \begin{pmatrix}
y_1 \\
y_2 \\
\vdots \\
y_m
\end{pmatrix} = \begin{pmatrix}
y_1 \\
y_2 \\
\vdots \\
y_m
\end{pmatrix}.$$ Now pre multiplying by $\text{Adj } A$ yields $\text{Det } A \begin{pmatrix}
y_1 \\
y_2 \\
\vdots \\
y_m
\end{pmatrix} = \begin{pmatrix}
y_1 \\
y_2 \\
\vdots \\
y_m
\end{pmatrix}$. Since $\text{Det } A$, as a Vondermonde determinant, is equal to a product of positive integers, each of which is less than or equal to $m$ and as $N$ is a $m!$-torsion free near-ring, it follows immediately that $y_i = 0$ for all $i$. \hfill \Box

3. Main results

Recently M. A. Öztürk and Y. B. Jun [7, Lemma 3.1] proved that in a 2-torsion free near-ring which admits a symmetric bi-additive mapping $D$ if the trace $d$ of $D$ is zero, then $D = 0$. Further, this result was generalized by K. H. Park and Y. S. Jung [9, Lemma 2.2] for permuting tri-additive mapping in 3!-torsion free near-ring in the year 2010. We have extended this result, as below, for permuting $n$-additive mapping in a $n!$-torsion free prime near-ring under some constraints.

**Theorem 3.1.** Let $N$ be $n!$-torsion free prime near-ring and $D$ be a permuting $n$-additive mapping of $N$ such that $D(N,N,\ldots,N) \subseteq Z$. If $d(x) = 0$ for all $x \in N$, then $D = 0$.

**Proof.** If $D = 0$, then we have nothing to do, if not then $D$ is a non zero permuting $n$-additive mapping of prime near-ring $N$ such that $D(N,N,\ldots,N) \subseteq Z$. Hence there exist $x_1,x_2,\ldots,x_n \in N$, all nonzero such that $D(x_1,x_2,\ldots,x_n) \neq 0$ and $D(x_1,x_2,\ldots,x_n) \in Z$. Since $D(x_1+x_1,x_2,\ldots,x_n) = D(x_1,x_2,\ldots,x_n) + D(x_1,x_2,\ldots,x_n) \in Z$, by Lemma 2.1(ii), $(N,+)$ is an abelian group. Hence
the trace $d(x) = D(x, x, \ldots, x)$ of permuting $n$-additive mapping $D$ can be expressed as:

$$
(3.1)
$$

where $x, y \in N$ and $h_k(x, y) = D(x, x, \ldots, x, y, y, \ldots, y, \underbrace{x, x, \ldots, x}_{(n-k)\text{-times}}, y, y, \ldots, y, k\text{-times})$. In particular by our hypothesis $d(\mu x + x_n) = 0$ where $1 \leq \mu \leq n - 1$. With the help of equation (3.1) we get

$$
0 = d(\mu x) + d(x_n) + \sum_{k=1}^{n-1} \binom{n}{k} h_k(\mu x, x_n)
$$

This yields that

$$
\mu y_1 + \mu^2 y_2 + \cdots + \mu^{n-2} y_{n-2} + \mu^{n-1} nD(x, x, \ldots, x, x_n) = 0,
$$

where $y_1, y_2, \ldots, y_{n-2} \in N$. By our hypothesis and Lemma 2.7, we deduce that

$$
(3.2)
$$

for all $x, x_n \in N$. Let $\nu(1 \leq \nu \leq n - 2)$ be any integer. By equation (3.2) we find that

$$
D(\nu x + x_{n-1}, \nu x + x_{n-1}, \ldots, \nu x + x_{n-1}, x_n) = 0.
$$

Expanding the above relation and using equation (3.2) again we obtain

$$
\nu z_1 + \nu^2 z_2 + \cdots + \nu^{n-3} z_{n-3} + \nu^{n-2} \binom{n}{2} D(x, x, \ldots, x, x_{n-1}, x_n) = 0,
$$

where $z_1, z_2, \ldots, z_{n-3} \in N$. By our hypothesis and Lemma 2.7, we conclude that $D(x, x, \ldots, x, x_{n-1}, x_n) = 0$ for all $x, x_{n-1}, x_n \in N$. Now if we continue the above process inductively, then we finally arrive at $D(x_1, x_2, \ldots, x_{n-1}, x_n) = 0$. This gives that $D = 0$, a contradiction. \[\square\]

In the year 1987 H. E. Bell [3, Theorem 2] proved that if a 2-torsion free zero symmetric prime near-ring $N$ admits a non zero derivation $D$ for which $D(N) \subseteq Z$, then $N$ is a commutative ring. Further, this result was generalized by K. H. Park [5, Theorem 3.1] in the year 2010 for permuting tri-derivation, who showed that if 3!-torsion free zero symmetric prime near-ring $N$ admits a non zero permuting tri-derivation $D$ for which $D(N, N, N) \subseteq Z$, then $N$ is a commutative ring. The following result shows that 2-torsion free and 3!-torsion free restrictions in the above results used by Bell and Park are superfluous. In fact, for permuting $n$-derivation in a prime near-ring $N$ we have obtained the following:
Theorem 3.2. Let $D$ be a non zero permuting $n$-derivation of prime near-ring $N$ such that $D(N,N,...,N) \subseteq Z$. Then $N$ is a commutative ring.

Proof. For all $x_1, x_2, \ldots, x_n \in N$, we have

\[(3.3) \quad D(x_1x_2',x_2,\ldots,x_n) = D(x_1,x_2,\ldots,x_n)x'_1 + x_1 D(x_1',x_2\ldots,x_n) \in Z.\]

Hence $x'_1[D(x_1,x_2,\ldots,x_n)x'_1 + x_1 D(x_1',x_2\ldots,x_n)] = \{D(x_1,x_2,\ldots,x_n)x'_1 + x_1 D(x_1',x_2\ldots,x_n)\}$ by the hypothesis and Lemma 2.4(i) we get $x_1 x_1 D(x_1,x_2,\ldots,x_n) = x_1 x_1' D(x_1,x_2,\ldots,x_n)$. This yields that $D(x_1,x_2,\ldots,x_n)$ \[(x'_1 x_1 - x_1 x'_1) = 0. \] Since $Z$ has no zero divisors, for each fixed $x_1 \in N$ either $(x'_1 x_1 - x_1 x'_1) = 0$ or $D(x_1,x_2,\ldots,x_n) = 0$ for all $x_1,x_2,\ldots,x_n \in N$. If first holds, then $x_1 \in Z$ if not, i.e., $D(x_1',x_2\ldots,x_n) = 0$, then equation(3.3) reduces to $D(x_1,x_2,\ldots,x_n) = (x_1,x_2,\ldots,x_n)$ for $D \neq 0$ and $D(x_1,x_2,\ldots,x_n) \subseteq Z$, by Lemma 2.2 $x'_1 \in Z$. Hence we conclude that $N \subseteq Z$. Thus we obtain that $N = Z$, i.e., $N$ is a commutative near-ring. If $N = \{0\}$, then $N$ is trivially a commutative ring. If $N \neq \{0\}$, then there exists $0 \neq x \in N$ and hence $x + x \in N = Z$. Now by Lemma 2.1(ii), we conclude that $N$ is a commutative near-ring. 

\[\square\]

Theorem 3.3. Let $N$ be a prime near-ring and $D_1$ and $D_2$ be any two non zero permuting $n$-derivations of $N$. If $[D_1(N,N,\ldots,N),D_2(N,N,\ldots,N)] = \{0\}$, then $(N,+)$ is an abelian group.

Proof. If both $z$ and $z + z$ commute element wise with $D_2(N,N,\ldots,N)$, then $zD_2(x_1,x_2,\ldots,x_n) = D_2(x_1,x_2,\ldots,x_n)z$ and $(z + z)D_2(x_1,x_2,\ldots,x_n) = D_2(x_1,x_2,\ldots,x_n)(z + z)$ for all $x_1,x_2,\ldots,x_n \in N$. In particular, $(z + z)D_2(x_1 + x_1',x_2,\ldots,x_n) = D_2(x_1 + x_1',x_2,\ldots,x_n)(z + z)$ for all $x_1,x_1',x_2,\ldots,x_n \in N$. From the previous equalities we get $zD_2(x_1 + x_1' - x_1 - x_1',x_2,\ldots,x_n) = 0$, i.e., $zD_2((x_1,x_1'),x_2,\ldots,x_n) = 0$. Putting $z = D_1(y_1,y_2,\ldots,y_n)$ we get $D_1(y_1,y_2,\ldots,y_n)D_2((x_1,x_1'),x_2,\ldots,x_n) = 0$. By Lemma 2.5(i) we conclude that $D_2((x_1,x_1'),x_2,\ldots,x_n) = 0$. Putting $w(x_1,x_1')$ in place of additive commutator $(x_1,x_1')$ where $w \in N$ we have $D_2(w(x_1,x_1'),x_2,\ldots,x_n) = 0$, i.e., $D_2(w,x_2,\ldots,x_n)(x_1,x_1') + wD_2((x_1,x_1'),x_2,\ldots,x_n) = 0$. Previous equality yields $D_2(w,x_2,\ldots,x_n)(x_1,x_1') = 0$. By Lemma 2.5(i) again we conclude that $(x_1,x_1') = 0$. Hence $(N,+)$ is an abelian group. \n
\[\square\]

Theorem 3.4. Let $N$ be a prime near-ring with non zero permuting $n$-derivations $D_1$ and $D_2$ such that

\[D_1(x_1,x_2,\ldots,x_n)D_2(y_1,y_2,\ldots,y_n) = -D_2(x_1,x_2,\ldots,x_n)D_1(y_1,y_2,\ldots,y_n)\]

for all $x_1,x_2,\ldots,x_n,y_1,y_2,\ldots,y_n \in N$. Then $(N,+)$ is an abelian group.

Proof. By our hypothesis we have, $D_1(x_1,x_2,\ldots,x_n)D_2(y_1,y_2,\ldots,y_n) + D_2(x_1,\ldots,x_n)D_1(y_1,y_2,\ldots,y_n) = 0$ for all $x_1,x_2,\ldots,x_n,y_1,y_2,\ldots,y_n \in N$. Replacing $y_1$ by $y_1 + y_1$ in previous equation we get $D_1(x_1,x_2,\ldots,x_n)D_2(y_1 + y_1 + \ldots + y_n) = 0$. Hence $(N,+)$ is an abelian group.
Since
Proof. Let $\alpha \in N \setminus \{1\}$. We define a derivation $d$ of $N$ by $d(a) = \alpha a - a \alpha$, for all $a \in N$. By Lemma 2.5(i) we conclude that $D_2((y_1, y_1), y_2, \ldots, y_n) = 0$. Putting $w(y_1, y_1)$ in place of $(y_1, y_1)$ where $w \in N$ in the previous equation and using Lemma 2.5(i); as used in the previous theorem, we conclude that $(N, +)$ is an abelian group.

**Corollary 3.1** ([1, Lemma 2.1]). Let $N$ be a prime near-ring with non zero derivations $d_1$ and $d_2$ such that $d_1(x)d_2(y) = -d_2(x)d_1(y)$ for all $x, y \in N$. Then $(N, +)$ is an abelian group.

**Theorem 3.5.** Let $D$ be a non zero permuting $n$-derivation of prime near-ring $N$. If $D(C, N, N, \ldots, N) = \{0\}$, then $(N, +)$ is an abelian group.

**Proof.** Since $D(c, r_2, \ldots, r_n) = 0$ for all $c \in C$ and for all $r_2, \ldots, r_n \in N$, $D(w, r_2, \ldots, r_n) = 0$ where $w \in N$, i.e., $wD(c, r_2, \ldots, r_n) + D(w, r_2, \ldots, r_n)c = 0$. In turn we get $D(w, r_2, \ldots, r_n)c = 0$ but $D \neq 0$, and therefore by Lemma 2.5(i); $c = 0$. Hence $(N, +)$ is an abelian group.

**Theorem 3.6.** Let $N$ be a semi prime near-ring and $D$ be a permuting $n$-derivation of $N$. If $D(x_1, x_2, \ldots, x_n)y_1 = x_1D(y_1, y_2, \ldots, y_n)$ for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in N$, then $D = 0$.

**Proof.** We have

\[(3.4) \quad D(x_1, x_2, \ldots, x_n)y_1 = x_1D(y_1, y_2, \ldots, y_n).\]

Putting $y_1z_1$ in place of $y_1$ in the above equation; where $z_1 \in N$, we get

\[D(x_1, x_2, \ldots, x_n)y_1z_1 = x_1D(y_1z_1, y_2, \ldots, y_n)\]
\[\quad = x_1D(y_1, y_2, \ldots, y_n)z_1 + x_1y_1D(z_1, y_2, \ldots, y_n).\]

By equation (3.4) we get $D(x_1, x_2, \ldots, x_n)y_1z_1 = D(x_1, x_2, \ldots, x_n)y_1z_1 + x_1y_1D(z_1, y_2, \ldots, y_n)$. This yields that $x_1y_1D(z_1, y_2, \ldots, y_n) = 0$. Now replacing $x_1$ by $D(z_1, y_2, \ldots, y_n)$ we get $D(z_1, y_2, \ldots, y_n)D(z_1, y_2, \ldots, y_n) = 0$. But since $N$ is a semi prime near-ring, we conclude that $D = 0$.

**Theorem 3.7.** Let $N$ be any prime near-ring and $D$ be any non-zero permuting $n$-derivation of $N$. If $K = \{a \in N \mid [D(N, N, \ldots, N), a] = \{0\}\}$, then

(i) $a \in K$ implies either $a \in Z$ or $d(a) = 0$,
(ii) $d(K) \subseteq Z$,
(iii) $K$ is a semigroup under multiplication,
(iv) If there exists an element $a \in K$ for which $d(a) \neq 0$ and $D(a^2, a, \ldots, a) \in Z$, then $(N, +)$ is an abelian group.
Proof. (i) We have

\[(3.5)\quad D(x_1, x_2, \ldots, x_n)a = aD(x_1, x_2, \ldots, x_n)\]

for all \(x_1, x_2, \ldots, x_n \in N\). Putting \(ax_1\) in place of \(x_1\) in the above equation and using Lemma 2.4(i) we get \(D(a, x_2, \ldots, x_n)x_1a + aD(x_1, x_2, \ldots, x_n)a = aD(a, x_2, \ldots, x_n)x_1 + aaD(x_1, x_2, \ldots, x_n).\) Using the equation (3.5), we get

\[D(a, x_2, \ldots, x_n)x_1a = aD(a, x_2, \ldots, x_n)x_1.\]

Now putting \(x_1y_1\) for \(x_1\) in the latter relation and using it again, we have \(D(a, x_2, \ldots, x_n)x_1[y_1, a] = 0\) where \(y_1 \in N\). This gives us \(D(a, x_2, \ldots, x_n)N[y_1, a] = \{0\}\). Since \(N\) is a prime near-ring, either \([a, y_1] = 0\) for all \(y_1 \in N\) or \(D(a, x_2, \ldots, x_n) = 0\) for all \(x_2, \ldots, x_n \in N\). If first holds, then \(a \in Z\), if not then \(D(a, x_2, \ldots, x_n) = 0\), and hence in particular, \(D(a, a, \ldots, a) = 0\) or \(d(a) = 0\).

(ii) From the above proof we observe that if \(a \in K\), then either \(a \in Z\) or \(d(a) = 0\). But \(d(a) = 0\) implies \(d(a) \in Z\). If \(d(a) \neq 0\), then we have \(a \in Z\). In this case we have \(D(xa, a, \ldots, a) = D(ax, a, \ldots, a)\) for all \(x \in N\). This yields that \(xD(a, a, \ldots, a) + D(x, a, \ldots, a)a = D(a, a, \ldots, a)x + aD(x, a, \ldots, a)\). This reduces to \(xD(a, a, \ldots, a) = (a, a, \ldots, a)x\), which shows that \(d(a) \in Z\) and thus \(d(K) \subseteq Z\).

(iii) Let \(a, b \in K\). Hence \(abD(r_1, r_2, \ldots, r_n) = D(r_1, r_2, \ldots, r_n)ab\) holds trivially. Associativity of \(N\) shows that \(K\) is a semigroup.

(iv) Consider \(D(a^2, a, \ldots, a) = aD(a, a, \ldots, a) + D(a, a, \ldots, a)a \in Z\). As \(d(a) = D(a, a, \ldots, a) \neq 0\) implies that \(a \in Z\) by (i). Hence \(D(a^2, a, \ldots, a) = D(a, a, \ldots, a)(a + a)\). By above proof (ii) we find that \(D(a, a, \ldots, a) \in Z \setminus \{0\}\) and hence using Lemma 2.2, \((a + a) \in Z\). By Lemma 2.1(ii) we conclude that \((N, +)\) is an abelian group. \(\Box\)

**Theorem 3.8.** Let \(N\) be a prime near-ring which admits a non zero permuting \(n\)-derivation \(D\) such that \(D(C, C, \ldots, C) \subseteq Z\). Then \(N\) is a commutative ring where \(C \neq \{0\}\).

**Proof.** For all \(c_1, c_1', \ldots, c_n \in C\), we get

\[(3.6)\quad D(c_1c_1', c_2, \ldots, c_n) = D(c_1, c_2, \ldots, c_n)c_1' + c_1D(c_1', c_2, \ldots, c_n) \in Z\]

and commuting this element with \(c_1'\) we arrive at \(D(c_1', c_2, \ldots, c_n)(c_1'c_1 - c_1c_1') = 0\) for all \(c_1, c_1', \ldots, c_n \in C\). Now by Lemma 2.1(i), we observe that for each \(c_1'\) either \(c_1'\) centralizes \(C\) or \(D(c_1', c_2, \ldots, c_n) = 0\). If first case holds for each element of \(C\), then \(C\) becomes commutative with respect to multiplication. On the other hand if second case holds, i.e., \(D(c_1', c_2, \ldots, c_n) = 0\), then equation(3.6) takes the form

\[(3.7)\quad D(c_1c_1', c_2, \ldots, c_n) = D(c_1, c_2, \ldots, c_n)c_1' \in Z\]

for all \(c_1, c_2, \ldots, c_n \in C\). By Lemmas 2.2 and 2.6, we conclude that \(c_1' \in Z\). Hence in this case also we conclude that \(c_1'\) centralizes \(C\). Hence in both cases we conclude that \(C\) is a commutative semi group with respect to multiplication.

Now we separate the proof in two cases:
Case I: Let $C \cap Z \neq \{0\}$. Then in this case it follows that if $C$ contains a non zero central element $w$, then we have $wxc = xwc = wcx$ for all $c \in C$ and for all $x \in N$. Hence we have $w(xc - cx) = 0$. By Lemma 2.1(i), we conclude that $c \in Z$, i.e., $C \subseteq Z$. For all $c \in C$ and for all $x, y \in N$ we have $xyc = yxc$ or $eyx = cxy$ since $xc \in C$. Lastly we get $c(xy - yx) = 0$. As $C \neq \{0\}$, by Lemma 2.1(i), $N$ becomes a commutative near-ring, i.e., $N = Z$. If $N = \{0\}$, then $N$ is trivially a commutative near-ring. If $N \neq \{0\}$, then there exists $t \in N \setminus \{0\}$. Hence $t + t \in N = Z$, and by Lemma 2.1(ii) we conclude that $N$ is a commutative ring.

Case II: Let $C \cap Z = \{0\}$. For this case in the light of equation (3.7) we claim that $D(c_1, c_2, \ldots, c_n) \neq 0$ for all $c_1, c_2, \ldots, c_1, c_{i+1}, \ldots, c_n = 0$ and all $c_i \in C \setminus \{0\}$. For each $c_i \in C$ and for all $c_1, c_2, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n \in C$, $D(c_1, c_2, \ldots, c_1) = D(c_1, c_2, \ldots, c_1)(c_1 + c_i)$ and hence by Lemmas 2.2 and 2.6, $2c_i \in Z$. Suppose that $2c_i \neq 0$ for all $c_i \in C \setminus \{0\}$. It is obvious that $x = \{xc \mid c \in C\} = \{0\}$ implies $x = 0$. This shows that for each $x \in N \setminus \{0\}$, there exists $c_x \in C$ such that $xc_x \neq 0$. Since $xc_x$ being an additive commutator also belongs to $C$, we have $2xc_x = x(2c_x)$ and by Lemma 2.2 we conclude that $x \in Z$. Hence $N = Z$, i.e., $N$ is a commutative near-ring. If $N = \{0\}$, then $N$ is trivially a commutative near-ring. If $N \neq \{0\}$, then there exists $p \in N \setminus \{0\}$ such that $p + p \in N = Z$. By Lemma 2.1(ii) we conclude that $N$ is a commutative ring. The only remaining possibility is that $C \cap Z = \{0\}$ and there exists $c_i \in C \setminus \{0\}$ such that $2c_i = 0$ and we complete our proof by showing that this leads to a contradiction. Suppose that $c_i \in C \setminus \{0\}$ and $2c_i = 0$. We have $D(c_1, c_2, \ldots, c_1, c_1, \ldots, c_n) = 3c_i D(c_1, c_2, \ldots, c_1, \ldots, c_n) = 0$. Since $2c_i^2 D(c_1, c_2, \ldots, c_1, \ldots, c_n) = 0$, we get $c_i^2 D(c_1, c_2, \ldots, c_1, \ldots, c_n) = 0$. This implies that $c_i^2 \in Z$ by Lemma 2.2. Since $C \cap Z = \{0\}$, $c_i^2 = 0$. Now $D(c_1, c_2, \ldots, x, \ldots, c_n) = xD(c_1, c_2, \ldots, c_1, \ldots, c_n) + D(c_1, c_2, \ldots, x, \ldots, c_n)$. Hence $x = D(c_1, c_2, \ldots, x, \ldots, c_n) = D(c_1, c_2, \ldots, x, \ldots, c_n)$. Left multiplying by $c_i$ we get $c_i^2 xD(c_1, c_2, \ldots, x, \ldots, c_n) = c_i xD(c_1, c_2, \ldots, x, \ldots, c_n)$. Finally we get $c_i xD(c_1, c_2, \ldots, x, \ldots, c_n) = 0$. This implies that $c_i N D(c_1, c_2, \ldots, c_1, \ldots, c_n) = \{0\}$, but primeness of $N$ yields that $D(c_1, c_1, \ldots, c_1, \ldots, c_n) = \{0\}$, and by Lemma 2.1(i), we conclude that $c_i = 0$, a contradiction.

References


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