BIMINIMAL CURVES IN 2-DIMENSIONAL SPACE FORMS

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Abstract. We study biminimal curves in 2-dimensional Riemannian manifolds of constant curvature.

Introduction

Elastic curves provide examples of classically known geometric variational problem. A plane curve is said to be an elastic curve if it is a critical point of the elastic energy, or equivalently a critical point of the total squared curvature [9].

In this paper, we study another geometric variational problem of curves in Riemannian 2-manifolds of constant curvature. The Euler-Lagrange equation studied in this paper is derived from the theory of biharmonic maps in Riemannian geometry.

A smooth map \( \phi : (M, g) \to (N, h) \) between Riemannian manifolds is said to be biharmonic if it is a critical point of the bienergy functional:

\[
E_2(\phi) = \int_M |\tau(\phi)|^2 \, dv_g,
\]

where \( \tau(\phi) = \text{tr} \nabla d\phi \) is the tension field of \( \phi \). Clearly, if \( \phi \) is harmonic, \( i.e., \tau(\phi) = 0 \), then \( \phi \) is biharmonic. A biharmonic map is said to be proper if it is not harmonic.

Chen and Ishikawa [3] studied biharmonic curves and surfaces in semi-Euclidean space (see also [6]). Caddeo, Montaldo and Piu [1] studied biharmonic curves on Riemannian 2-manifolds. They showed that biharmonic curves on Riemannian 2-manifolds of non-positive curvature are geodesics. Proper biharmonic curves on the unit 2-sphere are small circles of radius \( 1/\sqrt{2} \).

Next, Loubeau and Montaldo introduced the notion of biminimal immersion [10].

An isometric immersion \( \phi : (M, g) \to (N, h) \) is said to be biminimal if it is a critical point of the bienergy functional under all normal variations. Thus the biminimality is weaker than biharmonicity for isometric immersions, in general.
In this paper we study biminimal curves on Riemannian 2-manifolds of constant curvature. We shall give natural equations for biminimal curves explicitly in terms of Jacobi’s elliptic functions.

1. Preliminaries

1.1. Let $(M^m, g)$ and $(N^n, h)$ be Riemannian manifolds and $\phi : M \to N$ a smooth map. Then $\phi$ induces a vector bundle $\phi^*TN$ over $M$ by

$$\phi^*TN = \bigcup_{p \in M} T_{\phi(p)}N,$$

where $TN$ is the tangent bundle of $N$. The space of all smooth sections of $\phi^*TN$ is denoted by $\Gamma(\phi^*TN)$. A section of $\phi^*TN$ is called a vector field along $\phi$.

The Levi-Civita connection $\nabla^h$ of $(N, h)$ induces a unique connection $\nabla^\phi$ of $\phi^*TN$ which satisfies the condition

$$\nabla^\phi_X(V \circ \phi) = (\nabla^h_{\phi(X)}V) \circ \phi$$

for all $X \in \Gamma(TM)$ and $V \in \Gamma(\phi^*TN)$ (see [4, p. 4]).

The second fundamental form $\nabla d\phi$ is defined by

$$(\nabla d\phi)(X, Y) = \nabla^\phi_Xd\phi(Y) - d\phi(\nabla_XY), \quad X, Y \in \Gamma(TM).$$

Here $\nabla$ is the Levi-Civita connection of $(M, g)$. The tension field $\tau(\phi)$ is a section of $\phi^*TN$ defined by

$$\tau(\phi) = \text{tr} \nabla d\phi.$$

A smooth map $\phi$ is said to be harmonic if its tension field vanishes. It is well known that $\phi$ is harmonic if and only if $\phi$ is a critical point of the energy:

$$E(\phi) = \frac{1}{2} \int |d\phi|^2 dv_g$$

with respect to all compactly supported variations.

Now let $\phi : M \to N$ be a harmonic map. Then the Hessian $\mathcal{H}_\phi$ of $E$ is given by

$$\mathcal{H}_\phi(V, W) = \int h(J_\phi(V), W) dv_g, \quad V, W \in \Gamma(\phi^*TN).$$

Here the Jacobi operator $J_\phi$ is defined by

$$J_\phi(V) := \Delta_\phi V - R_\phi(V), \quad V \in \Gamma(\phi^*TN),$$

where $\Delta_\phi := -\sum_{i=1}^m (\nabla^{\phi}_{e_i} \nabla^{\phi}_{e_i} - \nabla_{\nabla^{\phi}_{e_i} e_i})$ and $R_\phi(V) = \sum_{i=1}^m R^N(V, d\phi(e_i))d\phi(e_i)$, and $\{e_i\}$ are the Riemannian curvature of $N$ and a local orthonormal frame field of $M$, respectively. For general theory of harmonic maps, we refer to Urakawa’s monograph [12].

Eells and Sampson [5] suggested to study polyharmonic maps. Polyharmonic maps of order 2 are frequently called biharmonic maps.
Definition 1.1. A smooth map \( \phi : (M, g) \to (N, h) \) is said to be biharmonic if it is a critical point of the bienergy functional:

\[
E_2(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 \, dv_g,
\]

with respect to all compactly supported variation.

The Euler-Lagrange equation of \( E_2 \) is

\[
\tau_2(\phi) := -J_\phi(\tau(\phi)) = 0.
\]

The section \( \tau_2(\phi) \) is called the bitension field of \( \phi \). For more informations on biharmonic maps, we refer to a survey \cite{montaldo-oniciuc} by Montaldo and Oniciuc.

If \( \phi \) is an isometric immersion, then \( \tau(\phi) = mH \), where \( H \) is the mean curvature vector field. Hence \( \phi \) is harmonic if and only if \( \phi \) is a minimal immersion. As is well known, an isometric immersion \( \phi : M \to N \) is minimal if and only if it is a critical point of the volume functional \( V \). The Euler-Lagrange equation of \( V \) is \( H = 0 \).

Motivated by this coincidence, the following notion was introduced by Loubau and Montaldo:

Definition 1.2 (\cite{loubau-montaldo}). An isometric immersion \( \phi : (M^m, g) \to (N^n, h) \) is called a biminimal immersion if it is a critical point of the bienergy functional \( E_2 \) with respect to all normal variation with compact support. Here, a normal variation means a variation \( \{\phi_t\} \) through \( \phi = \phi_0 \) such that the variational vector field \( V = d\phi_t/dt|_{t=0} \) is normal to \( M \).

The Euler-Lagrange equation of this variational problem is \( \tau_2(\phi)^\perp = 0 \). Here \( \tau_2(\phi)^\perp \) is the normal component of \( \tau_2(\phi) \). Since \( \tau(\phi) = mH \), the Euler-Lagrange equation is given explicitly by

\[
\left\{ \bar{\Delta}[\phi] - R_{\phi}(H) \right\}^\perp = 0.
\]

Obviously, every biharmonic immersion is biminimal, but the converse is not always true.

2. Biminimal curves

From now on we restrict our attention to unit speed curves in Riemannian 2-manifolds.

For a unit speed curve \( \gamma(s) \) in a Riemannian 2-manifold \( M \), its tension field is given by \( \tau(\gamma) = \nabla_{\gamma'}\gamma' \). Thus the bienergy of \( \gamma \) is the elastic energy

\[
E_2(\gamma) = \frac{1}{2} \int \kappa(s)^2 \, ds,
\]

where \( \kappa(s) \) is the signed curvature of \( \gamma \).

Here we recall the following fundamental result.
Lemma 2.1 ([10]). A unit speed curve $\gamma(s)$ in a Riemannian 2-manifold of Gaussian curvature $K$ is biminimal if and only if its signed curvature $\kappa(s)$ satisfies:

\begin{equation}
\kappa'' - \kappa^3 + \kappa K = 0.
\end{equation}

Note that $\gamma$ is biharmonic if and only if $\gamma$ is biminimal and additionally satisfies $\kappa \kappa' = 0$. Thus non-geodesic biharmonic curves have constant curvature.

Corollary 2.1. A non-geodesic curve in a Riemannian 2-manifold is biharmonic if and only if $\gamma$ is a Riemannian circle of signed curvature $\kappa$ satisfying $K = \kappa^2 > 0$. Thus proper biharmonic curves can exist only in constant positive curvature 2-manifolds.

Remark 1. Let $\gamma$ be a unit speed curve in Euclidean plane $\mathbb{R}^2$. Then $\gamma$ is an elastic curve if and only if its signed curvature satisfies

\[ \kappa'' + \frac{1}{2} (\kappa^3 - \lambda \kappa) = 0 \]

for some constant $\lambda$ [9]. Thus the Euler-Lagrange equation of the biminimal curve is different from the elastic curve equation.

3. Biminimal curves on Euclidean plane

First, we investigate biminimal curves on the Euclidean plane $\mathbb{R}^2$. In this case, the signed curvature $\kappa(s)$ is a solution to

\[ \kappa''(s) - \kappa(s)^3 = 0. \]

Multiplying $2\kappa'(s)$ to both hand sides of this ordinary differential equation, we get

\[ (\kappa')^2 = \frac{1}{2} (\kappa^4 + A) \]

for some constant $A$. Thus we obtain

\[ \int \frac{d\kappa}{\sqrt{\kappa^2 + A}} = \pm \frac{1}{\sqrt{2}} (s - s_0). \]

The left hand side of this equation is an elliptic integral of the first kind. Hence the signed curvature $\kappa(s)$ can be represented by Jacobi’s elliptic functions.

In our previous paper [8], we have solved the ordinary differential equation $\kappa'' = \kappa^3$. For our purpose, we recall the integration procedure given in [8].

Definition 3.1. For a positive constant $k$ such that $0 < k < 1$, the Jacobi’s sn-function $sn$ of modulus $k$ is defined by

\[ sn^{-1}(x; k) = \int_0^x \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}} \quad -1 \leq x \leq 1. \]
The sn-function is defined on the interval $-K(k) \leq x \leq K(k)$, where $K(k)$ is the complete elliptic integral of the first kind defined by
\[ K(k) := \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \]
The sn function is extended to the whole line $\mathbb{R}$ as a periodic function with fundamental period $4K(k)$. The cn function is defined by
\[ \text{cn}(x; k) = \sqrt{1 - \text{sn}^2(x; k)} \].

One can check the following integral formulas.
\[ \int_1^u \frac{du}{\sqrt{u^4 - 1}} = \frac{1}{\sqrt{2}} \text{cn}^{-1} \left( \frac{1}{u} \frac{1}{\sqrt{2}} \right), \]
\[ \int_1^u \frac{du}{\sqrt{u^4 + 1}} = K \left( \frac{1}{\sqrt{2}} \right) - \frac{1}{2} \text{cn}^{-1} \left( \frac{u^2 - 1}{u^2 + 1} \frac{1}{\sqrt{2}} \right). \]

3.1. $A = 0$. A simple and particular case is $A = 0$. In this case, $\kappa$ is an elementary function given explicitly by
\[ \kappa(s) = \pm \frac{\sqrt{2}}{s - s_0}. \]
The plane curve determined by this signed curvature is a logarithmic spiral. This case was discussed in [10].

3.1.1. $A > 0$. In this case we express $A = a^2$ with $a > 0$. Put $\kappa = \sqrt{au}$, then by (3.1), we have
\[ \int_{\sqrt{\kappa}}^{\kappa} \frac{d\kappa}{\sqrt{\kappa^4 + a^4}} = \frac{1}{\sqrt{a}} \int_1^u \frac{du}{\sqrt{u^4 + 1}} \]
\[ = \frac{1}{\sqrt{a}} \left\{ K \left( \frac{1}{\sqrt{2}} \right) - \frac{1}{2} \text{cn}^{-1} \left( \frac{u^2 - 1}{u^2 + 1} \frac{1}{\sqrt{2}} \right) \right\}. \]
Thus we obtain
\[ \kappa(s) = \pm \sqrt{a} \left( 1 + \text{cn}(\nu(s); 1/\sqrt{2}) \right)^{-\frac{1}{2}}, \]
where
\[ \nu(s) = \mp 2a(s - s_0) + 2K(1/\sqrt{2}). \]

3.1.2. $A < 0$. In this case we express $A = -a^2$ with $a > 0$. Put $\kappa = \sqrt{au}$ as before, then by (3.1) we get
\[ \int_{\sqrt{\kappa}}^{\kappa} \frac{d\kappa}{\sqrt{\kappa^4 - a^4}} = \frac{1}{\sqrt{a}} \int_1^u \frac{du}{\sqrt{u^4 - 1}} \]
\[ = \frac{1}{\sqrt{a}} \left\{ \frac{1}{\sqrt{2}} \text{cn}^{-1} \left( \frac{1}{u} \frac{1}{\sqrt{2}} \right) \right\}. \]
From we get the following formula:

\[ \kappa(s) = \frac{\sqrt{a}}{(c_n(\sqrt{a}(s - s_0)); 1/\sqrt{2})}. \]

Note that \( c_n \) is an even function.

**Theorem 3.1.** Let \( \gamma(s) \) be a Frenet curve in Euclidean plane \( \mathbb{R}^2 \). Then \( \gamma \) is biminimal if and only if it is determined by one of the following natural equations.

1. \( \kappa(s) = \pm \frac{\sqrt{2}}{s - s_0}. \)
   In this case \( \gamma \) is a logarithmic spiral.

2. \( \kappa(s) = \pm \sqrt{a} \left( \frac{1 + c_n(\nu(s); 1/\sqrt{2})}{1 - c_n(\nu(s); 1/\sqrt{2})} \right)^{1/2}, \)
   with \( \nu(s) = \mp 2\sqrt{2a(s - s_0)} + 2K(1/\sqrt{2}), \) or

3. \( \kappa(s) = \frac{\sqrt{a}}{c_n(\sqrt{a}(s - s_0); 1/\sqrt{2})}. \)

**4. Biminimal curves on the 2-sphere and the hyperbolic plane**

In this section we study biminimal curves in space forms of curvature \( c \neq 0 \).

Multiplying \( 2\kappa' \) to the biminimal equation

\[ \kappa''(s) - \kappa(s)^3 + c\kappa(s) = 0, \]

we obtain

\[ (\kappa')^2 - \frac{1}{2}\kappa^4 + c\kappa^2 = d, \]

where \( d \) is a constant. From this equation, we have

\[ \int \frac{d\kappa}{\sqrt{\kappa^4 - 2c\kappa + 2d}} = \int \frac{ds}{\sqrt{2}} = \frac{1}{\sqrt{2}(s - s_0)}. \]

The left hand side of this equation is an elliptic integral.

**4.1.** \( c^2 - 2d > 0 \). In this case, we can put \( r = \sqrt{c^2 - 2d} > 0 \). Then we have

\[ \int \frac{d\kappa}{\sqrt{\kappa^4 - 2c\kappa + 2d}} = \int \frac{d\kappa}{\sqrt{(\kappa^2 - c + r)(\kappa^2 - c - r)}}. \]

In this case, the positivity of \( (\kappa')^2 \) implies

\[ \kappa^2 > c + r \quad \text{or} \quad 0 < \kappa^2 < c - r. \]

We have three possibilities.
4.1.1. $c < 0$ and $d > 0$. Since $d > 0$, we can put
\[ a^2 = -c + r > 0, \quad b^2 = -c - r > 0. \]
Equivalently, we have
\[ a^2 + b^2 = -2c, \quad a^2 b^2 = 2d. \]
Hence we get
\[ \kappa^4 - 2c\kappa^2 + 2d = (\kappa^2 + a^2)(\kappa^2 - b^2). \]
Note that, in this case, the positivity condition $\kappa^2 > c + r$ is satisfied. By using the following integral formula
\[ \int_0^x \frac{dx}{\sqrt{(x^2 + a^2)(x^2 + b^2)}} = \frac{1}{a} \csc^{-1} \left( \frac{b}{\sqrt{b^2 + x^2}} \right), \quad b \leq a, \]
we have
\[ \kappa(s) = b \left\{ \frac{1 - \csc^2 \left( \frac{a(s-s_0)}{\sqrt{2}} ; \frac{\sqrt{a^2-b^2}}{a} \right)}{\csc^2 \left( \frac{a(s-s_0)}{\sqrt{2}} ; \frac{\sqrt{a^2-b^2}}{a} \right)} \right\}^{\frac{1}{2}}. \]

4.1.2. $c > 0$ and $d > 0$. In this case, we can put
\[ a^2 = c + r > 0, \quad b^2 = c - r > 0. \]
Equivalently, we have
\[ a^2 + b^2 = 2c, \quad a^2 b^2 = 2d. \]
Hence we get
\[ \kappa^4 - 2c\kappa^2 + 2d = (\kappa^2 - a^2)(\kappa^2 - b^2). \]
The positivity condition (3.4) is rewritten as
\[ \kappa^2 > a^2 \text{ or } 0 < \kappa^2 < b^2. \]
Comparing this condition with the following integral formulas.
\[ \int_x^\infty \frac{dx}{\sqrt{(x^2 - a^2)(x^2 - b^2)}} = \frac{1}{a} \text{sn}^{-1} \left( \frac{a}{x}, \frac{b}{a} \right), \quad 0 < b < a \leq x, \]
\[ \int_0^x \frac{dx}{\sqrt{(a^2 - x^2)(b^2 - x^2)}} = \frac{1}{a} \text{sn}^{-1} \left( \frac{x}{a}, \frac{b}{a} \right), \quad 0 \leq |x| \leq b < a. \]
Then we obtain
\[ \kappa(s) = \frac{a}{\text{sn} \left( \frac{a(s-s_0)}{\sqrt{2}} ; \frac{b}{a} \right)} \text{ or } \]
\[ \kappa(s) = b \text{sn} \left( \frac{a(s-s_0)}{\sqrt{2}} ; \frac{b}{a} \right), \]
respectively.
4.1.3. $d < 0$. In this case, we can put
\[ a^2 = -c + r, \quad b^2 = c + r. \]
Equivalently,
\[ a^2 - b^2 = -2c, \quad a^2 b^2 = -2d. \]
Thus we have
\[ \kappa^4 - 2c\kappa^2 + 2d = (\kappa^2 + a^2)(\kappa^2 - b^2). \]
By using the integral formula
\[ \int_b^x \frac{dx}{\sqrt{(x^2 + a^2)(x^2 - b^2)}} = \frac{1}{\sqrt{a^2 + b^2}} \text{cn}^{-1}\left(\frac{b}{x}; \frac{a}{\sqrt{a^2 + b^2}}\right), \quad b \leq x, \]
we get
\[ \kappa(s) = \frac{b}{\text{cn}\left(\frac{\sqrt{a^2 + b^2}}{\sqrt{2}}(s - s_0); \frac{a}{\sqrt{a^2 + b^2}}\right)}. \]

4.2. $c^2 - 2d = 0$. In this case, the ordinary equation is reduced to
\[ \int \frac{d\kappa}{\kappa^2 - c} = \frac{1}{\sqrt{2}}(s - s_0). \]
Thus we obtain
\[ \kappa(s) = -\sqrt{c} \tanh\left(\frac{\sqrt{c}(s - s_0)/\sqrt{2}}{1}\right), \quad c > 0, \]
\[ \kappa(s) = \sqrt{-c} \tan\left(\frac{\sqrt{-c}(s - s_0)/\sqrt{2}}{1}\right), \quad c < 0. \]

4.3. $c^2 - 2d < 0$. Since $2d > c^2 > 0$, we may put $2d = \alpha^4 \quad (\alpha > 0)$. Then we can express $c$ as
\[ c = -\alpha^2 \cos(2\theta). \]
Because $|c/\sqrt{2d}| = |c/\alpha^2| \leq 1$. Then by using the integral formula
\[ \int_0^x \frac{dx}{\sqrt{x^4 + 2\alpha^2 \cos(2\theta)x^2 + \alpha^4}} = \frac{1}{2\alpha} \text{cn}^{-1}\left(\frac{\alpha^2 - x^2}{\alpha^2 + x^2}; \sin \theta\right), \]
we obtain
\[ \kappa(s) = \alpha \left(\frac{1 - \text{cn}\left(\frac{\sqrt{2\alpha}(s - s_0)}{\sin \theta}\right)}{1 + \text{cn}\left(\frac{\sqrt{2\alpha}(s - s_0)}{\sin \theta}\right)}\right)^{\frac{1}{2}}. \]

**Theorem 4.1.** Let $\gamma(s)$ be a unit speed curve in a Riemannian 2-manifold $M^2(c)$ of constant curvature $c \neq 0$. If $\gamma$ is biminimal and not a geodesic. Then the signed curvature $\kappa$ of $\gamma$ is given by (4.4), (4.7), (4.8), (4.10), (4.11), (4.12), or (4.13).
5. Concluding remarks

Submanifolds with harmonic mean curvature ($\Delta \mathbf{H} = 0$) or normal harmonic mean curvature ($\Delta^\perp \mathbf{H} = 0$) have been studied extensively. Here $\Delta^\perp$ is the rough Laplacian of the normal bundle (and called the normal Laplacian). More generally, submanifolds with property $\Delta \mathbf{H} = \lambda \mathbf{H}$ or $\Delta^\perp \mathbf{H} = \lambda \mathbf{H}$ have been studied extensively by many authors (See references in [2], [7]). Analogously, we may generalize the notion of biminimal immersion to the following one:

**Definition 5.1** ([10]). An isometric immersion $\phi : M \to N$ is called a $\lambda$-biminimal immersion if it is a critical point of the functional:

$$E_{2,\lambda}(\phi) = E_2(\phi) + \lambda E(\phi), \quad \lambda \in \mathbb{R}.$$  

The Euler-Lagrange equation for $\lambda$-biminimal immersions is

$$\tau_2(\phi)^\perp = \lambda \tau(\phi).$$

More explicitly,

$$\{\bar{\Delta}_\phi \mathbf{H} - \mathcal{R}_\phi (\mathbf{H})\}^\perp = -\lambda \mathbf{H}$$

or equivalently

$$J_{\phi}(\mathbf{H})^\perp = -\lambda \mathbf{H}.$$  

**Corollary 5.1.** A non-geodesic curve $\gamma$ in a Riemannian 2-manifold is $\lambda$-biminimal if and only if

$$\kappa'' - \kappa^3 + \kappa (K - \lambda) = 0.$$  

Thus, by replacing $K$ by $c - \lambda$ in (2.1), one can obtain natural equations for $\lambda$-biminimal curves in the space form of curvature $c$.

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