ON (ɛ)-LORENTZIAN PARA-SASAKIAN MANIFOLDS

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ABSTRACT. In this paper we study (ɛ)-Lorentzian para-Sasakian manifolds and show its existence by an example. Some basic results regarding such manifolds have been deduced. Finally, we study conformally flat and Weyl-semisymmetric (ɛ)-Lorentzian para-Sasakian manifolds.

1. Introduction

In [1] Bejancu and K. L. Duggal introduced (ɛ)-Sasakian manifolds. Also XuFeng and Xiaoli [11] showed that every (ɛ)-Sasakian manifold must be a real hypersurface of some indefinite Kähler manifold. Further, in [6] R. Kumar, R. Rani and R. Nagaich study (ɛ)-Sasakian manifolds. Since Sasakian manifolds with indefinite metric play significant role in Physics [5], our natural trend is to study various contact manifolds with indefinite metric. Recently, in 2009, U. C. De, Avijit Sarkar [4] study (ɛ)-Kenmotsu manifolds. In 1989, K. Matsumoto [7] introduced the notion of Lorenzian para-Sasakian manifolds. I. Mihai and R. Rosca [9] defined the same notion independently and several authors [8, 10] studied LP-Sasakian manifolds. In this paper we like to introduce (ɛ)-Lorentzian para-Sasakian manifolds with indefinite metric which also include usual LP-Sasakian manifold. The present paper is organized as follows:

Section 1 is introductory. In Section 2, we define (ɛ)-LP-Sasakian manifolds and give an example of such a manifold. We also give some basic results of such a manifold in the same section. In Section 3, we study conformally flat (ɛ)-LP-Sasakian manifolds. Finally, we consider Weyl-semisymmetric (ɛ)-LP-Sasakian manifolds.

2. (ɛ)-Lorentzian para-Sasakian manifolds

An n-dimensional differentiable manifold is called (ɛ)-Lorentzian para-Sasakian manifold if the following conditions hold:

\[ \phi^2 = I + \eta (X) \xi, \quad \eta(\xi) = -1, \]

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(2.2) \[ g(\xi, \xi) = \epsilon, \quad \eta(X) = \epsilon g(X, \xi), \]
(2.3) \[ g(\phi X, \phi Y) = g(X, Y) + \epsilon \eta(X) \eta(Y), \]
where \( \epsilon \) is 1 or \(-1\) according as \( \xi \) is space-like or time-like vector field. Also in \((\epsilon)\)-Lorentzian para-Sasakian manifold, we have
(2.4) \[ (\nabla_X \phi) Y = g(X, Y) \xi + \epsilon \eta(Y) X + 2\epsilon \eta(X) \eta(Y), \]
where \( \nabla \) denotes the operator of covariant differentiation with respect to the Lorentzian metric \( g \).

**Definition 2.1.** An \((\epsilon)\)-LP-Sasakian manifold will be called a manifold of quasi-constant curvature if the curvature tensor \( \hat{R} \) of type \((0, 4)\) satisfies the condition
(2.5) \[
\hat{R}(X, Y, Z, W) = a [g(Y, Z) g(X, W) - g(X, Z) g(Y, W)] \\
+ b [g(X, W) T(Y) T(Z) - g(X, Z) T(Y) T(W) \\
+ g(Y, Z) T(X) T(W) - g(Y, W) T(X) T(Z)],
\]
where \( \hat{R}(X, Y, Z, W) = g(R(X, Y) Z, W), \ R \) is the curvature tensor of type \((1, 3); a, b \) are scalar functions and \( \rho \) is a unit vector field defined by
(2.6) \[ g(X, \rho) = T(X). \]

The notion of quasi-constant curvature for Riemannian manifolds were given by Chen and Yano [2].

**Definition 2.2.** An \((\epsilon)\)-LP-Sasakian manifold will be called an \(\eta\)-Einstein manifold if the Ricci tensor \( S \) of type \((0, 2)\) satisfies
(2.7) \[ S(X, Y) = a g(X, Y) + b \eta(X) \eta(Y), \]
where \( a \) and \( b \) are scalar functions.

**Definition 2.3.** A type of Riemannian manifold whose curvature tensor \( \hat{R} \) of type \((0, 4)\) satisfies the condition
(2.8) \[ \hat{R}(X, Y, Z, W) = F(Y, Z) F(X, W) - F(X, Z) F(Y, W), \]
where \( F \) is a symmetric tensor of type \((0, 2)\) is called a special manifold with the associated symmetric tensor \( F \) and is denoted by \( \psi(F) \).

In 1956, S. S. Chern [3] study such type of manifolds. These manifolds are important for the following reasons:

Firstly, for possessing some remarkable properties relating to curvature and characteristic classes and secondly, for containing a manifold of quasi-constant curvature [2].

**Definition 2.4.** An \((\epsilon)\)-LP-Sasakian manifold will be called Weyl-semisymmetric if it satisfies \((R, (X, Y) C)(Y, Z) W = 0, \) where \( R(X, Y) \) denotes the curvature operator and \( C(Y, Z) W \) is the Weyl-conformal curvature tensor.
Lemma 2.1. An \((\epsilon)-\text{contact metric manifold}\) is an \((\epsilon)-\text{LP-Sasakian manifold} if and only if
\[
\nabla_X \xi = \epsilon \phi X.
\]

Proof. Let the manifold be an \((\epsilon)-\text{Lorentzian para-Sasakian manifold}\). Then from the equation (2.4) it follows that
\[
\nabla_X \phi Y - \phi \nabla_X Y = g(X, Y) \xi + \epsilon \eta(Y) X + 2\epsilon \eta(X) \eta(Y) \xi.
\]
Putting \(Y = \xi\), we get
\[
-\phi \nabla_X \xi = -\epsilon (X + \eta(X) \xi),
\]
or,
\[
\phi \nabla_X \xi = \epsilon \phi^2(X),
\]
which implies,
\[
\nabla_X \xi = \epsilon \phi(X).
\]
Conversely, let the above relation holds. Now the fundamental 2-form \(\Phi\) of the \((\epsilon)-\text{almost contact metric structure}\) is defined by \([5]\)
\[
\Phi(X, Y) = g(X, \phi Y)
\]
for all vector fields \(X, Y \in \chi(M)\). Now since \(\eta \wedge \phi\) is up to a constant factor the volume element of the manifold, it is parallel with respect to \(\nabla\), i.e., \(\nabla_X (\eta \wedge \phi) = 0\). Hence we have
\[
(\nabla_X \eta)(Y) \Phi(Z, W) + \eta(Y)(\nabla_X \Phi)(Z, W) + (\nabla_X \eta)(Z) \Phi(W, Y)
\]
\[
+ \eta(Z)(\nabla_X \Phi)(W, Y) + (\nabla_X \eta)(W) \Phi(Y, Z) + (\nabla_X \eta)(W) \Phi(Y, Z) + \eta(W) (\nabla_X \Phi)(Y, Z) = 0.
\]
Putting \(W = \xi\), we get
\[
(\nabla_X \Phi) Y = \epsilon g(\Phi \nabla_X \xi, Y) \xi + \eta(Y) \Phi \nabla_X \xi,
\]
Now using the value of \(\nabla_X \xi\), we have
\[
(\nabla_X \phi) Y = g(X, Y) \xi + \epsilon \eta(Y) X + 2\epsilon \eta(X) \eta(Y) \xi.
\]
Hence the manifold is an \((\epsilon)-\text{Lorentzian para-Sasakian manifold}\). \(\Box\)

Example. Consider the 3-dimensional manifold \(M = [x, y, z] \in \mathbb{R}^3, z \neq 0\), where \((x, y, z)\) are the standard coordinates in \(\mathbb{R}^3\). The vector fields
\[
e_1 = e^z \frac{\partial}{\partial y}, e_2 = e^z \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right), e_3 = \frac{\partial}{\partial z}
\]
are linearly independent at each point of \(M\). Let \(g\) be the Lorentzian metric defined by
\[
g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0,
g(e_1, e_1) = g(e_2, e_2) = \epsilon, \quad g(e_3, e_3) = -\epsilon.
\]
Let \( \eta \) be the 1-form defined by \( \eta(Z) = g(Z, e_3) \) for any \( Z \in \chi(M) \). Let \( \phi \) be the \((1,1)\) tensor field defined by \( \phi e_1 = -e_1, \phi e_2 = -e_2, \phi e_3 = 0 \). Then using the linearity of \( \phi \) and \( g \), we have

\[
\eta(e_3) = -1, \quad \phi^2(Z) = Z + \eta(Z) \xi, \quad \text{and} \quad g(\phi Z, \phi W) = g(Z, W) + \epsilon \eta(Z) \eta(W)
\]

for any \( Z, W \in \chi(M) \). Let \( \nabla \) be the Levi-Civita connection with respect to the Lorentzian metric \( g \). Then we have

\[
[e_1, e_2] = 0, \quad [e_1, e_3] = -\epsilon e_1, \quad [e_2, e_3] = -\epsilon e_2.
\]

The Riemannian connection \( \nabla \) of the Lorentzian metric \( g \) is given by

\[
2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g([X, Y], Z) + g(Z, [X, Y])
\]

which is known as Koszul’s formula.

From Koszul’s formula, we have

\[
\nabla_{e_1} e_3 = -\epsilon e_1, \quad \nabla_{e_2} e_3 = -\epsilon e_2, \quad \nabla_{e_3} e_3 = 0,
\]

\[
\nabla_{e_1} e_1 = -\epsilon e_3, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_3} e_1 = 0.
\]

From the above result it can be easily seen that the manifold satisfies

\[
\nabla_X \xi = \epsilon \phi X
\]

for \( \xi = e_3 \). Hence the manifold under consideration is an \((\epsilon)\)-Lorentzian para-Sasakian manifold.

**Lemma 2.2.** In an \((\epsilon)\)-Lorentzian para-Sasakian manifold (2.10) \((\nabla_X \eta)(Y) = g(\phi X, Y)\).

**Proof.**

\[
(\nabla_X \eta)(Y) = \nabla_X \eta(Y) - \eta(\nabla_X Y) = \epsilon \nabla_X g(Y, \xi) - \epsilon g(\nabla_X Y, \xi) - \epsilon g(Y, \nabla_X \xi) + \epsilon g(Y, \nabla_X \xi).
\]

Using the value of \( \nabla_X \xi \), we have

\[
(\nabla_X \eta)(Y) = g(\phi X, Y). \quad \square
\]

**Lemma 2.3.** In an \((\epsilon)\)-Lorentzian para-Sasakian manifold (2.11) \( R(X, Y) \xi = \eta(Y) X - \eta(X) Y \).

**Proof.**

\[
R(X, Y) \xi = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]} \xi = \nabla_X (\epsilon \phi Y) - \nabla_Y (\epsilon \phi X) - \epsilon \phi ([X, Y]).
\]

The above relation after simplification gives

\[
R(X, Y) \xi = \eta(Y) X - \eta(X) Y. \quad \square
\]
Note. From the equation (2.11) it follows that in an \((\epsilon)\)-Lorentzian para-Sasakian manifold,
\[(2.12)\]
\[R(\xi, X) Y = \epsilon g(X, Y) \xi - \eta(Y) X.\]

Also in an \((\epsilon)\)-Lorentzian para-Sasakian manifold
\[(2.13)\]
\[\eta(R(X, Y) Z) = \epsilon (g(Y, Z) \eta(X) - g(X, Z) \eta(Y)).\]

Lemma 2.4. In an \((\epsilon)\)-Lorentzian para-Sasakian manifold
\[(2.14)\]
\[S(X, \xi) = (n - 1) \eta(X).\]

Proof. From the equation (2.13) we have
\[g(R(X, Y) Z, \xi) = \epsilon g(Y, Z) g(X, \xi) - \epsilon g(X, Z) g(Y, \xi).\]

Putting \(Y = Z = e_i\), where \(\{e_i\}\) is an orthonormal basis of the tangent space at each point of the manifold, and taking summation over \(i\) where \(i = 1, 2, \ldots, n\), we get
\[S(X, \xi) = (n - 1) \eta(X).\]

3. Conformally flat \((\epsilon)\)-Lorentzian para-Sasakian manifold

The Weyl conformal curvature tensor \(C\) of type \((1, 3)\) of an \(n\)-dimensional Riemannian manifold is given by
\[(3.1)\]
\[C(X, Y) Z = R(X, Y) Z - \frac{1}{(n - 2)} [S(Y, Z) X - S(X, Z) Y + g(Y, Z) QX - g(X, Z) QY] + \frac{r}{(n - 1)(n - 2)} [g(Y, Z) X - g(X, Z) Y],\]

where \(Q\) is the Ricci operator defined by \(g(QX, Y) = S(X, Y)\) and \(r\) is the scalar curvature. Let us suppose that the manifold is conformally flat. Then from the above equation, we have
\[(3.2)\]
\[g(R(X, Y) Z, W) = \frac{1}{(n - 2)} [S(Y, Z) g(X, W) - S(X, Z) g(Y, W) + g(Y, Z) S(X, W) - g(X, Z) S(Y, W)] - \frac{r}{(n - 1)(n - 2)} [g(Y, Z) g(X, W) - g(X, Z) g(Y, W)].\]

Putting \(W = \xi\) and using the equation (2.14), the above equation gives
\[(3.3)\]
\[\epsilon \eta(R(X, Y) Z) = \frac{1}{(n - 2)} [\epsilon S(Y, Z) \eta(X) - \epsilon S(X, Z) \eta(Y) + (n - 1) g(Y, Z) \eta(X) - (n - 1) g(X, Z) \eta(Y)] - \frac{r}{(n - 1)(n - 2)} [\epsilon g(Y, Z) \eta(X) - \epsilon g(X, Z) \eta(Y)].\]
In view of the equation (2.13) and $\epsilon^2 = 1$, the above equation yields

\begin{equation}
S(Y, Z) \eta(X)
= S(X, Z) \eta(Y) + \left(\frac{r}{n-1} - \epsilon\right) (g(Y, Z) \eta(X) - g(X, Z) \eta(Y)).
\end{equation}

For $X = \xi$, we get

\begin{equation}
S(Y, Z) = \left(\frac{r}{n-1} - \epsilon\right) g(Y, Z) - \left(\frac{r \epsilon + n - n^2}{n-1}\right) \eta(Y) \eta(Z).
\end{equation}

Hence we can state the following:

**Theorem 3.1.** An $(2n + 1)$-dimensional coformally flat $(\epsilon)$-Lorentzian para-Sasakian manifold is an $\eta$-Einstein manifold.

Using the equation (3.5) in (3.2), we get

\begin{equation}
g(R(X, Y) Z, W)
= \frac{1}{n-2} \left(\frac{2r}{n-1} - 2\epsilon\right) g(Y, Z) g(X, W)
- \left(\frac{2r}{n-1} - 2\epsilon\right) g(X, Z) g(Y, W) - \left(\frac{r \epsilon + n - n^2}{n-1}(n-2)\right)
[\eta(Y) \eta(Z) g(X, W) - \eta(X) \eta(Z) g(Y, W)]
+ \eta(X) \eta(W) g(Y, Z) - \eta(Y) \eta(W) g(X, Z)
- \eta(X) \eta(W) g(Y, Z) - \eta(Y) \eta(Z) g(X, W)].
\end{equation}

The above relation can be written as

\begin{equation}
g(R(X, Y) Z, W)
= \frac{r - 2n \epsilon + 2\epsilon}{(n-1)(n-2)} \left[g(Y, Z) g(X, W) - g(X, Z) g(Y, W)\right]
- \left(\frac{r \epsilon + n - n^2}{(n-1)(n-2)}\right) [\eta(X) \eta(Z) g(Y, W) + \eta(Y) \eta(W) g(X, Z)]
- \eta(X) \eta(W) g(Y, Z) - \eta(Y) \eta(Z) g(X, W)].
\end{equation}

In view of Definition (2.1) and the above relation we have the following:

**Theorem 3.2.** An $n$-dimensional coformally flat $(\epsilon)$-Lorentzian para-Sasakian manifold is of quasi-constant curvature.

It is also proved that a $\psi(F)_n$ contains a manifold of quasi-constant curvature as a subclass:

Let

\begin{equation}
F(X, Y) = \sqrt{a} g(X, Y) + \frac{b}{\sqrt{a}} T(X) T(Y).
\end{equation}

Now from the equation (2.5) we know that

\begin{equation}
\end{equation}
Therefore the manifold of quasi-constant curvature is a \( \psi(F)_n \).

From the above condition and Theorem 3.2 we have the following:

**Theorem 3.3.** A conformally flat \( \epsilon \)-Lorentzian para-Weyl-semisymmetric Sasakian manifold is a \( \psi(F)_n \).

4. Weyl-semisymmetric \( \epsilon \)-Lorentzian para-Sasakian manifolds

An \( \epsilon \)-Lorentzian para-Sasakian manifold is said to be Weyl-semisymmetric if

\[ R.C = 0. \]

From the equation (3.1), we get

\[
g(C(X,Y)Z,\xi) = g(R(X,Y)Z,\xi) - \frac{1}{n-2} [g(Y,Z)S(X,\xi) - g(X,Z)S(Y,\xi) + S(Y,Z)g(X,\xi)]
\]

(4.1)

From the above equation, we have

\[
\eta(C(X,Y)Z) = \frac{1}{(n-2)^2} \left( \frac{r}{n-1} - \epsilon \right) [g(Y,Z)\eta(X) - \eta(Y)g(X,Z) + S(X,Z)\eta(Y)].
\]

(4.2)

Putting \( Z = \xi \), in the above equation, we have

\[
\eta(C(X,Y)\xi) = 0.
\]

(4.3)

Again putting \( X = \xi \) in the equation (4.2), we get

\[
\eta(C(\xi,Y)Z) = \frac{1}{n-2} \left( \frac{r}{n-1} - \epsilon \right) [g(Y,Z) - \epsilon \eta(Y)\eta(Z)] - S(Y,Z) + (n-1)\eta(Y)\eta(Z).
\]

(4.4)

If the manifold is Weyl-semisymmetric, then we have

\[
g[R(\xi,Y)C(U,V)W,\xi] - g[C(R(\xi,Y)U,V)W,\xi] - g[C(U,R(\xi,Y)V,W),\xi] - g[C(U,V)R(\xi,Y)W,\xi] = 0.
\]

(4.5)

From the equation (2.12), we have

\[
g[R(\xi,X)Y,\xi] = g(X,Y) - \epsilon \eta(Y\eta(X)).
\]

(4.6)

Using the equation (4.6) in (4.5), we get
\[ g(Y, C(U, V)W) - \epsilon \eta(C(U, V)W)\eta(Y) \]
\[ - g[C(\epsilon g(Y, U) - \eta(U) Y, V)W, \xi] \]
\[ - g[C(U, \epsilon g(Y, V) - \eta(V) Y)W, \xi] \]
\[ - g[C(U, V)(\epsilon g(Y, W) - \eta(W) Y), \xi] = 0. \]

From the above equation, we have
\[ - \dot{C}(U, V, W, Y) + \eta(Y)\eta[C(U, V)W) \]
\[ - \epsilon \eta(U)\eta(C(Y, V)W) - \epsilon \eta(V)\eta(C(U, Y)W) \]
\[ - \epsilon \eta(W)\eta(C(U, V)Y) + g(Y, U)\eta(C(\xi, V)W) \]
\[ + g(Y, V)\eta(C(U, \xi)W) + g(Y, W)\eta(C(U, V)\xi) = 0, \]
where \( \dot{C}(U, V, W, Y) = g(C(U, V)W, Y) \).

Putting \( Y = U \), we get
\[ - \dot{C}(U, V, W, U) + \eta(U)\eta[C(U, V)W) \]
\[ (V)\eta(C(U, U)W) \]
\[ - \epsilon \eta(W)\eta(C(U, V)U) + g(U, U)\eta(C(\xi, V)W) \]
\[ + g(U, V)\eta(C(U, \xi)W) + g(U, W)\eta(C(U, V)\xi) = 0. \]

Again putting \( U = e_i \), where \( \{e_i\} \) is an orthonormal basis of the tangent space at each point of the manifold, and taking summation over \( i \) where \( i = 1, 2, \ldots, n \), we get
\[ \sum_{i=1}^{n} \dot{C}(e_i, V, W, e_i) = 0 \]
and using (4.3) in (4.9), we have
\[ \eta(C(\xi, V)W) = 0. \]

Using the equation (4.3) and (4.10) in (4.8), we get
\[ - \dot{C}(U, V, W, Y) + \eta(Y)\eta[C(U, V)W) \]
\[ - \epsilon \eta(U)\eta(C(Y, V)W) - \epsilon \eta(V)\eta(C(U, Y)W) \]
\[ - \epsilon \eta(W)\eta(C(U, V)Y) = 0. \]

Using the equation (4.2) in (4.11), we get
\[ - \dot{C}(U, V, W, Y) - \frac{\eta(W)}{n - 2}\left\{ \frac{\epsilon r}{n - 1} - 1 \right\} g(Y, V)\eta(U) \]
\[ - g(U, Y)\eta(V) - \epsilon (S(Y, V)\eta(U) - S(Y, U)\eta(V)) \]
\[ - \frac{(\epsilon - 1)}{n - 2}\left\{ \frac{\epsilon r}{n - 1} - 1 \right\} [g(U, W)\eta(V)\eta(Y) \]
\[ - g(V, W)\eta(U)\eta(Y) - S(U, W)\eta(Y)(V) \]
\[ + S(V, W)\eta(U)\eta(Y)] = 0. \]
From the equation (4.10), we have from (4.4)

\begin{equation}
S(Y, Z) = \left( \frac{r}{n-1} - \epsilon \right) g(Y, Z) - \left( \frac{r \epsilon}{n-1} - n \right) \eta(Y) \eta(Z).
\end{equation}

Using the equation (4.13) in (4.12)

\begin{equation}
C(U, V, W, Y) = 0.
\end{equation}

From the above equation we see that $R.C = 0$ implies that $C = 0$. Hence using this condition with the help of Theorem 3.2 we have the following:

**Theorem 4.1.** A $n$-dimensional Weyl-semisymmetric $(\epsilon)$-Lorentzian para-Sasakian manifold is of quasi-constant curvature.

Theorem 3.3 and (4.14) leads the following:

**Corollary 4.1.** A $n$-dimensional Weyl-semisymmetric $(\epsilon)$-Lorentzian para-Sasakian manifold is a $(F)$.

**Application.** $(\epsilon)$-Lorentzian para-Sasakian manifolds are used in the theory of Relativity and Newton's law of gravitational field.

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