ON DEGENERATE $q$-BERNOULLI POLYNOMIALS

TAEKYUN KIM

Abstract. In this paper, we introduce the degenerate Carlitz $q$-Bernoulli numbers and polynomials and give some interesting identities and properties of these numbers and polynomials which are derived from the generating functions and $p$-adic integral equations.

1. Introduction

Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will, respectively, denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. Let $\nu_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = \frac{1}{p}$. When one talks of $q$-extension, $q$ is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, we assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we assume $|q-1|_p < p^{-\nu_p}$ so that $q^x = \exp(x \log q)$ for $|x|_p < 1$. We use the notation $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q \to 1} [x]_q = x$.

In [2], L. Carlitz considered $q$-Bernoulli numbers as follows:

$$
(1.1) \quad \beta_{0,q} = 1, \quad q(q\beta_q + 1)^n - \beta_{n,q} = \begin{cases} 
1, & \text{if } n = 1, \\
0, & \text{if } n > 1,
\end{cases}
$$

with the usual convention about replacing $\beta^n_q$ by $\beta_{n,q}$. The $q$-Bernoulli polynomials are defined by

$$
(1.2) \quad \beta_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} \beta_{l,q} q^l [x]_q^{n-l} \quad \text{(see [2, 8])}.
$$

In [4, 3], L. Carlitz defined the degenerate Bernoulli polynomials which are given by the generating function to be

$$
(1.3) \quad \frac{t}{(1 + \lambda t)^{\frac{q}{p}} - 1}(1 + \lambda t)^{\frac{q}{p}} = \sum_{n=0}^{\infty} \beta_n(x|\lambda) \frac{t^n}{n!} \quad \text{(see [2, 5])}.
$$

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When $x = 0$, $\beta_n(\lambda) = \beta_n(0|\lambda)$ are called the degenerate Bernoulli numbers. Note that $\lim_{\lambda \to 0} \beta_n(x|\lambda) = B_n(x)$, where $B_n(x)$ are the ordinary Bernoulli polynomials (see [1-12]). Let $Ud(Z_p)$ be the space of uniformly differentiable functions on $Z_p$. For $f \in Ud(Z_p)$, the $p$-adic $q$-integral on $Z_p$ is defined by

\[(1.4) \quad I_q(f) = \int_{Z_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x \quad (\text{see [8]})\]

The Carlitz’s $q$-Bernoulli polynomials can be represented by $p$-adic $q$-integrals on $Z_p$ as follows:

\[(1.5) \quad \int_{Z_p} [x+y]_q^n d\mu_q(y) = \beta_{n,q}(x) \quad (n \geq 0)\]

Thus, by (1.4), we get

\[(1.6) \quad \int_{Z_p} e^{[x+y]_q t} d\mu_q(y) = \sum_{n=0}^{\infty} \beta_{n,q}(x) \frac{t^n}{n!} \quad (\text{see [8]})\]

From (1.6), we can derive the following equation:

\[(1.7) \quad \beta_{m,q}(x) = \frac{1}{(1-q)^m} \sum_{j=0}^{m} \binom{m}{j} (-1)^j q^{jx} \frac{j+1}{j+1_q} \quad (m \geq 0)\]

In this paper, we introduce the degenerate Carlitz $q$-Bernoulli numbers and polynomials and give some interesting identities and properties of the numbers and polynomials which are derived from the generating functions and $p$-adic integral equations on $Z_p$.

2. Degenerate Carlitz $q$-Bernoulli numbers and polynomials

In this section, we assume that $\lambda, t \in \mathbb{C}_p$ with $0 < |\lambda|_p \leq 1$, $|t|_p = p^{-\frac{1}{m+1}}$. Then, as $|\lambda|_p < p^{-\frac{1}{m+1}}$, $|\log(1 + \lambda)|_p = |\lambda|_p$ and hence $|\frac{1}{\lambda} \log(1 + \lambda)|_p = |t|_p < p^{-\frac{1}{m+1}}$ and now it makes sense to take the limit as $\lambda \to 0$.

In the viewpoint of (1.3), we consider the degenerate Carlitz $q$-Bernoulli polynomials which are given by the generating function to be

\[(2.1) \quad \int_{Z_p} (1 + \lambda t)^{\frac{[x+y]_q}{\lambda}} d\mu_q(y) = \sum_{n=0}^{\infty} \beta_{n,q}(x|\lambda) \frac{t^n}{n!}\]

When $x = 0$, $\beta_{n,q}(\lambda) = \beta_{n,q}(0|\lambda)$ are called the degenerate Carlitz $q$-Bernoulli numbers.

Now, we observe that

\[(2.2) \quad \int_{Z_p} (1 + \lambda t)^{\frac{[x+y]_q}{\lambda}} d\mu_q(y) = \sum_{n=0}^{\infty} \int_{Z_p} \left( \frac{[x+y]_q}{\lambda} \right)^n \mu_q(y) \lambda^n t^n\]

\[= \sum_{n=0}^{\infty} \int_{Z_p} \left( \frac{[x+y]_q}{\lambda} \right)^n \mu_q(y) \lambda^n \frac{t^n}{n!},\]
where \(\frac{[x+y]_q}{\lambda} = \frac{[x+y]_q}{\lambda} \times \frac{[x+y]_q}{\lambda} - 1 \times \cdots \times \frac{[x+y]_q}{\lambda} - n + 1\).

Now, we define \([x+y]_{n,\lambda}\) as \([x+y]_{0,\lambda} = 1\),

\[(2.3) \quad [x+y]_{n,\lambda} = [x+y]_q ([x+y]_q - \lambda) \cdots ([x+y]_q - (n-1)\lambda) \quad (n \geq 1).\]

Therefore, by (2.1), (2.2) and (2.3), we obtain the following theorem.

**Theorem 2.1.** For \(n \geq 0\), we have

\[
\int_{\mathbb{Z}_p} [x+y]_{n,\lambda} d\mu_q(y) = \beta_{n,q}(x|\lambda).
\]

Let \(S_1(n,m)\) be the Stirling numbers of the first kind which are defined by

\[(x)_n = \sum_{l=0}^{n} S_1(n,l) x^l, \quad (n \geq 0).\]

Then, by (2.2), we get

\[
(2.4) \quad \int_{\mathbb{Z}_p} \left(\frac{[x+y]_q}{\lambda}\right)_n d\mu_q(y) = \sum_{l=0}^{n} S_1(n,l) \lambda^{n-l} \beta_{l,q}(x).
\]

Therefore, by (2.2) and (2.4), we obtain the following theorem.

**Theorem 2.2.** For \(n \geq 0\), we have

\[
\beta_{n,q}(x|\lambda) = \sum_{l=0}^{n} S_1(n,l) \lambda^{n-l} \beta_{l,q}(x).
\]

Note that \(\lim_{\lambda \to 0} \beta_{n,q}(x|\lambda) = \beta_{n,q}(x)\).

**Corollary 2.3.** For \(n \geq 0\), we have

\[
\beta_{n,q}(x|\lambda) = \sum_{l=0}^{n} \sum_{j=0}^{l} \frac{S_1(n,l)}{(1-q)^j} (-1)^j q^j x^j \lambda^{n-l}.
\]

We observe that

\[
(1 + \lambda t)^{[x+y]_q} = e^{[x+y]_q \log(1+\lambda t)} = \sum_{n=0}^{\infty} \left(\frac{[x+y]_q}{\lambda}\right)^n \frac{1}{n!} (\log(1 + \lambda t))^n
\]

\[
(2.5) \quad = \sum_{m=0}^{\infty} \left(\frac{[x+y]_q}{\lambda}\right)^m \frac{1}{m!} \sum_{n=m}^{\infty} S_1(n,m) \frac{\lambda^{n-m}}{n!}
\]

\[
= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \lambda^{n-m} S_1(n,m) [x+y]_q^m\right) \frac{t^n}{n!}.
\]
Thus, by (2.5), we get
\[
\int_{\mathbb{R}} (1 + \lambda t)^{x+y} d\mu_q(y) = \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \lambda^{n-m} S_1(n,m) \right) \left( \int_{\mathbb{R}} [x+y]^m q^m d\mu_q(x) \right) \frac{t^n}{n!} \\
= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \lambda^{n-m} S_1(n,m) \beta_{m,q}(x) \right) \frac{t^n}{n!}.
\]
Replacing \( t \) by \( \frac{1}{\lambda} (e^{\lambda t} - 1) \) in (2.1), we get
\[
\int_{\mathbb{R}} e^{x+y} t^{x+y} d\mu_q(y) = \sum_{m=0}^{\infty} \beta_{m,q}(x|\lambda) \frac{1}{m!} \frac{1}{\lambda^m} (e^{\lambda t} - 1)^m \\
(2.6) \quad = \sum_{m=0}^{\infty} \beta_{m,q}(x|\lambda) \lambda^{-m} \sum_{n=m}^{\infty} S_2(n,m) \frac{\lambda^n t^n}{n!} \\
= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \beta_{m,q}(x|\lambda) \lambda^{n-m} S_2(n,m) \right) \frac{t^n}{n!},
\]
where \( S_2(n,m) \) are the Stirling numbers of the second kind.

We note that the left hand side of (2.6) is given by
\[
\int_{\mathbb{R}} e^{x+y} t^{x+y} d\mu_q(y) = \sum_{n=0}^{\infty} \left( \beta_{n,q}(x) \lambda^n \right) \frac{t^n}{n!} \\
(2.7) \quad = \sum_{n=0}^{\infty} \beta_{n,q}(x) \frac{t^n}{n!}.
\]
Therefore, by (2.6) and (2.7), we obtain the following theorem.

**Theorem 2.4.** For \( n \geq 0 \), we have
\[
\beta_{n,q}(x) = \sum_{m=0}^{n} \beta_{m,q}(x|\lambda) \lambda^{n-m} S_2(n,m).
\]

Note that
\[
(1 + \lambda t)^{x+y} = (1 + \lambda t)^{x+y} \frac{x+y}{x+y} \\
= \left( \sum_{m=0}^{\infty} [x]_{m,\lambda} \frac{t^m}{m!} \right) \left( \sum_{l=0}^{\infty} q^l \frac{[y]^l q^l}{l!} \right) \\
= \left( \sum_{m=0}^{\infty} [x]_{m,\lambda} \frac{t^m}{m!} \right) \left( \sum_{k=0}^{\infty} \lambda^{-k} q^{-k} [y]^k S_1(k,l) \frac{t^k}{k!} \right) \\
(2.8) \quad = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \sum_{l=0}^{k} [x]_{n-k,\lambda} \lambda^{k-l} q^{k-l} [y]^l S_1(k,l) \frac{t^n}{n!} \right) \frac{t^n}{n!}.
\]
Thus, by (2.8), we get

\[
\sum_{n=0}^{\infty} \beta_{n,q}(x|\lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \sum_{l=0}^{k} [x]_{n-k,\lambda} \lambda^{k-l}q^{lx} \int_{\mathbb{Z}} [y]_q^l d\mu_q(y) S_1(k,l) \binom{n}{k} \right) \frac{t^n}{n!}
\]

Therefore, by (2.3), we obtain the following theorem.

**Theorem 2.5.** For \( n \geq 0 \), we have

\[
\beta_{n,q}(x|\lambda) = \sum_{k=0}^{n} \sum_{l=0}^{k} [x]_{n-k,\lambda} \lambda^{k-l}q^{lx} \beta_{l,q} S_1(k,l) \binom{n}{k} \frac{t^n}{n!}
\]

For \( r \in \mathbb{N} \), we define the degenerate Carlitz \( q \)-Bernoulli polynomials of order \( r \) as follows:

\[
\int_{\mathbb{Z}} \cdots \int_{\mathbb{Z}} (1 + \lambda t)^{\sum_{i=1}^{r} x_i} d\mu_q(x_1) \cdots d\mu_q(x_r) = \sum_{n=0}^{\infty} \beta^{(r)}_{n,q}(x|\lambda) \frac{t^n}{n!}
\]

We observe that

\[
\int_{\mathbb{Z}} \cdots \int_{\mathbb{Z}} (1 + \lambda t)^{\sum_{i=1}^{r} x_i} d\mu_q(x_1) \cdots d\mu_q(x_r)
\]

\[
= \sum_{m=0}^{\infty} \lambda^{-m} \int_{\mathbb{Z}} \cdots \int_{\mathbb{Z}} [x_1 + \cdots + x_r + x]^m d\mu_q(x_1) \cdots d\mu_q(x_r) \frac{1}{m!} (\log(1 + \lambda t))^m
\]

\[
= \sum_{m=0}^{\infty} \beta^{(r)}_{m,q}(x|\lambda) \lambda^{-m} \sum_{n=m}^{\infty} S_1(n,m) \frac{\lambda^n}{n!} \frac{t^n}{n!}
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \lambda^{n-m} \beta^{(r)}_{m,q}(x) S_1(n,m) \right) \frac{t^n}{n!}
\]

where \( \beta^{(r)}_{m,q}(x) \) are the Carlitz \( q \)-Bernoulli polynomials of order \( r \).

Therefore, by (2.10) and (2.11), we obtain the following theorem.

**Theorem 2.6.** For \( n \geq 0 \), we have

\[
\beta^{(r)}_{n,q}(x|\lambda) = \sum_{m=0}^{n} \lambda^{n-m} \beta^{(r)}_{m,q}(x) S_1(n,m).
\]
Replacing \( t \) by \( \frac{1}{\lambda} (e^{\lambda t} - 1) \) in (2.10), we have

\[
\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{[x_1 + \cdots + x_r + x_1]t} d\mu_q(x_1) \cdots d\mu_q(x_r)
= \sum_{m=0}^{\infty} \beta_m^{(r)}(x|\lambda) \frac{1}{m!} \lambda^{-m} (e^{\lambda t} - 1)^m
\]

\( (2.12) \)

\[
= \sum_{m=0}^{\infty} \beta_m^{(r)}(x|\lambda) \lambda^{-m} \sum_{n=m}^{\infty} S_2(n,m) \frac{\lambda^n t^n}{n!}
= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \lambda^{n-m} \beta_m^{(r)}(x|\lambda) S_2(n,m) \right) \frac{t^n}{n!}.
\]

The left hand side of (2.12) is given by

\[
\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{[x_1 + \cdots + x_r + x_1]t} d\mu_q(x_1) \cdots d\mu_q(x_r) = \sum_{n=0}^{\infty} \beta_n^{(r)}(x|\lambda) \frac{t^n}{n!}.
\]

By comparing the coefficients on the both sides of (2.12) and (2.13), we obtain the following theorem.

**Theorem 2.7.** For \( n \geq 0 \), we have

\[
\beta_n^{(r)}(x) = \sum_{m=0}^{n} \lambda^{n-m} S_2(n,m) \beta_m^{(r)}(x|\lambda).
\]

We recall that

\[
\int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x = \lim_{N \to \infty} \frac{1}{[dp^N]_q} \sum_{x=0}^{dp^N-1} f(x) q^x,
\]

where \( d \in \mathbb{N} \) and \( f \in UD(\mathbb{Z}_p) \).

Now, we observe that

\[
(2.14) \quad \beta_n(x|\lambda) = \sum_{l=0}^{n} S_1(n,l) \lambda^{n-l} \int_{\mathbb{Z}_p} [x + y]^l q d\mu_q(y),
\]

and

\[
(2.15) \quad \int_{\mathbb{Z}_p} [x + y]^l q d\mu_q(y) = \frac{1}{[m]_q} \sum_{i=0}^{m-1} q^i [m]_q^i \int_{\mathbb{Z}_p} \left[ \frac{x + i}{m} + y \right]^l d\mu_q(y) = [m]_q^{l-1} \sum_{i=0}^{m-1} q^i \beta_l, q^i \left( \frac{x + i}{m} \right),
\]

where \( l \in \mathbb{Z}_{\geq 0} \) and \( m \in \mathbb{N} \).
Therefore, by (2.14) and (2.15), we obtain the following theorem.

**Theorem 2.8.** For $n \geq N \geq 0$, $m \in \mathbb{N}$, we have

\[
\beta_{n,q}(x|\lambda) = \sum_{l=0}^{m-1} \sum_{i=0}^{m-1} S_1(n,l) \lambda^{m-l} |m|_q^{l-1} q^i \beta_{l,q}^{m}(x+i|\lambda).
\]

From (1.4), we note that

\[
qI_q(f_1) - I_q(f) = (q - 1)f(0) + \frac{q - 1}{\log q} f'(0),
\]

where $f'(0) = \left. \frac{df(x)}{dx} \right|_{x=0}$.

By (2.16), we get

\[
q\beta_{n,q}(x+1|\lambda) - \beta_{n,q}(x|\lambda) = (q - 1)\lambda^n \left( \frac{[x]_q}{\lambda} \right) + \sum_{l=1}^{n} \lambda^{n-l}[x]_q^{l-1} q^x,
\]

where $n \in \mathbb{N}$.

Therefore, by (2.17), we obtain the following theorem.

**Theorem 2.9.** For $n \geq 0$, we have

\[
q\beta_{n,q}(x+1|\lambda) - \beta_{n,q}(x|\lambda) = (q - 1)\lambda^n \left( \frac{[x]_q}{\lambda} \right) + \sum_{l=1}^{n} \lambda^{n-l}[x]_q^{l-1} q^x.
\]

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**References**


Taekyun Kim  
Department of Mathematics  
Kwangwoon University  
Seoul 139-701, Korea  
\textit{E-mail address:} tkkim@kw.ac.kr