COMMUTATIVE $p$-SCHUR RINGS OVER NON-ABELIAN GROUPS OF ORDER $p^3$

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Abstract. Recently, it was proved that every $p$-Schur ring over an abelian group of order $p^3$ is Schurian. In this paper, we prove that every commutative $p$-Schur ring over a non-abelian group of order $p^3$ is Schurian.

1. Introduction

Let $H$ be a finite group. We denote by $\mathbb{C}H$ the group algebra of $H$ over the complex number field $\mathbb{C}$. For a nonempty subset $T \subseteq H$, we set $T := \sum_{t \in T} t$, which is treated as an element of $\mathbb{C}H$.

A subalgebra $A$ of the group algebra $\mathbb{C}H$ is called a Schur ring over $H$ if there exists a partition $\text{Bsets}(A) := \{T_0, T_1, \ldots, T_r\}$ of $H$ satisfying the following conditions:

(i) $\{T_i \mid T_i \in \text{Bsets}(A)\}$ is a linear basis of $A$;
(ii) $T_0 = \{1_H\}$;
(iii) $T_i^{-1} := \{t^{-1} \mid t \in T_i\} \in \text{Bsets}(A)$ for all $T_i \in \text{Bsets}(A)$.

A Schur ring $A$ over a $p$-group $H$ is called a $p$-Schur ring if the size of every element in $\text{Bsets}(A)$ is a power of $p$, where $p$ is a prime.

Let $G$ be a subgroup of $\text{Sym}(H)$ containing the left regular representation of $H$. We denote by $T_0 = \{1_H\}, T_1, \ldots, T_r$ the orbits of the stabilizer $G_{1_H}$. The transitivity module $V(H, G_{1_H})$ of $G$ is the vector space spanned by $\{T_i \mid 0 \leq i \leq r\}$. It was proved in [16] that $V(H, G_{1_H})$ is a Schur ring over $H$.

Customarily, a Schur ring $A$ over $H$ is called Schurian if $A$ is the transitivity module $V(H, G_{1_H})$ of some group $G$ containing the left regular representation of $H$.

A family of Schur rings which are not Schurian was given in [16, Theorem 26.4]. It is known that every Schur ring over a cyclic $p$-group is Schurian (see [12]). In 1979, M. Klin conjectured that every Schur ring over a cyclic group is

Received October 1, 2013; Revised March 27, 2014.
2010 Mathematics Subject Classification. Primary 20B05, 20B25.
Key words and phrases. $p$-Schur ring, Schurian, Cayley scheme.
This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2013R1A1A2005349).

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But, it was proved in [3] that there exist non-Schurian Schur rings over cyclic groups. In [15], Spiga and Wang proved that every $p$-Schur ring over an elementary abelian $p$-group of rank 3 is Schurian. Recently, Kim showed that every $p$-Schur ring over an abelian group of order $p^3$ is Schurian (see [10]). In this paper, we focus on $p$-Schur rings over non-abelian groups of order $p^3$. The following example is a non-Schurian $7$-Schur ring over a non-abelian group of order $7^3$. We conjecture that such examples can be constructed for each prime $p \geq 7$.

**Example 1.1.** Let $H = \langle a, b \mid a^{7^2} = b^7 = 1, ab = ba^8 \rangle$ be a non-abelian group of order $7^3$. Then a partition $B_{\text{sets}}(A)$ of $H$ determines a non-commutative $7$-Schur ring which is not Schurian, where

$$B_{\text{sets}}(A) = \{ \{ t \} \mid t \in \langle a^{7^2}, b \rangle \} \cup \{ a^i(ba^{7^2})^j(a^{7^2})^i \mid 0 \leq i \leq 6 \} \cup \{ a^i(ba^7)^2(a^{7^2})^i \mid 0 \leq i \leq 6 \} \cup \{ a^i(ba^7)^3(a^{7^2})^i \mid 0 \leq i \leq 6 \} \cup \{ a^i(ba^7)^4(a^{7^2})^i \mid 0 \leq i \leq 6 \} \cup \{ a^i(ba^7)^5(a^{7^2})^i \mid 0 \leq i \leq 6 \} \cup \{ a^i(ba^7)^6(a^{7^2})^i \mid 0 \leq i \leq 6 \} \cup \{ a^i(ba^7)^7(a^{7^2})^i \mid 0 \leq i \leq 6 \}.$$

So we restrict our attention on commutative $p$-Schur rings. The following is our main theorem.

**Theorem 1.2.** Every commutative $p$-Schur ring over a non-abelian group of order $p^3$ is Schurian.

Note that every $2$-Schur ring over a group of order 8 is commutative and Schurian (see [6]).

This paper is organized as follows. In Section 2, we review notations and known facts about Schur rings. In Section 3, we give a proof of the main theorem.

### 2. Preliminaries

Throughout this paper, we use the notations given in [12].

Let $A$ be a Schur ring over $H$. We say that a subgroup $K$ of $H$ is an $A$-subgroup if $K \subseteq A$. For each $A$-subgroup $E$ of $H$, one can define a subring $A_E$ by setting $A_E = A \cap CE$. It is easy to see that $A_E$ is a Schur ring over $E$ and $B_{\text{sets}}(A_E) = \{ T \mid T \in B_{\text{sets}}(A), T \subseteq E \}$.

For a group $H$, we denote by $R_H$ the set of all binary relations on $H$ that invariant with respect to the left regular representation of $H$. Then the mapping

$$2^H \rightarrow R_H \ (T \mapsto R_H(T)),$$

where $R_H(T) = \{ \{ h, ht \} \mid h \in H, t \in T \}$, is a bijection. If $A$ is a Schur ring over $H$, then the pair

$$C(A) = (H, R_H(B_{\text{sets}}(A))),$$
where \( R_H(Bsets(A)) = \{ R_H(T) \mid T \in Bsets(A) \} \), is called a Cayley (association) scheme over \( H \). (See [18] for association schemes.)

Let \( C = (H, R) \) be a Cayley scheme. For each \( r \in R \), we set \( r(1_H) = \{ h \in H \mid (1_H, h) \in r \} \). Then the vector space spanned by \( \{ r(1_H) \mid r \in R \} \) is a Schur ring over \( H \).

**Theorem 2.1** ([11]). The correspondence \( A \mapsto C(A), C(A) \mapsto A \) induces a bijection between the Schur rings and Cayley schemes over the group \( H \) that preserves the natural partial orders on these sets.

The following propositions are results in [16, 18].

**Proposition 2.2.** Let \( A \) be a Schur ring over \( H \). If \( T \in Bsets(A) \), then the stabilizer \( St(T) := \{ h \in H \mid Th = T = hT \} \) is an \( A \)-subgroup of \( H \).

**Proposition 2.3.** Let \( A \) be a Schur ring over \( H \) and \( m \) an element of \( H \). If \( T, \{ m \} \in Bsets(A) \), then \( Tm = \{ tm \mid t \in T \} \) lies in \( Bsets(A) \).

**Proposition 2.4.** Let \( A \) be a \( p \)-Schur ring over a group \( H \) of order \( p^m \). Then

(i) the group \( O_\theta(A) := \{ h \in H \mid \{ h \} \in Bsets(A) \} \) is a non-trivial \( A \)-subgroup;

(ii) the group \( O^\theta(A) := \{ \{ T^{-1}T \mid T \in Bsets(A) \} \} \) is a proper \( A \)-subgroup;

(iii) there exists a series \( H_0 = \{ 1_H \} < H_1 < \cdots < H_m = H \) of \( A \)-subgroups such that \( |H_{i+1}/H_i| = p \) for \( i = 0, 1, \ldots, m-1 \).

**Proposition 2.5** ([8]). Let \( A \) be a Schur ring over an abelian group \( H \) of order \( p^m \). If there exists \( T \in Bsets(A) \) with size \( p^{m-1} \), then \( Bsets(A) = Bsets(A_{O^\theta(A)}) \cup \{ T^{(i)} \mid 1 \leq i \leq p-1 \} \), where \( T^{(i)} = \{ t^i \mid t \in T \} \).

The following lemma follows straightforwardly from Propositions 2.4 and 2.5.

**Lemma 2.6.** Let \( A \) be a \( p \)-Schur ring over a group \( H \) of order \( p^2 \). Then \( Bsets(A) \) is either \( \{ h \mid h \in H \} \) or \( \{ \{ e \}, T \mid e \in E, T \in (H/E) \setminus \{ E \} \} \) for some subgroup \( E \) of \( H \).

**Lemma 2.7** ([9]). Let \( A \) be a commutative \( p \)-Schur ring over a group \( H \) of order \( p^3 \) and \( L \) an \( A \)-subgroup of order \( p^2 \). Then \( \{ |T| \mid T \in Bsets(A) \setminus Bsets(A_L) \} \) is either \( \{ p \} \) or \( \{ p^2 \} \).

Let \( H \) be a group and \( L \) a subgroup of \( H \). We denote by \( H/L \) the set of left cosets. For \( h \in H \) we define a permutation \( h_R \) as follows:

\[ h_R(x) = hx \text{ for each } x \in H. \]

For \( h \in H \) and \( e \in H/L \) we define a permutation \( h_e \) as follows:

\[ h_e(x) = \begin{cases} h_R(x) & \text{if } x \in e, \\ x & \text{otherwise.} \end{cases} \]

A relative \((m, n, k, \lambda)\)-difference set (RDS) in a finite group \( G \) of order \( mn \) relative to a subgroup \( N \) of order \( n \) is a \( k \)-subset \( R \) of \( G \) such that every element
\(g \in G \setminus N\) has exactly \(\lambda\) representations \(g = r_1r_2^{-1}\) with \(r_1, r_2 \in R\) and no non-identity element of \(N\) has such a representation.

**Proposition 2.8** ([4, 7, 14]). Let \(R\) be a \((p, p, p, 1)\)-RDS in \(G\), where \(p\) is an odd prime. Then \(G\) is elementary abelian.

A function \(f : \mathbb{F}_p \to \mathbb{F}_p\) is called **planar** if \(f(x + a) - f(x)\) is a permutation function of \(\mathbb{F}_p\) for each \(a \neq 0\). It is known that a planar function over \(\mathbb{F}_p\) with odd prime \(p\) can be written as the form of a quadratic polynomial (see [4, 14]).

**Proposition 2.9** ([13]). A function \(f\) is planar if and only if the set \(R = \{(x, f(x)) \in \mathbb{F}_p \times \mathbb{F}_p \mid x \in \mathbb{F}_p\}\) is a \((p, p, p, 1)\)-RDS in \(\mathbb{F}_p \times \mathbb{F}_p\) relative to \(\{0\} \times \mathbb{F}_p\).

### 3. \(p\)-Schur rings over non-abelian groups of odd prime-cube order

Let \(p\) be an odd prime. It is well known that there are exactly two non-abelian groups of order \(p^3\) up to isomorphism, namely

\[H_1 = \langle a, b \mid a^p = b^p = 1, ab = ba^{p+1}\rangle\]
\[H_2 = \langle a, b, c \mid a^p = b^p = c^p = 1, [a, b] = c, [a, c] = [b, c] = 1\rangle.

**Remark 3.1.** (i) Every \(\varphi \in \text{Aut}(H_1)\) is a mapping defined by \(a \mapsto a^ib^j\) and \(b \mapsto a^{im}b^{jm}\), where \(i \in \mathbb{Z}_{p^2}, i \neq 0 \text{ (mod } p)\) and \(j, m \in \mathbb{Z}_p\) (see [5, Section 1.5.1]).

(ii) Every \(\varphi \in \text{Aut}(H_2)\) is a mapping defined by \(a \mapsto a^ib^jc^k, b \mapsto a^ib^mc^n\) and \(c \mapsto c^s\), where \(i, j, k, l, m, n, s \in \mathbb{Z}_p\) and \(s = im - jl \neq 0\) (see [5, Section 1.5.3]).

For the remainder of this section, we assume that \(\mathcal{A}\) is a commutative \(p\)-Schur ring over \(H_i\) \((i = 1, 2)\). For convenience, we often omit the subindex \(i\) of \(H_i\).

**Lemma 3.2.** Let \(\mathcal{A}\) be a commutative \(p\)-Schur ring over \(H\). Then there exists a series of \(\mathcal{A}\)-subgroups of \(H\). Moreover, by replacing the generators if necessary, it is one of the following types:

\(\text{(Type(1))} \{1\} \triangleleft \langle a^i \rangle < \langle a \rangle \triangleleft H_1,\)
\(\text{(Type(2))} \{1\} \triangleleft \langle a^ib^j \rangle < \langle a^i, b \rangle \triangleleft H_1,\)
\(\text{(Type(3))} \{1\} \triangleleft \langle a^ic^j \rangle < \langle a, c \rangle \triangleleft H_2,\)
where \(i, j \in \mathbb{Z}_p\).

**Proof.** By Proposition 2.4(iii), there exists a series of \(\mathcal{A}\)-subgroups of \(H\), i.e.,
\(\{1\} < L < M < H.\)

When \(H = H_1\), \(M\) is either \(\langle a^ib^j \rangle \) \((i \neq 0)\) or \(\langle a^i, b \rangle\). If \(M = \langle a^ib^j \rangle\), then replacing the generator \(a^ib^j\) by \(a\), we have Type(1). If \(M = \langle a^i, b \rangle\), then we have Type(2).

When \(H = H_2\), \(M\) is either \(\langle a, c \rangle\) or \(\langle b, c \rangle\). If \(M = \langle b, c \rangle\), then using an automorphism of \(H\) \((b \mapsto a, a \mapsto b^{p-1}, c \mapsto c)\), we have Type(3). \(\Box\)

**Lemma 3.3.** Let \(\mathcal{A}\) be a commutative \(p\)-Schur ring over \(H\). Suppose that there exists an element \(T \in \text{Bsets}(\mathcal{A})\) with size \(p^2\). Then \(\mathcal{A}\) is Schurian.
Proof. By Proposition 2.4(iii), there exists an $A$-subgroup $L$ of order $p^2$. By Lemma 2.7, every element of $\text{Bsets}(A) \setminus \text{Bsets}(A_L)$ has size $p^2$.

First of all, we claim that each element of $\text{Bsets}(A) \setminus \text{Bsets}(A_L)$ belongs to $(H/L \setminus \{L\})$. Suppose $hl_1, hl_2 \in T$, where $l_1, l_2 \in L, h \in H \setminus L$ and $j \neq 1$. Then $(h^{-1}l_1)^{-1}hl_2 \in T^{-1}T$. By Proposition 2.4(ii), this is a contradiction.

By Lemma 2.6, we divide our consideration into two cases.

(i) $\text{Bsets}(A_L) = \{\{l\} \mid l \in L\}$.

Define a subgroup $G$ of $\text{Sym}(H)$ by $\langle l_f, h_{R} \mid l \in L, h \in H, f \in H/L \rangle$. Clearly, $G$ contains the left regular representation of $H$. It is easy to see that, for given $l_f, h_{R}$, we have $h_{R}^{-1}l_{f}h_{R} = l'_{f}$, for some $l' \in L, f' \in H/L$. So, we can check $G_1 = \langle l_f \mid l \in L, f \in (H/L) \setminus \{L\} \rangle$. Thus, the set of orbits of $G_1$ is $\text{Bsets}(A)$.

(ii) $\text{Bsets}(A_L) = \{\{e\}, T \mid T \in (L/E) \setminus \{E\}, e \in E\}$, where $E$ is an $A$-subgroup of order $p$.

For fixed $e \in E, f \in L \setminus E$ and $g \in H$, we set $x := (1 \ e \ \ldots \ \ e^{p-1})$, $y := (1 \ f \ \ldots \ f^{p-1})(e \ fe \ \ldots \ f^{p-1}e)$, $z := (1 \ z_2 \ \ldots \ z_p)$, where $z_2 = (1 \ g \ \ldots \ g^{p-1})(f \ gf \ \ldots \ g^{p-1}f)$, $z_2 = (e \ ge \ \ldots \ g^{p-1}e)(f \ gf \ \ldots \ g^{p-1}f)$. It is known that $(x, y, z)$ is a Sylow $p$-subgroup of $\text{Sym}(H)$ (see Exercise 2.6.10 of [2]). This implies that $\text{Bsets}(A)$ is the set of orbits of a Sylow $p$-subgroup of $\text{Sym}(H)$. \hfill $\Box$

By Lemma 3.3, from now on, we assume that every element of $\text{Bsets}(A)$ has at most size $p$.

**Lemma 3.4.** Let $A$ be a commutative $p$-Schur ring over $H$ such that $|O_p(A)| = p$ and $L$ an $A$-subgroup of order $p^2$. If $T$ is an element of $\text{Bsets}(A) \setminus \text{Bsets}(A_L)$ such that $\text{St}(T) = \{1\}$, then $T^{-1}T = pI + \sum_{T' \in I} T'$, where $I = \text{Bsets}(A_L) \setminus \{\{h\} \mid h \in O_p(A)\}$.

**Proof.** Since $p \geq 3$, we have $T^{-1}T = pI + \sum_{T' \in \text{Bsets}(A_L) \setminus \{\{1\}\}} c_{T'}T'$. Since $\text{St}(T) = \{1\}$, we have $c_{T'} = 0$ for each $T' \in O_p(A) \setminus \{1\}$. Thus, we have $T^{-1}T = pI + \sum_{T' \in I} c_{T'}T'$.

We claim that $c_{T'} = 1$ for each $T' \in I$.

First of all, we show that $T = \{bx_0, bx_1, \ldots, bx_{p-1}e^{p-1}\}$, where $b \in H \setminus L$, $c \in L \setminus O_p(A)$, $x_i \in O_p(A)$. By Proposition 2.4(ii), all elements of $T$ belong to a coset in $H/L$, i.e., $T = \{ba_0, ba_1, \ldots, ba_{p-1}\}$, where $b \in H \setminus L, a_i \in L$.

Suppose that, for distinct $i, j, a_i$ and $a_j$ belong to the same coset in $L/O_p(A)$. Then $a_i, d = a_j$ for some $d \in O_p(A)$. By Proposition 2.3, we have $Td = T$, a contradiction.

Next, we calculate $T^{-1}T = (x_0^{-1}b^{-1} + c^{-1}x_0^{-1}b^{-1} + \cdots + cx_{p-1}^{-1}b^{-1})(bx_0 + bx_1c + \cdots + bx_{p-1}e^{p-1})$. Using the fact that $L$ is abelian, we can check that every element of $I$ should appear in $T^{-1}T$. \hfill $\Box$
Thus, the size of $T$ implies that $c_{T'} = 1$ for each $T' \in I$. \hfill \Box

**Lemma 3.5.** Let $A$ be a commutative $p$-Schur ring over $H$ and $L$ an $A$-subgroup of order $p^2$. If there exists $T \in \text{Bsets}(A) \setminus \text{Bsets}(AL)$ such that $\text{St}(T) \neq \{1\}$, then $O^0(A)$ is the center of $H$.

**Proof.** We consider three types of $A$-subgroup series. Fix an element $T \in \text{Bsets}(A) \setminus \text{Bsets}(AL)$ such that $\text{St}(T) \neq \{1\}$.

In Type(1), we have $\text{St}(T) = \langle a^p \rangle$. We claim that, for each element of $\text{Bsets}(A) \setminus \text{Bsets}(AL)$, its stabilizer is $\langle a^p \rangle$. Without loss of generality, we can assume $T = \langle a^p b \rangle$ by replacing the generators if necessary. Then we have $T = p[a^pb]$. This implies $O^0(A) = \langle a^p \rangle$.

In Type(2), we claim $\text{St}(T) = \langle a^p \rangle$. Suppose $\text{St}(T) = \langle a^p b^j \rangle$ for some $j \neq 0$. Then we can put $T = \langle a^p b^j \rangle a$. Since $A$ is commutative, we have $\langle a^p b^j \rangle [a \cdot a^p b^j] = a^p b^j [a b^j]$ by the direct computation. This is a contradiction. Thus, we have $\text{St}(T) = \langle a^p \rangle$. This implies $O^0(A) = \langle a^p \rangle$.

Type(3) is similar to the second one. \hfill \Box

Now we divide our consideration into cases depending on $|O^0(A)|$. By Proposition 2.4(ii), we have $|O^0(A)| = p$ or $p^2$.

**Proposition 3.6.** If $A$ is a commutative $p$-Schur ring over $H$ satisfying one of the following conditions:

1. $|O^0(A)| = p$,
2. $|O_0(A)| = |O^0(A)| = p^2$,

then $A$ is Schurian.

**Proof.** If $A$ satisfies condition(1), then $A$ is Schurian by the main theorem of [17].

If $A$ satisfies condition(2), then $O^0(A)$ is either cyclic or elementary abelian. Suppose $O^0(A)$ is elementary abelian. By [1, Lemma 3.3], $\text{Bsets}(A)$ has elements with size $p^2$, a contradiction. Thus, $O^0(A)$ is cyclic. By the main theorem of [17], $A$ is Schurian. \hfill \Box

**Lemma 3.7.** Let $A$ be a commutative $p$-Schur ring over $H$ such that $|O_0(A)| = p$ and $|O^0(A)| = p^2$. Then $O^0(A)$ is elementary abelian.

**Proof.** Fix an element $T \in \text{Bsets}(A) \setminus \text{Bsets}(A_{O_0^0(A)})$ with size $p$. By Lemma 3.5, we have $\text{St}(T) = \{1\}$. By Lemma 3.4, we have $T = pL + \sum_{T' \in I} T'$, where $I = \text{Bsets}(A_{O_0^0(A)}) \setminus \text{Bsets}(A_{O_0^0(A)})$. This implies that there exists a $(p, p, 1)$-RDS in $O^0(A)$. By Proposition 2.8, $O^0(A)$ is elementary abelian. \hfill \Box

**Proposition 3.8.** Let $A$ be a commutative $p$-Schur ring over $H$ such that $|O_0(A)| = p$ and $|O^0(A)| = p^2$. Then $A$ is Schurian.

**Proof.** By Lemma 3.3, we assume that every element of $\text{Bsets}(A)$ has at most size $p$. By Lemma 3.7, $O^0(A)$ is elementary abelian. Then $A$-subgroup series is either Type(2) or Type(3). We fix an element $T \in \text{Bsets}(A) \setminus \text{Bsets}(A_{O_0^0(A)})$. 


In the case of Type(2), we have Bsets\(\mathcal{A}_{(a^p,b)}\) = \{\{h\} | h \in \langle a^pb^i\rangle\} \cup \{L | L \in ((a^p,b)/\langle a^pb^i\rangle) \setminus \langle a^pb^i\rangle\}.

First of all, we assume \(j \neq 0\). Without loss of generality, we can put \(\Omega_0(\mathcal{A}) = \langle a^pb^i\rangle\). Since \(\text{St}(\mathcal{T}) = \{1\}\), we can assume \(T = \{axy_0, ax_1y_0, ax_2y_0, \ldots, ax_{p-1}y_0\}\), where \(y_0 \in \Omega_0(\mathcal{A})\). Since \(\mathcal{A}\) is commutative, it must be satisfied \((a^pb)^mT = (a^pb)^m\) for each \(1 \leq m \leq p-1\). Thus, all \(x_i\) are same, i.e., \(T = \{ab^1, ab^2a^p, \ldots, ab^iap^p\}\) for some \(j\). This implies \(\text{St}(\mathcal{T}) = \langle a^p\rangle\), a contradiction.

Next, we assume \(j = 0\). Then we have \(\Omega_0(\mathcal{A}) = \langle a^p\rangle\). By Lemma 3.4, we can assume \(T = \{ay_0, a^2y_0, a^3y_0, \ldots, a^{p-1}y_0\}\), where \(y_0 \in \Omega_0(\mathcal{A})\). This implies that there exists a \((p,p,1)\)-RDS in \(\langle a^p, b\rangle\). By Proposition 2.9, we have \(T = \{ab^1a^{pf(i)} \mid 0 \leq i \leq p-1\}\), where \(f(i)\) is a planar function.

Replacing the generator \(a\) by \(a^{pf(0)}\), we can assume \(f(i)\) such that \(f(0) = 0\). By the same argument, i.e., replacing \(b\) by \(ba^{pf(1)}\), we also assume \(f(1) = 0\).

It is well known that \(f(x)\) is a quadratic polynomial. So we assume that \(f(x) = dx^2 + ex\). It is easy to see that \(f(i + 1) - f(i) = 2di\) for each \(i \in \mathbb{F}_p\).

Now we define \(\gamma \in \text{Aut}(\mathcal{H})\) by \(a \mapsto ab\) and \(b \mapsto (a^p)^{2d}b\). Then \(P := \langle hR \mid h \in \mathcal{H}\rangle \rtimes \langle \gamma \rangle\) is a subgroup of \(\text{Sym}(\mathcal{H})\). Using \(f(i + 1) - f(i) = 2di\), we can check \(\gamma(ab^ia^{pf(i)}) = ab^{i+1}a^{pf(i+1)}\). Thus, it follows that the set of orbits of \(P_1\) is Bsets\((\mathcal{A})\).

In the case of Type(3), we have Bsets\(\mathcal{A}_{(a,c)}\) = \{\{h\} \in \langle a^pc^i\rangle\} \cup \{L | L \in \langle a, c\rangle \setminus \langle a^pc^i\rangle\} \cup \{\langle a^pc^i\rangle\}\}. Using the fact that \(c\) corresponds to \(a^p\) in Type(2), we can induce \(f(i + 1) - f(i) = 2di\) as mentioned in Type(2). Defining \(\gamma \in \text{Aut}(\mathcal{H})\) by \(a \mapsto ac^2\), \(b \mapsto ba\) and \(c \mapsto c\), we can check that \(\mathcal{A}\) is Schurian. \(\square\)

In conclusion, it is proved that every commutative \(p\)-Schur ring over a non-abelian group of order \(p^3\) is Schurian.

**Acknowledgements.** The author would like to thank Professor I. Ponomarenko for correcting the errors of his presentation at the 59th KPPY combinatorics seminar. He also thanks anonymous referees for their valuable comments.

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