APPROXIMATION METHODS FOR A COMMON MINIMUM-NORM POINT OF A SOLUTION OF VARIATIONAL INEQUALITY AND FIXED POINT PROBLEMS IN BANACH SPACES

N. SHAHZAD AND H. ZEGEYE

Abstract. We introduce an iterative process which converges strongly to a common minimum-norm point of solutions of variational inequality problem for a monotone mapping and fixed points of a finite family of relatively nonexpansive mappings in Banach spaces. Our theorems improve most of the results that have been proved for this important class of nonlinear operators.

1. Introduction

Let $E$ be a real Banach space with dual $E^*$. We denote by $J$ the normalized duality mapping from $E$ into $2^{E^*}$ defined for each $x \in E$ by

$$Jx := \{f^* \in E^* : \langle x, f^* \rangle = ||x||^2 = ||f^*||^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between members of $E$ and $E^*$. It is well known that $E$ is smooth if and only if $J$ is single-valued and if $E$ is uniformly smooth, then $J$ is uniformly continuous on bounded subsets of $E$. Moreover, if $E$ is a reflexive and strictly convex real Banach space with a strictly convex dual, then $J^{-1}$ is single valued, one-to-one, surjective, and it is the duality mapping from $E^*$ into $E$ and thus $JJ^{-1} = I_{E^*}$ and $J^{-1}J = I_E$ (see [17]). If $E = H$, a real Hilbert space, then the duality mapping becomes the identity map on $H$.

Let $E$ be a smooth real Banach space with dual $E^*$. Let the Lyapunov functional $\phi : E \times E \rightarrow \mathbb{R}$, introduced by Alber [1], be defined by

$$\phi(y, x) = ||y||^2 - 2\langle y, Jx \rangle + ||x||^2 \quad \text{for } x, y \in E,$$

Received April 7, 2013; Revised May 21, 2013.
2010 Mathematics Subject Classification. 47H05, 47H09, 47H10, 47J05, 47J25.
Key words and phrases. monotone mappings, relatively nonexpansive mappings, strong convergence, variational inequality problems.

©2014 Korean Mathematical Society
where $J$ is the normalized duality mapping from $E$ into $2^{E^*}$. It is obvious from the definition of the function $\phi$ that
\begin{equation}
(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2 \quad \text{for } x, y \in E.
\end{equation}
We observe that in a Hilbert space $H$, (1.1) reduces to $\phi(x, y) = \|x - y\|^2$ for $x, y \in H$.

Let $E$ be a reflexive, strictly convex and smooth Banach space and let $C$ be a nonempty, closed and convex subset of $E$. The generalized projection mapping, introduced by Alber [1], is a mapping $\Pi_C : E \to C$ that assigns an arbitrary point $x \in E$ to the minimizer, $\bar{x}$, of $\phi(\cdot, x)$ over $C$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem
\begin{equation}
(1.3)
\phi(\bar{x}, x) = \min \{ \phi(y, x) : y \in C \}.
\end{equation}
If $E$ is a Hilbert space, then $\Pi_C = P_C$ is the metric projection of $H$ onto $C$.

In fact, we have the following result.

Lemma 1.1 ([1]). Let $C$ be a nonempty closed and convex subset of a real reflexive, strictly convex, and smooth Banach space $E$ and let $x \in E$. Then there exists a unique element $x_0 \in C$ such that $\phi(x_0, x) = \min \{ \phi(z, x) : z \in C \}$.

Let $C$ be a nonempty subset of a real Banach space $E$ with dual $E^*$. A mapping $A : C \to E^*$ is said to be monotone if for each $x, y \in C$, the following inequality holds:
\begin{equation}
(1.4)
\langle x - y, Ax - Ay \rangle \geq 0.
\end{equation}
$A$ is said to be $\gamma$-inverse strongly monotone if there exists a positive real number $\gamma$ such that
\begin{equation}
(1.5)
\langle x - y, Ax - Ay \rangle \geq \gamma \|Ax - Ay\|^2 \quad \text{for all } x, y \in C.
\end{equation}
If $A$ is $\gamma$-inverse strongly monotone, then it is Lipschitz continuous with constant $\frac{1}{\gamma}$, i.e., $\|Ax - Ay\| \leq \frac{1}{\gamma} \|x - y\|$ for all $x, y \in C$, and it is called strongly monotone if there exists $k > 0$ such that
\begin{equation}
(1.6)
\langle x - y, Ax - Ay \rangle \geq k \|x - y\|^2 \quad \text{for all } x, y \in C.
\end{equation}
Clearly, the class of monotone mappings includes the class of strongly monotone and the class of $\gamma$-inverse strongly monotone mappings.

Suppose that $A$ is a monotone mapping from $C$ into $E^*$. The variational inequality problem is formulated as finding
\begin{equation}
(1.7)
\text{a point } u \in C \text{ such that } \langle v - u, Au \rangle \geq 0 \quad \text{for all } v \in C.
\end{equation}
The set of solutions of the variational inequality problem is denoted by $VI(C, A)$.

Variational inequality problems are related to the convex minimization problems, the zero of monotone mappings and the complementarity problems. Consequently, many researchers (see, e.g., [4, 8, 9, 11, 20, 21]) have made efforts to obtain iterative methods for approximating solutions of variational inequality problems in the setting of Hilbert spaces or Banach spaces.
If $E = H$, a real Hilbert space, Iiduka, Takahashi and Toyoda [6] introduced the following projection algorithm:

$$x_0 = w \in C, \ x_{n+1} = P_C(x_n - \alpha_n Ax_n) \text{ for any } n \geq 0,$$

where $P_C$ is the metric projection from $H$ onto $C$ and $\{\alpha_n\}$ is a sequence of positive real numbers. They proved that the sequence $\{x_n\}$ generated by (1.8) converges weakly to an element of $VI(C, A)$ provided that $A$ is a $\gamma$-inverse strongly monotone mapping.

When $E$ is a 2-uniformly convex and uniformly smooth Banach space, Iiduka and Takahashi [5] introduced the following iteration scheme for finding a solution of the variational inequality problem for a $\gamma$-inverse strongly monotone mapping $A$:

$$x_{n+1} = \Pi_CJ^{-1}(Jx_n - \alpha_n Ax_n) \text{ for any } n \geq 0,$$

where $\Pi_C$ is the generalized projection from $E$ onto $C$, $J$ is the normalized duality mapping from $E$ into $E^*$ and $\{\alpha_n\}$ is a sequence of positive real numbers. They proved that the sequence $\{x_n\}$ generated by (1.9) converges weakly to an element of $VI(C, A)$ provided that $VI(C, A) \neq \emptyset$ and $A$ satisfies $||Ax|| \leq ||Ax - Ap||$ for all $x \in C$ and $p \in VI(C, A)$.

We note that the convergence results obtained above are weak convergence. To obtain strong convergence, Iiduka and Takahashi [4], studied the following iterative scheme, in a 2-uniformly convex and uniformly smooth Banach space $E$, for a variational inequality problem for a $\gamma$-inverse strongly monotone mapping $A$:

$$x_0 = w \in K, \text{ chosen arbitrary,}$$
$$y_n = \Pi_CJ^{-1}(Jx_n - \alpha_n Ax_n)$$
$$C_n = \{z \in E : \phi(z, y_n) \leq \phi(z, x_n)\},$$
$$Q_n = \{z \in E : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\},$$
$$x_{n+1} = \Pi_{C_n \cap Q_n}(x_0), \text{ for each } n \geq 0,$$

where $\Pi_{C_n \cap Q_n}$ is the generalized projection from $E$ onto $C_n \cap Q_n$, $J$ is the normalized duality mapping from $E$ into $E^*$ and $\{\alpha_n\}$ is a positive real sequence satisfying certain conditions. They proved that the sequence $\{x_n\}$ converges strongly to an element of $A^{-1}(0)$.

**Remark 1.2.** We remark that the computation of $x_{n+1}$ in Algorithms (1.10) requires the computations of $C_n$, $Q_n$ and $C_n \cap Q_n$ for each $n \geq 1$.

**Remark 1.3.** We note that, as it is mentioned in [24], if $C$ is a subset of a real Banach space $E$ and $A : C \to E^*$ is a monotone mapping satisfying $||Ax|| \leq ||Ax - Ap||$, $\forall x \in C$ and $p \in VI(C, A)$, then $VI(C, A) = A^{-1}(0) = \{p \in C : Ap = 0\}$.

Let $T$ be a mapping from $C$ into itself. We denote by $F(T)$ the fixed points set of $T$. A point $p$ in $C$ is said to be an asymptotic fixed point of $T$ (see [14]) if $C$ contains a sequence $\{x_n\}$ which converges weakly to $p$ such that
lim_{n \to \infty} ||x_n - Tx_n|| = 0. The set of asymptotic fixed points of $T$ will be denoted by $\hat{F}(T)$. A mapping $T$ from $C$ into itself is said to be nonexpansive if $||Tx - Ty|| \leq ||x - y||$ for each $x, y \in C$, and is called relatively nonexpansive if (R1) $F(T) \neq \emptyset$; (R2) $\phi(p, Tx) \leq \phi(p, x)$ for $x \in C$ and (R3) $F(T) = \hat{F}(T)$.

If $E$ is a uniformly smooth and uniformly convex real Banach space, and $A \subset E \times E^*$ is a maximal monotone mapping with $A^{-1}(0) \neq \emptyset$, then the resolvent $Q_r := (J + rA)^{-1}J$, for $r > 0$, is relatively nonexpansive (see [13]).

If $E = H$, a real Hilbert space, then the class of relatively nonexpansive mappings contains the class of nonexpansive mappings with $F \neq \emptyset$ (see, eg, [25]).

In [3], Iduka and Takahashi studied the following iterative scheme for a common point of fixed point set of nonexpansive mapping and solution set of a variational inequality problem for a $\gamma$-inverse strongly monotone mapping $A$ in a Hilbert space $H$:

\begin{equation}
\begin{aligned}
&\alpha_n w + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n), \ n \geq 0, \\
&\{x_0 = w \in C
\end{aligned}
\end{equation}

where $\{\alpha_n\}$ is a sequences satisfying certain conditions. They proved that the sequence $\{x_n\}$ converges strongly to an element of $F := F(S) \cap VI(C, A)$ provided that $F \neq \emptyset$.

In addition, many authors have considered the problem of finding a common element of the fixed point set of relatively nonexpansive mapping and the solution set of a variational inequality problem for a $\gamma$-inverse strongly monotone mapping $A$ (see, e.g., [10, 16, 18, 20, 21, 23]) which is nearest to the initial point $x_0 = w$.

However, we notice that it is quite often to seek a minimum-norm solution of a given nonlinear problem. A point $\bar{x} \in C$, where $C$ is a nonempty, closed and convex subset of a real Hilbert space $H$, is called a minimum-norm solution of a nonlinear problem with solution $F \neq \emptyset$ if and only if there exists $\bar{x} \in F$ satisfying the property that

\begin{equation}
||\bar{x}|| = \min\{||x|| : x \in F\},
\end{equation}

that is, $\bar{x}$ is the nearest point projection of the origin onto $F$.

A typical example is the least-squares solution to the constrained linear inverse problem

\begin{equation}
\begin{aligned}
&Ax = b, \\
&x \in C,
\end{aligned}
\end{equation}

where $A$ is a bounded linear operator from $H$ into another real Hilbert space $H_1$ and $b$ is a given point in $H_1$. The least-squares solution to (1.13) is the solution of the following minimization problem with the minimum equal to zero:

\begin{equation}
\min_{x \in C} \frac{1}{2}||Ax - b||^2.
\end{equation}

Let $\Omega$ denote the (closed convex) solution set of (1.13) (or equivalently (1.14)). Then, in this case, $\Omega$ has a unique element $\bar{x}$ if and only if it is a solution of
the following variational inequality:
\[ \bar{x} \in C \text{ such that } \langle A^* (A\bar{x} - b), x - \bar{x} \rangle \geq 0, \quad x \in C, \]
where \( A^* \) is the adjoint of \( A \). In addition, we observe that inequality (1.15) can be rewritten as
\[ \bar{x} \in C, \quad \langle \bar{x} - \gamma A^* (A\bar{x} - b) - \bar{x}, x - \bar{x} \rangle \leq 0, \quad x \in C, \]
where \( \gamma > 0 \) is any positive scalar. In the terminology of projection, we see that (1.16) is equivalent to the fixed point equation
\[ \bar{x} = P_C (\bar{x} - \gamma A^* (A\bar{x} - b)). \]
It is not hard to see that for \( 0 < \gamma < \frac{2}{\|A\|^2} \), the mapping \( x \to P_C (x - \gamma A^* (Ax - b)) \) is nonexpansive. Therefore, finding the least-squares solution of the constrained linear inverse problem (1.13) is equivalent to finding the minimum-norm fixed point of the nonexpansive mapping \( x \to P_C (x - \gamma A^* (Ax - b)) \).

Let \( C \) be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space \( E \). Let \( T_i : C \to C \), for \( i = 1, 2, \ldots, N \), be relatively nonexpansive mapping and \( A : C \to E^* \) be a continuous monotone mapping with \( F := \bigcap_{i=1}^{N} F(T_i) \cap VI(C, A) \neq \emptyset \).

It is our purpose in this paper to introduce an iterative scheme (see (3.1)) which converges strongly, to a minimum-norm (with respect to the generalized projection) point of \( F \), that is, to a point \( x^* \in F \) such that \( x^* = \Pi_F (0) \). Our theorems improve most of the results that have been proved for this important class of nonlinear mappings.

2. Preliminaries

Let \( E \) be a Banach space and let \( S(E) = \{ x \in E : \|x\| = 1 \} \). Then a Banach space \( E \) is said to be smooth provided that the limit
\[ \lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}, \]
exists for each \( x, y \in S(E) \). The norm of \( E \) is said to be uniformly smooth if the limit (2.1) is attained uniformly for \( (x, y) \) in \( S(E) \times S(E) \) (see [17]).

The modulus of convexity of \( E \) is the function \( \delta_E : (0, 2] \to [0, 1] \) defined by
\[ \delta_E(\epsilon) := \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| = \|y\| = 1; \ \epsilon = \|x - y\| \right\}. \]
\( E \) is called uniformly convex if and only if \( \delta_E(\epsilon) > 0 \) for every \( \epsilon \in (0, 2] \).

In the sequel, we shall make use of the following lemmas.

**Lemma 2.1** ([22]). Let \( C \) be a nonempty closed and convex subset of a real reflexive, strictly convex, and smooth Banach space \( E \). If \( A : C \to E^* \) is continuous monotone mapping, then \( VI(C, A) \) is closed and convex.
Lemma 2.2 ([13]). Let $E$ be a strictly convex and smooth Banach space, let $C$ be a nonempty, closed and convex subset of $E$, and let $T$ be a relatively nonexpansive mapping from $C$ into itself. Then $F(T)$ is closed and convex.

Lemma 2.3 ([1]). Let $K$ be a nonempty closed and convex subset of a real reflexive, strictly convex, and smooth Banach space $E$ and let $x \in E$. Then

$$\forall y \in K, \quad \phi(y, \Pi_K x) + \phi(\Pi_K x, x) \leq \phi(y, x).$$

Lemma 2.4 ([8]). Let $E$ be a real smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of $E$. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \to 0$ as $n \to \infty$, then $x_n - y_n \to 0$ as $n \to \infty$.

Lemma 2.5 ([1]). Let $E$ be a reflexive strictly convex and smooth Banach space with $E^*$ as its dual. Then

$$V(x, x^*) + 2 \langle J^{-1}x^* - x, y^* \rangle \leq V(x, x^* + y^*)$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Lemma 2.6 ([1]). Let $C$ be a convex subset of a real smooth Banach space $E$. Let $x \in E$. Then $x_0 = \Pi_C x$ if and only if

$$\langle z - x_0, Jx - Jx_0 \rangle \leq 0, \quad \forall z \in C.$$

Lemma 2.7 ([20]). Let $E$ be a uniformly convex Banach space and $B_R(0)$ be a closed ball of $E$. Then, there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that

$$||\alpha_0 x_0 + \alpha_1 x_1 + \cdots + \alpha_N x_N||^2 \leq \sum_{i=0}^{N} \alpha_i ||x_i||^2 - \alpha_i \alpha_j g(||x_i - x_j||)$$

for $\alpha_i \in (0, 1)$ such that $\sum_{i=0}^{N} \alpha_i = 1$ and $x_i \in B_R(0) := \{x \in E : ||x|| \leq R\}$, for some $R > 0$.

Let $C$ be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space $E$. Let $A \subseteq E \times E^*$ be a monotone mapping satisfying

(2.2) \[ D(A) \subset C \subset \cap_{r>0} J^{-1}R(J+rA). \]

Then we have the following lemmas.
Lemma 2.8 ([2]). Let $E$ be a smooth and strictly convex Banach space, $C$ be a nonempty closed convex subset of $E$, and $A \in E \times E^*$ a monotone operator satisfying (2.2). Let $Q_r$ be the resolvent of $A$ defined by $Q_r = (J + rA)^{-1}J$, for $r > 0$ and $\{r_n\}$ a sequence of $(0, \infty)$ such that $\lim_{n \to \infty} r_n = \infty$. If $\{x_n\}$ is a bounded sequence of $C$ such that $Q_{r_n}x_n \rightharpoonup z$, then $z \in A^{-1}(0)$.

Lemma 2.9 ([7]). Let $E$ be a smooth and strictly convex Banach space, $C$ be a nonempty, closed and convex subset of $E$, and $A \in E \times E^*$ a monotone operator satisfying (2.2) and $A^{-1}(0)$ is nonempty. Let $Q_r$ be the resolvent of $A$ defined by $Q_r = (J + rA)^{-1}J$ for $r > 0$. Then for each $r > 0$,
\[
\phi(p, Q_r x) + \phi(Q_r x, x) \leq \phi(p, x)
\]
for all $p \in A^{-1}(0)$ and $x \in C$.

Lemma 2.10 ([19]). Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:
\[
a_{n+1} \leq (1 - \beta_n)a_n + \beta_n \delta_n, \quad n \geq n_0,
\]
where $\{\beta_n\} \subset (0, 1)$ and $\{\delta_n\} \subset R$ satisfying the following conditions:
\[
\lim_{n \to \infty} \beta_n = 0, \quad \sum_{n=1}^{\infty} \beta_n = \infty, \quad \text{and} \quad \limsup_{n \to \infty} \delta_n \leq 0.
\]
Then, $\lim_{n \to \infty} a_n = 0$.

Lemma 2.11 ([12]). Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:
\[
a_{m_k} \leq a_{m_k+1} \quad \text{and} \quad a_k \leq a_{m_k+1}.
\]
In fact, $m_k = \max\{j \leq k: a_j < a_{j+1}\}$.

3. Main result

We now prove the following theorem.

Theorem 3.1. Let $C$ be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space $E$. Let $A : C \to E^*$ be a continuous monotone mapping satisfying (2.2) and $||Ax|| \leq ||Ax - Ap||$, $\forall x \in C$ and $p \in VI(C, A)$. Let $T_i : C \to C$, $i = 1, 2, \ldots, N$, be a finite family of relatively nonexpansive mappings. Assume that $F := \cap_{i=1}^{N} F(T_i) \cap VI(C, A)$ is nonempty. Let $\{x_n\}$ be a sequence generated by
\[
x_0 \in C, \text{ chosen arbitrarily,}
\]
\[
y_n = \Pi_C[(1 - \alpha_n)(J + r_n A)^{-1}Jx_n],
\]
\[
x_{n+1} = \Pi_C J^{-1}(\beta_n Jy_n + \sum_{i=1}^{N} \beta_i JT_i y_n), \quad \forall n \geq 0,
\]
(3.1)
where \( \alpha_n \in (0,1) \), \( \{\beta_i\}_{i=0}^{N} \subseteq [c,d] \subset (0,1) \) and \( \{r_n\} \) a sequence of \( (0,\infty) \) satisfying: \( \lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{i=0}^{N} \beta_i = 1 \) and \( \lim_{n \to \infty} r_n = \infty \). Then \( \{x_n\} \) converges strongly to the minimum-norm element of \( F \).

Proof. Let \( p \in \Pi_F(0) \) and \( w_n := (J + r_nA)^{-1}Jx_n := Q_{T_n}x_n \). Then, since by Remark 1.3 we have that \( p \in A^{-1}(0) \), from Lemma 2.9 we get that

\[
\phi(p, w_n) = \phi(p, Q_{T_n}x_n) \leq \phi(p, x_n).
\]

Now from (3.1), Lemma 2.3 and property of \( \phi \) and (3) we get that

\[
\phi(p, y_n) = \phi(p, \Pi_C(1 - \alpha_n)w_n) \leq \phi(p, (1 - \alpha_n)w_n)
\]

\[
= \phi(p, J^{-1}(\alpha_nJ0 + (1 - \alpha_n)Jw_n))
\]

\[
= ||p||^2 - 2\langle p, \alpha_nJ0 + (1 - \alpha_n)Jw_n \rangle + ||\alpha_nJ0 + (1 - \alpha_n)Jw_n||^2
\]

\[
\leq ||p||^2 - 2\alpha_n\langle p, J0 \rangle - 2\langle 1 - \alpha_n \rangle \langle p, Jw_n \rangle
\]

\[
+ \alpha_n||J0||^2 + (1 - \alpha_n) ||Jw_n||^2
\]

\[
= \alpha_n \phi(p, 0) + (1 - \alpha_n) \phi(p, w_n)
\]

(3.2)

Moreover, from (3.1), Lemma 2.3, Lemma 2.7, relatively nonexpansiveness of \( T \) and (3.2) we get that

\[
\phi(p, x_{n+1}) = \phi(p, \Pi_C J^{-1}(\beta_0Jy_n + \sum_{i=1}^{N} \beta_iJT_iy_n))
\]

\[
\leq \phi(p, J^{-1}(\beta_0Jy_n + \sum_{i=1}^{N} \beta_iJT_iy_n))
\]

\[
= ||p||^2 - 2\langle p, \beta_0Jy_n + \sum_{i=1}^{N} \beta_iJT_iy_n \rangle + ||\beta_0Jy_n + \sum_{i=1}^{N} \beta_iJT_iy_n||^2
\]

\[
\leq ||p||^2 - 2\beta_0\langle p, Jy_n \rangle - 2\sum_{i=1}^{N} \beta_i\langle p, JT_iy_n \rangle
\]

\[
+ \beta_0||Jy_n||^2 + \sum_{i=1}^{N} \beta_i||JT_iy_n||^2 - \beta_0\beta_i g(||Jy_n - T_iy_n||)
\]

(3.3)

\[
= \beta_0 \phi(p, y_n) + \sum_{i=1}^{N} \beta_i \phi(p, T_iy_n) - \beta_0\beta_i g(||Jy_n - T_iy_n||)
\]

\[
\leq \beta_0 \phi(p, y_n) + (1 - \beta_0) \phi(p, y_n) - \beta_0\beta_i g(||Jy_n - T_iy_n||)
\]

(3.4)

\[
\leq \alpha_n \phi(p, 0) + (1 - \alpha_n) \phi(p, x_n)
\]
for each $i \in \{1, 2, \ldots, N\}$. Thus, by induction,

$$\phi(p, x_{n+1}) \leq \max\{\phi(p, 0), \phi(p, x_0)\}, \forall n \geq 0,$$

which implies that $\{x_n\}$ is bounded and hence $\{y_n\}$ and $\{w_n\}$ are bounded. Now let $z_n = (1 - \alpha_n)w_n$. Then we note that $y_n = \Pi_C z_n$. Furthermore, since $\alpha_n \to 0$ we get that

$$z_n - w_n = \alpha_n(-w_n) \to 0 \text{ as } n \to \infty. \tag{3.5}$$

Thus, using Lemma 2.3, Lemma 2.5 and property of $\phi$ we obtain that

$$\phi(p, y_n) \leq \phi(p, z_n) = V(p, Jz_n)$$

$$\leq V(p, Jz_n - \alpha_n(J0 - Jp)) - 2(z_n - p, -\alpha_n(J0 - Jp))$$

$$= \phi(p, J^{-1}(\alpha_nJp + (1 - \alpha_n)Jw_n) + 2\alpha_n(z_n - p, J0 - Jp))$$

$$\leq \alpha_n\phi(p, p) + (1 - \alpha_n)\phi(p, w_n) + 2\alpha_n(z_n - p, J0 - Jp)$$

$$= (1 - \alpha_n)(\phi(p, w_n) + 2\alpha_n(z_n - p, J0 - Jp))$$

$$\leq (1 - \alpha_n)\phi(p, x_n) + 2\alpha_n(z_n - p, J0 - Jp). \tag{3.6}$$

Furthermore, from (3.3) and (3.6) we have that

$$\phi(p, x_{n+1}) \leq (1 - \alpha_n)\phi(p, x_n) + 2\alpha_n(z_n - p, J0 - Jp)$$

$$- \beta_0\beta_1 g(||Jy_n - JT_i y_n||) \tag{3.7}$$

$$\leq (1 - \alpha_n)\phi(p, x_n) + 2\alpha_n(z_n - p, J0 - Jp). \tag{3.8}$$

Now, we consider two cases:

**Case 1.** Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\phi(p, x_n)\}$ is non-increasing. In this situation, $\{\phi(p, x_n)\}$ is convergent. Then from (3.7) we have that

$$\beta_0\beta_1 g(||Jy_n - JT_i y_n||) \to 0. \tag{3.9}$$

which implies, by the property of $g$ that

$$Jy_n - JT_i y_n \to 0 \text{ as } n \to \infty, \tag{3.10}$$

and hence, since $J^{-1}$ is uniformly continuous on bounded sets we obtain that

$$y_n - T_i y_n \to 0 \text{ as } n \to \infty \tag{3.11}$$

for each $i \in \{1, 2, \ldots, N\}$.

Furthermore, Lemma 2.3, property of $\phi$ and the fact that $\alpha_n \to 0$ as $n \to \infty$, imply that

$$\phi(y_n, \Pi_C z_n) \leq \phi(y_n, z_n)$$

$$= \phi(y_n, J^{-1}(\alpha_nJ0 + (1 - \alpha_n)Jw_n))$$

$$\leq \alpha_n\phi(y_n, 0) + (1 - \alpha_n)\phi(w_n, w_n)$$

$$\leq \alpha_n\phi(y_n, 0) + (1 - \alpha_n)\phi(w_n, w_n) \to 0 \text{ as } n \to \infty, \tag{3.12}$$
and hence
\[(3.13)\quad y_n - z_n \to 0 \text{ as } n \to \infty.\]

Since \(\{z_n\}\) is bounded and \(E\) is reflexive, we choose a subsequence \(\{z_{n_i}\}\) of \(\{z_n\}\) such that \(z_{n_i} \to z\) and \(\limsup_{n \to \infty} (z_{n_i} - p, Jw - Jp) = \lim_{i \to \infty} (z_{n_i} - p, J0 - Jp)\).

Then, from (3.5) and (3.13) we get that
\[(3.14)\quad y_{n_i} \to z, \quad w_{n_i} \to z \text{ as } i \to \infty.\]

Thus, since \(T\) satisfies condition (R3) we obtain from (3.11) that \(z \in F(T_i)\) for each \(i \in \{1, 2, \ldots, N\}\) and hence \(z \in \cap_{i=1}^N F(T_i)\). Furthermore, since \(w_{n_i} \to z\), Lemma 2.8 implies that \(z \in A^{-1}(0)\) and hence by Remark 1.3 we have that \(z \in VI(C, A)\).

Then from the above discussions we obtain that \(z \in F = \cap_{i=1}^N F(T_i) \cap VI(C, A)\). Therefore, by Lemma 2.6, we immediately obtain that
\[
\limsup_{n \to \infty} (z_n - p, J0 - Jp) = \lim_{i \to \infty} (z_{n_i} - p, J0 - Jp) = (z - p, J0 - Jp) \leq 0.
\]

It follows from Lemma 2.10 and (3.8) that \(\phi(p, x_n) \to 0\) as \(n \to \infty\). Consequently, by Lemma 2.4 we get that \(x_n \to p\).

**Case 2.** Suppose that there exists a subsequence \(\{n_i\}\) of \(\{n\}\) such that
\[
\phi(p, x_{n_i}) < \phi(p, x_{n_{i+1}})
\]
for all \(i \in \mathbb{N}\). Then, by Lemma 2.11, there exist a nondecreasing sequence \(\{m_k\} \subset \mathbb{N}\) such that \(m_k \to \infty\), \(\phi(p, x_{m_k}) \leq \phi(p, x_{m_{k+1}})\) and \(\phi(p, x_k) \leq \phi(p, x_{m_{k+1}})\) for all \(k \in \mathbb{N}\). Then from (3.7) and the fact that \(\alpha_n \to 0\) we obtain that
\[
g(||Jy_{m_k} - JT_i y_{m_k}||) \to 0, \text{ as } k \to \infty
\]
for each \(i \in \{1, 2, \ldots, N\}\). Thus, using the same proof as in Case 1, we obtain that \(y_{m_k} - T_i y_{m_k} \to 0, y_{m_k} - z_{m_k} \to 0,\) as \(k \to \infty,\) and hence we obtain that
\[(3.15)\quad \lim_{k \to \infty} \sup_{k \to \infty} (z_{m_k} - p, J0 - Jp) \leq 0.
\]

Then from (3.8) we have that
\[(3.16)\quad \phi(p, x_{m_{k+1}}) \leq (1 - \alpha_{m_k}) \phi(p, x_{m_k}) + 2\alpha_{m_k} (z_{m_k} - p, J0 - Jp).
\]

Since \(\phi(p, x_{m_k}) \leq \phi(p, x_{m_{k+1}})\), (3.16) implies that
\[
\alpha_{m_k} \phi(p, x_{m_k}) \leq \phi(p, x_{m_k}) - \phi(p, x_{m_{k+1}}) + 2\alpha_{m_k} (z_{m_k} - p, J0 - Jp)
\]
\[
\leq 2\alpha_{m_k} (z_{m_k} - p, J0 - Jp).
\]

In particular, since \(\alpha_{m_k} > 0\), we get
\[
\phi(p, x_{m_k}) \leq 2(z_{m_k} - p, J0 - Jp).
\]

Then, from (3.15) we obtain that \(\phi(p, x_{m_k}) \to 0\) as \(k \to \infty\). This together with \(3.16\) gives \(\phi(p, x_{m_{k+1}}) \to 0\) as \(k \to \infty\). But \(\phi(p, x_k) \leq \phi(p, x_{m_{k+1}})\) for all \(k \in \mathbb{N}\), thus we obtain that \(x_k \to p\). Therefore, from the above two cases, we
can conclude that \( \{x_n\} \) converges strongly to \( p \) which is the minimum-norm element of \( F \) and the proof is complete. \( \square \)

If in Theorem 3.1, we assume that \( N = 1 \), then we get the following corollary.

**Corollary 3.2.** Let \( C \) be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space \( E \). Let \( A : C \to E^* \) be a continuous monotone mapping satisfying (2.2) and \( ||Ax|| \leq ||Ax-Ap||, \forall x \in C \) and \( p \in VI(C, A) \). Let \( T : C \to C \) be a relatively nonexpansive mapping. Assume that \( F := F(T) \cap VI(C, A) \) is nonempty. Let \( \{x_n\} \) be a sequence generated by

\[
\begin{align*}
\{x_n\} & \text{ chosen arbitrarily,} \\
y_n & = \Pi_C[(1-\alpha_n)(J + r_n A)^{-1}Jx_n], \\
x_{n+1} & = \Pi_C[J^{-1}(\beta Jy_n + (1-\beta)JT_y), \forall n \geq 0, \\
\end{align*}
\]

where \( \alpha_n \in (0, 1), \beta \in [c, d] \subset (0, 1) \) and \( \{r_n\} \) a sequence of \((0, \infty)\) satisfying: \( \lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \) and \( \lim_{n \to \infty} r_n = \infty \). Then \( \{x_n\} \) converges strongly to the minimum-norm element of \( F \).

If in Theorem 3.1, we assume that \( T_i \equiv I \), for \( i = 1, 2, \ldots, N \), identity map on \( C \), then we get the following corollary.

**Corollary 3.3.** Let \( C \) be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space \( E \). Let \( A : C \to E^* \) be a continuous monotone mapping satisfying (2.2) and \( ||Ax|| \leq ||Ax-Ap||, \forall x \in C \) and \( p \in VI(C, A) \). Assume that \( VI(C, A) \) is nonempty. Let \( \{x_n\} \) be a sequence generated by

\[
\begin{align*}
\{x_n\} & \text{ chosen arbitrarily,} \\
y_n & = \Pi_C[(1-\alpha_n)(J + r_n A)^{-1}Jx_n], \\
x_{n+1} & = \Pi_C[J^{-1}(\beta Jy_n + (1-\beta)JT_y), \forall n \geq 0, \\
\end{align*}
\]

where \( \alpha_n \in (0, 1), \) and \( \{r_n\} \) a sequence of \((0, \infty)\) satisfying: \( \lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \) and \( \lim_{n \to \infty} r_n = \infty \). Then \( \{x_n\} \) converges strongly to the minimum-norm element of \( VI(C, A) \).

We make use of Remark 1.3 to restate the above theorem.

**Theorem 3.4.** Let \( C \) be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space \( E \). Let \( A : C \to E^* \) be a continuous monotone mapping satisfying (2.2) and \( ||Ax|| \leq ||Ax-Ap||, \forall x \in C \) and \( p \in VI(C, A) \). Let \( T_i : C \to C, i = 1, 2, \ldots, N, \) be a finite family of relatively nonexpansive mappings. Assume that \( F := \cap_{i=1}^{N} F(T_i) \cap A^{-1}(0) \) is nonempty. Let \( \{x_n\} \) be a sequence generated by

\[
\begin{align*}
\{x_n\} & \text{ chosen arbitrarily,} \\
y_n & = \Pi_C[(1-\alpha_n)(J + r_n A)^{-1}Jx_n], \\
x_{n+1} & = \Pi_C[J^{-1}(\beta Jy_n + \sum_{i=1}^{N} \beta_i JT_i y_n), \forall n \geq 0. \\
\end{align*}
\]
where \( \alpha_n \in (0, 1), \{ \beta_i \}_{i=0}^N \subset [c, d] \subset (0, 1) \) and \( \{ r_n \} \) a sequence of \((0, \infty)\) satisfying: \( \lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^\infty \alpha_n = \infty, \sum_{i=0}^N \beta_i = 1 \), and \( \lim_{n \to \infty} r_n = \infty \). Then \( \{ x_n \} \) converges strongly to the minimum-norm element of \( F \).

A monotone mapping \( A \subseteq E \times E^* \) is said to be maximal monotone if its graph is not properly contained in the graph of any monotone mapping. We know that if \( A \) is maximal monotone mapping, then \( A^{-1}(0) \) is closed and convex (see [17] for more details). The following lemma is well-known.

**Lemma 3.5 ([15])**. Let \( E \) be a smooth, strictly convex and reflexive Banach space, let \( C \) be a nonempty, closed and convex subset of \( E \) and let \( A \subseteq E \times E^* \) be a monotone mapping. Then \( A \) is maximal if and only if \( R(J + rA) = E^* \) for all \( r > 0 \).

We note from the above lemma that if \( A \) is maximal, then it satisfies condition (2.2) and hence we have the following corollary.

**Corollary 3.6.** Let \( C \) be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space \( E \). Let \( A : C \rightarrow E^* \) be a maximal monotone mapping. Let \( T_i : C \rightarrow C, i = 1, 2, \ldots, N, \) be a finite family of relatively nonexpansive mappings. Assume that \( F := \cap_{i=1}^N F(T_i) \cap A^{-1}(0) \) is nonempty. Let \( \{ x_n \} \) be a sequence generated by

\[
\begin{align*}
\{ x_0 \in C, \text{ chosen arbitrarily,} \\
y_n &= \Pi_C \left[ (1 - \alpha_n)(J + r_nA)^{-1}Jx_n \right], \\
x_{n+1} &= \Pi_C J^{-1}(\beta_nJy_n + \sum_{i=0}^N \beta_iJT_iy_n), \quad \forall n \geq 0,
\end{align*}
\]

where \( \alpha_n \in (0, 1), \{ \beta_i \}_{i=0}^N \subset [c, d] \subset (0, 1) \) and \( \{ r_n \} \) a sequence of \((0, \infty)\) satisfying: \( \lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^\infty \alpha_n = \infty, \sum_{i=0}^N \beta_i = 1 \), and \( \lim_{n \to \infty} r_n = \infty \). Then \( \{ x_n \} \) converges strongly to the minimum-norm element of \( F \).

If in Corollary 3.6, we assume that \( T_i \equiv I, \) for \( i = 1, 2, \ldots, N, \) identity map on \( C \), then we get the following corollary.

**Corollary 3.7.** Let \( C \) be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space \( E \). Let \( A : C \rightarrow E^* \) be a maximal monotone mapping. Assume that \( A^{-1}(0) \) is nonempty. Let \( \{ x_n \} \) be a sequence generated by

\[
\begin{align*}
\{ x_0 \in C, \text{ chosen arbitrarily,} \\
x_{n+1} &= \Pi_C \left[ (1 - \alpha_n)(J + r_nA)^{-1}Jx_n \right], \quad \forall n \geq 0,
\end{align*}
\]

where \( \alpha_n \in (0, 1) \) and \( \{ r_n \} \) a sequence of \((0, \infty)\) satisfying: \( \lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^\infty \alpha_n = \infty \) and \( \lim_{n \to \infty} r_n = \infty \). Then \( \{ x_n \} \) converges strongly to the minimum-norm element of \( A^{-1}(0) \).

If in Theorem 3.1, we assume that \( A \equiv 0 \), then the assumption that \( E \) be 2-uniformly convex may be relaxed. In fact, we have the following corollary.
Corollary 3.8. Let $C$ be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space. Let $T_i : C \to C$, $i = 1, 2, \ldots, N$, be a finite family of relatively nonexpansive mappings. Assume that $F := \cap_{i=1}^N F(T_i)$ is nonempty. Let $\{x_n\}$ be a sequence generated by

$$
\begin{align*}
    x_0 &\in C, \text{ chosen arbitrarily}, \\
    y_n &= \Pi_C[(1 - \alpha_n)x_n], \\
    x_{n+1} &= \Pi_C[J^{-1}(\beta_0Jy_n + \sum_{i=1}^N \beta_iJT_iy_n)), \forall n \geq 0,
\end{align*}
$$

(3.22)

where $\alpha_n \in (0, 1)$ and $\{\beta_i\}_{i=0}^N \subset [c, d] \subset (0, 1)$ satisfying: $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$, $\sum_{i=0}^N \beta_i = 1$. Then $\{x_n\}$ converges strongly to the minimum-norm element of $F$.

If in Corollary 3.8, we assume that $N = 1$, then we get the following corollary.

Corollary 3.9. Let $C$ be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space. Let $T : C \to C$, be a relatively nonexpansive mappings. Assume that $F(T)$ is nonempty. Let $\{x_n\}$ be a sequence generated by

$$
\begin{align*}
    x_0 &\in C, \text{ chosen arbitrarily}, \\
    y_n &= \Pi_C[(1 - \alpha_n)x_n], \\
    x_{n+1} &= \Pi_C[J^{-1}(\beta Jy_n + (1 - \beta)JT_0y_n)), \forall n \geq 0,
\end{align*}
$$

(3.23)

where $\beta \in (0, 1)$ and $\alpha_n \in (0, 1)$ satisfying: $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to the minimum-norm element of $F(T)$.

If $E = H$, a real Hilbert space, then $E$ is uniformly convex and uniformly smooth real Banach space. In this case, $J = I$, identity map on $H$ and $\Pi_C = \Pi_C$, projection mapping from $H$ onto $C$. Thus, the following corollaries hold.

Corollary 3.10. Let $C$ be a nonempty, closed and convex subset of a real Hilbert space $H$. Let $A : C \to H$ be a continuous monotone mapping satisfying (2.2) and $||Ax|| \leq ||Ax - Ap||$, $\forall x \in C$ and $p \in VI(C, A)$. Let $T_i : C \to C$, $i = 1, 2, \ldots, N$, be a finite family of nonexpansive mappings. Assume that $F := \cap_{i=1}^N F(T_i) \cap VI(C, A)$ is nonempty. Let $\{x_n\}$ be a sequence generated by

$$
\begin{align*}
    x_0 &\in C, \text{ chosen arbitrarily}, \\
    y_n &= \Pi_C[(1 - \alpha_n)(I + r_nA)^{-1}x_n], \\
    x_{n+1} &= \Pi_C(\beta_0y_n + \sum_{i=1}^N \beta_iT_iy_n), \forall n \geq 0,
\end{align*}
$$

(3.24)

where $\alpha_n \in (0, 1)$, $\{\beta_i\}_{i=0}^N \subset [c, d] \subset (0, 1)$ and $\{r_n\}$ a sequence of $(0, \infty)$ satisfying: $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$, $\sum_{i=0}^N \beta_i = 1$, and $\lim_{n \to \infty} r_n = \infty$. Then $\{x_n\}$ converges strongly to the minimum-norm element of $F$.

Corollary 3.11. Let $C$ be a nonempty, closed and convex subset of a real Hilbert space $H$. Let $A : C \to H$ be a maximal monotone mapping. Let $T_i : C \to C$, $i = 1, 2, \ldots, N$, be a finite family of nonexpansive mappings.
Assume that $F := \cap_{i=1}^{N} F(T_i) \cap A^{-1}(0)$ is nonempty. Let $\{x_n\}$ be a sequence generated by

$$
\begin{align*}
  x_0 &\in C, \text{ chosen arbitrarily,} \\
  y_n &= P_C\left[(1 - \alpha_n)(I + r_nA)^{-1}x_n\right], \\
  x_{n+1} &= P_C(\beta_0y_n + \sum_{i=1}^{N} \beta_i T_i y_n), \quad \forall n \geq 0,
\end{align*}
$$

where $\alpha_n \in (0, 1)$, $\{\beta_i\}_{i=0}^{N} \subset [c, d] \subset (0, 1)$ and $\{r_n\}$ a sequence of $(0, \infty)$ satisfying $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{i=0}^{N} \beta_i = 1$, and $\lim_{n \to \infty} r_n = \infty$. Then $\{x_n\}$ converges strongly to the minimum-norm element of $F$.

4. Application

In this section, we study the problem of finding a minimizer of a continuously Fréchet differentiable convex functional which has minimum-norm in Banach spaces. The following is deduced from Corollary 3.7.

**Theorem 4.1.** Let $E$ be a uniformly convex and uniformly smooth real Banach space. Let $f$ be a continuously Fréchet differentiable convex functional on $E$ and $\nabla f$ is maximal monotone with $F := (\nabla f)^{-1}(0) = \{z \in E : f(z) = \min_{y \in E} f(y)\} \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by

$$
\begin{align*}
  x_0 &\in C \text{ chosen arbitrarily,} \\
  x_{n+1} &= \Pi_C((1 - \alpha_n)(J + r_n \nabla f)^{-1}Jx_n),
\end{align*}
$$

where $\alpha_n \in (0, 1)$ and $\{r_n\}$ a sequence of $(0, \infty)$ satisfying $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n \to \infty} r_n = \infty$. Then $\{x_n\}$ converges strongly to the minimum-norm element of $F$.

**Remark 4.2.** Our theorems improve most of the results that have been proved for these important class of non-linear mappings. In particular, Corollary 3.3 improves Theorem 3.1 of [5] and hence results of [6] in the sense that our convergence is strong in a more general class of continuous monotone mappings in a more general Banach spaces provided that $A$ satisfies (2.2).

Moreover, Corollary 3.7 improves Theorem 3.3 of [4] in the sense that our convergence is valid in a more general Banach spaces that does not require computations of $C_n$, $Q_n$ and $C_n \cap Q_n$ for each $n \geq 0$ provided that $A$ is maximal monotone mapping.

In addition, Corollary 3.2 improves Theorem 3.1 of [3] in the sense that our convergence is for a more general class of relatively nonexpansive and continuous monotone mappings in a more general Banach spaces provided that $A$ satisfies (2.2).

**Acknowledgements.** The authors thank the referee for his valuable comments and suggestions, which improved the presentation of this manuscript.
References


N. Shahzad
Department of Mathematics
King Abdulaziz University
P.O.B. 80203, Jeddah 21589, Saudi Arabia
E-mail address: nshahzad@kau.edu.sa

H. Zegeye
Department of Mathematics
University of Botswana
Pvt. Bag 00704, Gaborone, Botswana
E-mail address: habtuz@yahoo.com