A NOTE ON THE GENERALIZED MYERS THEOREM FOR FINSLER MANIFOLDS

BING-YE WU

Abstract. In this note we establish a generalized Myers theorem under line integral curvature bound for Finsler manifolds.

1. Introduction

The celebrated Myers theorem in global Riemannian geometry says that if a Riemannian manifold $M$ satisfies $\text{Ric}(v) \geq (n-1)a > 0$ for all unit vector $v$, then $M$ is compact and

$$\text{diam}(M) \leq \frac{\pi}{\sqrt{a}}.$$

There are many generalizations of Myers theorem (see e.g., [2, 3, 7]). In [7] the author proved the following result.

Theorem 1.1. Let $(M, g)$ be an $n$-dimensional complete Riemannian manifold. Then for any $\delta > 0$, $a > 0$, there exists $\epsilon = \epsilon(n, a, \delta)$ satisfying the following. If for any $p \in M$ and each minimal geodesic $\gamma$ emanating from $p$, the Ricci curvature satisfies

$$\int_{\gamma} \max\{(n-1)a - \text{Ric}(\gamma'(t)), 0\} dt \leq \epsilon(n, a, \delta),$$

then $M$ is compact with

$$\text{diam}(M) \leq \frac{\pi}{\sqrt{a}} + \delta.$$

Myers theorem has also been generalized to Finsler manifolds [1]. In this note we shall prove the following result which generalizes Theorem 1.1.

Theorem 1.2. Let $(M, F)$ be an $n$-dimensional forward complete Finsler manifold. If there is $\Lambda > 0$ such that for any $p \in M$ and each minimal geodesic $\gamma$
emanating from p, the Ricci curvature satisfies
\[
\int_{\gamma} \max\{(n-1)a - \text{Ric}(\gamma'(t)), 0\} \, dt \leq \Lambda,
\]
then M is compact with
\[
\text{diam}(M) \leq \frac{\pi}{\sqrt{a}} + \frac{\Lambda}{(n-1)a}.
\]

2. Finsler geometry

In this section we briefly recall some fundamental materials of Finsler geometry, and for details one is referred to see [1, 4, 5, 6]. Let \((M, F)\) be a Finsler \(n\)-manifold with Finsler metric \(F : TM \to [0, \infty)\). Let \((x, y) = (x^i, y^i)\) be local coordinates on \(TM\), and \(\pi : TM \setminus 0 \to M\) the natural projection. Unlike in the Riemannian case, most Finsler quantities are functions of \(TM\) rather than \(M\).

The fundamental tensor \(g_{ij}\) is defined by
\[
g_{ij}(x, y) := \frac{1}{2} \frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j}.
\]
Let \(R^i_{\ jkl}\) be the first Chern curvature tensor, and \(R^i_{\ jkl} := g_{js} R_{s \ jkl}^{\ *}\). Write \(g_y = g_{ij}(x, y) dx^i \otimes dx^j, R_y = R_{ijkl}(x, y) dx^i \otimes dx^j \otimes dx^k \otimes dx^l\). For a tangent plane \(P \subset T_x M\), let
\[
K(P, y) = K(y; u) := \frac{R_y(y, u, u, y)}{g_y(y, u) g_y(u, u) - [g_y(y, u)]^2},
\]
where \(y, u \in P\) are tangent vectors such that \(P = \text{span}\{y, u\}\). We call \(K(P, y)\) the flag curvature of \(P\) with flag pole \(y\). Let
\[
\text{Ric}(y) = \sum_i K(y; e_i),
\]
where \(\{e_1, \ldots, e_n\}\) is a \(g_y\)-orthogonal basis for the corresponding tangent space. We call \(\text{Ric}(y)\) the Ricci curvature of \(y\).

Let \(V = \psi \partial / \partial x^i\) be a non-vanishing vector field on an open subset \(U \subset M\). One can introduce a Riemannian metric \(\tilde{g} = g_V\) and a linear connection \(\nabla^V\) (called Chern connection) on the tangent bundle over \(U\) as follows:
\[
\nabla^V_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} := \Gamma^k_{ij}(x, v) \frac{\partial}{\partial x^k},
\]
where \(\Gamma^k_{ij}(x, v)\) are the Chern connection coefficients.

The Legendre transformation \(l : TM \to T^* M\) is defined by
\[
l(Y) = \begin{cases} g_Y(Y, \cdot), & Y \neq 0 \\ 0, & Y = 0. \end{cases}
\]
Now let \(f : M \to \mathbb{R}\) be a smooth function on \(M\). The gradient of \(f\) is defined by \(\nabla f = l^{-1}(df)\). Thus we have
\[
df(X) = g_V(f, X), \quad X \in TM.
\]
Let \( U = \{ x \in M : \nabla f \mid_x \neq 0 \} \). We define the Hessian \( H(f) \) of \( f \) on \( U \) as follows:
\[
H(f)(X, Y) := XY(f) - \nabla_X Y(f), \quad \forall X, Y \in TM \mid_U.
\]
It is known that \( H(f) \) is symmetric, and it can be rewritten as (see [6])
\[
H(f)(X, Y) = g_{\nabla X} (\nabla_{\nabla X} f, Y).
\]
It should be noted that the notion of Hessian defined here is different from that in [4]. In that case \( H(f) \) is in fact defined by
\[
H(f)(X, X) := X \cdot X \cdot f - \nabla_X X(f), \quad \forall X \in TM \mid U.
\]
and there is no definition for \( H(f)(X, Y) \) if \( X \neq Y \). The advantage of our definition is that \( H(f) \) is a symmetric bilinear form and we can treat it by using the theory of symmetric matrix.

For any fixed \( p \in M \) let \( r = d_F(p, \cdot) \) be the distance function from \( p \) induced by Finsler metric \( F \), and \((r, \theta)\) the polar coordinates on \( M \setminus C(p) \), where \( C(p) \) is the cut loci of \( p \). The following lemma is crucial to prove Theorem 1.2.

**Lemma 2.1.** Let \( h = h(r, \theta) = \text{trace}_{g_{e_r}} H(r) \). Then \( \lim_{r \to 0^+} h = +\infty \), and
\[
\frac{dh}{dr} \leq -\text{Ric}(\nabla r) - \frac{h^2}{n-1}.
\]

**Proof.** Let \( E_1, \ldots, E_n \) be the local \( g_{e_r} \)-orthonormal frame fields on \( M \setminus C(p) \). We have the following equality where \( r \) is smooth (see (5.1) of [6]):
\[
\frac{d}{dr} \text{trace}_{g_{e_r}} H(r) = -\text{Ric}(\nabla r) - \sum_{i,j} (H(r)(E_i, E_j))^2.
\]
Note that \( \nabla r \) is a geodesic field, and thus \( H(r)(\nabla r, \cdot) = 0 \), which together with above equality and Schwartz inequality we clearly have the desired inequality. On the other hand, for sufficiently small \( \epsilon \) let \( b \) be the upper bound of flag curvature on \( B_p(\epsilon) \), then by Hessian comparison theorem [6] it follows that
\[
h(r, \theta) \geq (n-1)ct_b(r) = \begin{cases} (n-1)\sqrt{b} \cdot \cotan(\sqrt{br}), & b > 0 \\ \frac{n-1}{b}, & b = 0 \\ (n-1)\sqrt{-b} \cdot \cotanh(\sqrt{-br}), & b < 0 \end{cases}, \quad \forall r < \epsilon,
\]
and consequently, \( \lim_{r \to 0^+} h = +\infty \).

**3. Proof of Theorem 1.2**

Now let us complete the proof of Theorem 1.2. For any fixed \( p, q \in M \) let \( \gamma : [0, L] \to M \) be the minimal unit-speeded geodesic from \( p \) to \( q \) with \( L = r(q) = d_F(p, q) \). Let \( h = h(r, \theta) \) be defined by Lemma 2.1, and consider \( f = f(t) := h(\gamma(t)) \), then \( f \) is smooth on \((0, L)\). By Lemma 2.1 one has
\[
f'(t) \leq -\text{Ric}(\gamma'(t)) - \frac{f(t)^2}{n-1}.
\]
and consequently,

\[
(\arccot \left( \frac{f}{(n-1)\sqrt{a}} \right))' = -\frac{1}{(n-1)\sqrt{a}} \frac{f'}{1 + \frac{f^2}{(n-1)^2 a}}
\]

(1)

\[
\geq \frac{\text{Ric}(\gamma') + \frac{f^2}{\pi - 1}}{(1 + \frac{f^2}{(n-1)^2 a})(n-1)\sqrt{a}}
\]

\[
= \frac{\text{Ric}(\gamma') - (n-1)a + (n-1)a \left(1 + \frac{f^2}{(n-1)^2 a}\right)}{(1 + \frac{f^2}{(n-1)^2 a})(n-1)\sqrt{a}}
\]

\[
\geq -\frac{1}{(n-1)\sqrt{a}} \max\{(n-1)a - \text{Ric}(\gamma'), 0\} + \sqrt{a}.
\]

For any small \(\varepsilon > 0\) integrating (1) on \((\varepsilon, L-\varepsilon)\) we get

\[
\pi - \arccot \left( \frac{f(\varepsilon)}{(n-1)\sqrt{a}} \right) > \arccot \left( \frac{f(L-\varepsilon)}{(n-1)\sqrt{a}} \right) - \arccot \left( \frac{f(\varepsilon)}{(n-1)\sqrt{a}} \right)
\]

\[
\geq -\frac{1}{(n-1)\sqrt{a}} \int_{\varepsilon}^{L-\varepsilon} \max\{(n-1)a - \text{Ric}(\gamma'(t)), 0\} dt + (L-2\varepsilon)\sqrt{a}.
\]

On the other hand, \(\lim_{t \to +0} f(t) = +\infty\) by Lemma 2.1, and thus

\[
\lim_{t \to +0} \arccot \left( \frac{f(t)}{(n-1)\sqrt{a}} \right) = 0.
\]

Now let \(\varepsilon \to +0\) in (2) it follows that

\[
\pi \geq -\frac{1}{(n-1)\sqrt{a}} \int_{0}^{L} \max\{(n-1)a - \text{Ric}(\gamma'(t)), 0\} dt + L\sqrt{a}
\]

\[
\geq -\frac{\Lambda}{(n-1)\sqrt{a}} + L\sqrt{a},
\]

and consequently,

\[
L \leq \frac{\pi}{\sqrt{a}} + \frac{\Lambda}{(n-1)a}.
\]

So we complete the proof.

References


Department of Mathematics
Minjiang University
Fuzhou 350108, Fujian, P. R. China
E-mail address: bingyewu@yahoo.cn