CHOW STABILITY OF CANONICAL GENUS 4 CURVES

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Abstract. In this paper, we give sufficient conditions on a canonical genus 4 curve for it to be Chow (semi)stable.

1. Introduction

A Deligne-Mumford stable curve is a complete connected curve $C$ having ample dualising sheaf $\omega_C$ and admitting only nodes as singularities. An $n$-canonical curve $C \subset \mathbb{P}^N$ is a Deligne-Mumford stable curve of arithmetic genus $g$ embedded by the complete linear system $|\omega_C^{\otimes n}|$ where $N = (2n-1)(g-1) - 1$ if $n \geq 2$, and $N = g - 1$ if $n = 1$.

Let $\text{Chow}_{g,n}$ be the closure of the locus of the Chow forms of $n$-canonical curves of arithmetic genus $g$ in the Chow variety of algebraic cycles of dimension 1 and degree $2g-2$ in $\mathbb{P}^N$. The natural action of $\text{SL}_{N+1}$ on $\mathbb{P}^N$ induces an action on $\text{Chow}_{g,n}$. Denote the corresponding GIT (Geometric Invariant Theory) quotient space by $\text{Chow}_{g,n} / \text{SL}_{N+1}$. To understand this quotient space as a parameter space of curves with some geometric properties, we need to find Chow stability conditions.

Mumford showed that, for $n \geq 5$ and $g \geq 2$, the Chow stable curves are precisely Deligne-Mumford stable curves and there is no strictly Chow semistable curve (cf. [14]). This implies that the quotient space is precisely the moduli space of Deligne-Mumford stable curves $\overline{M}_g$.

The cases when $n = 3$ and $g \geq 3$ were concerned by Schubert in [16]. He proved that a 3-canonical curve of genus $g \geq 3$ is Chow stable if and only if it is pseudo-stable and also showed that there is no strictly Chow semistable curve, and thus the quotient space is the moduli space of pseudo-stable curves $\overline{M}_g^{\text{ps}}$.

A pseudo-stable curve is a complete connected curve $C$ satisfying the following properties.

- $\omega_C$ is ample,
- it admits at worst nodes and ordinary cusps as singularities, and

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it has no elliptic components meeting the rest at one point.

Hyeon and Lee proved that, when $n = 3$ and $g = 2$, the pseudo-stable curves are indeed Chow semistable and completely classified the strictly Chow semistable points in [10]. They also concerned the case $n = 2$ and $g = 3$. Hassett and Hyeon studied for the case when $n = 2$ and $g \geq 4$ in [8] and the cases when $n = 4$ and general $g$ were studied by Hyeon and Morrison in [12].

The purpose of this paper is to study the cases when $n = 1$ and $g = 4$. More precisely, we want to give sufficient conditions on a canonical genus 4 curve for it to be Chow stable or semistable. To do this, we use the Hilbert-Mumford criterion (cf. Theorem 2.2). Our main results are presented in Section 3.2. We show that any irreducible curve in Chow$_{4,1}$ with mild singularities is Chow stable (cf. Theorem 3.8). For reducible curves, we prove that a general curve in Chow$_{4,1}$ with two irreducible components is Chow stable except when it is a union of two elliptic curves meeting at three points (cf. Theorems 3.10 and 3.11).

After appearing the preliminary version of this paper, Casalaina-Martin, Jensen, and Laza (cf. [2], Theorem 3.1) classified Chow stable and semistable points in Chow$_{4,1}$ by using the GIT analysis for cubic threefolds. Our results are partial but we make a direct computation of the stability conditions on Chow$_{4,1}$.

Throughout this paper, we use the following notations and conventions.

- We work over an algebraically closed field $k$ of characteristic zero.
- A curve is a connected, complete scheme of pure dimension 1.
- For a curve $C$, the genus $g(C)$ of $C$ is its arithmetic genus and we write $\omega_C$ for its dualising sheaf.
- We say that a point $p \in C$ is a singular point of type $A_n$ if

$$\mathcal{O}_{C,p} \simeq k[[x, y]]/(y^2 - x^{n+1}).$$

In particular, a node (resp. ordinary cusp) is a singular point of type $A_1$ (resp. $A_2$).

- For a polynomial $P(m)$ of degree $n$ in $m$, we denote by n.l.c.$P(m)$ for the coefficient of $\frac{1}{n!}m^n$ in $P(m)$.

2. Chow stability and canonical embedding

In this section, we review some basic facts for Chow stability.

2.1. Chow stability

A weighted flag $F$ of $\mathbb{P}^n$ consists of a choice of coordinates $X_0, \ldots, X_n$ of $\mathbb{P}^n$ and a sequence of integers $r_0 \geq \cdots \geq r_n = 0$.

Let $F$ be a weighted flag of $\mathbb{P}^n$ as above and $X$ be a variety in $\mathbb{P}^n$ of dimension $r$. Let $\alpha : \tilde{X} \to X$ be a proper birational morphism. Let us define an ideal sheaf $\mathcal{I}(X)$ of $\mathcal{O}_{\tilde{X} \times \mathbb{A}^1}$ by

$$\mathcal{I}(X) \cdot [\alpha^*\mathcal{O}_X(1) \otimes \mathcal{O}_{\mathbb{A}^1}] = \text{the subsheaf generated by } t^i X_i, \ i = 1, \ldots, n.$$
It is well known that \( \chi(\mathcal{O}_{\tilde{X} \times \mathbb{A}^1}(m)/\mathcal{I}(X)^m \mathcal{O}_{\tilde{X} \times \mathbb{A}^1}(m)) \) is a polynomial of degree \( r + 1 \) for \( m \gg 0 \) (cf. [14], Proposition 2.1). Define

\[
e_F(X) := \text{n.l.c.} \chi(\mathcal{O}_{\tilde{X} \times \mathbb{A}^1}(m)/\mathcal{I}(X)^m \mathcal{O}_{\tilde{X} \times \mathbb{A}^1}(m)).
\]

Lemma 5.6 in [14] shows that \( e_F(X) \) does not depend on \( \alpha \).

For a Chow cycle \( X = \sum a_i Y_i \) where \( Y_i \) are subvarieties of \( \mathbb{P}^n \) of dimension \( r \) and \( a_i \) are nonnegative integer, define

\[
e_F(X) := \sum a_i e_F(Y_i).
\]

**Definition 2.1.** The natural action of \( SL_{n+1} \) on \( \mathbb{P}^n \) induces an action on the Chow variety of \( \mathbb{P}^n \). We say that a Chow cycle \( X \) in \( \mathbb{P}^n \) is Chow stable (resp. semistable, unstable) if its Chow from is GIT stable (resp. semistable, unstable) under the action of \( SL_{n+1} \) on the Chow variety of \( \mathbb{P}^n \).

The following theorem is the Hilbert-Mumford criterion which is very useful to determine GIT stability.

**Theorem 2.2** ([15], Theorem 2.1). Let \( X \) be a Chow cycle of dimension \( r \) in \( \mathbb{P}^n \). Then \( X \) is Chow semistable (resp. Chow stable) if and only if

\[
e_F(X) - \frac{r + 1}{n + 1} \deg X \sum r_i \leq 0 \quad \text{(resp. < 0)}
\]

for any weighted flag \( F \) of \( \mathbb{P}^n \).

### 2.2. Criteria for Chow stability

We now review some methods for determining Chow stability. For more detail, we refer to [14, 15, 16].

Let \( L_i \subset \mathbb{P}^n \) be the linear subspace defined by \( X_i = \cdots = X_n = 0 \) and let \( P_{L_i} : \mathbb{P}^n - L_i \rightarrow \mathbb{P}^{n-1} \) be the natural projection along \( L_i \).

**Definition 2.3.** Let \( C \subset \mathbb{P}^n \) be an irreducible reduced curve in \( \mathbb{P}^n \) with \( C \not\subset L_i \). Let \( \alpha_{L_i} : \tilde{C} \rightarrow \mathbb{P}^{n-1} \) be the morphism extending the composition of \( P_{L_i} \) and the normalization \( \alpha : \tilde{C} \rightarrow C \). Define

\[
\deg P_{L_i}(C) := \begin{cases} 
(\deg \alpha_{L_i})(\deg \alpha_{L_i}(\tilde{C})) & \text{if } \alpha_{L_i}(\tilde{C}) \text{ is a curve} \\
0 & \text{otherwise}
\end{cases}
\]

and

\[
e_i = e^F_i(C) := \deg C - \deg P_{L_i}(C).
\]

For a Chow cycle \( C = \sum a_j C_j \) where \( C_j \) is a 1 dimensional subvariety of \( \mathbb{P}^n \) and \( a_j \) are nonnegative integer, assume that \( C_j \not\subset L_i \) for all \( j \). Define

\[
e_i = e^F_i(C) := \sum a_j e^F_i(C_j).
\]

From the definition, \( e_0 = 0 \) and \( e_n = \deg C \) if \( C \not\subset L_n \).
Proposition 2.4 ([14], Corollary 4.11). Let $C \subset \mathbb{P}^n$ be a curve such that each irreducible component of $C$ does not contain in $L_n$. Then, for any sequence $0 = s_0 < \cdots < s_l = n$, it is satisfied that

$$e_F(C) \leq \sum_{i=0}^{l-1} (r_{s_i} - r_{s_{i+1}})(e_{s_i} + e_{s_{i+1}}).$$

Let $C$ be an irreducible reduced curve in $\mathbb{P}^n$ and let $\alpha : \tilde{C} \to C$ be the normalization of $C$. Pick a point $p$ in $\tilde{C}$ and let $s$ and $t$ be generators of the maximal ideals of $\mathcal{O}_{\tilde{C}, p}$ and $\mathcal{O}_C$, respectively. For the natural valuation $v_p$ on $\mathcal{O}_{\tilde{C}, p}$, set $\text{ord}_p X_i := v_p(\alpha^* X_i)$. Recall that $\mathcal{I}(C)$ be the ideal sheaf of $\mathcal{O}_{\tilde{C} \times \mathbb{A}^1}$ defined by

$$\mathcal{I}(C) \cdot [\alpha^* \mathcal{O}_{\mathbb{C}}(1) \otimes \mathcal{O}_{\mathbb{A}^1}] = \text{the subsheaf generated by } t^r \alpha^* X_i, \quad i = 1, \ldots, n.$$

For each $p \in \tilde{C}$, $\mathcal{I}(C)_{p \times \{0\}} \subset \mathcal{O}_{\tilde{C} \times \mathbb{A}^1, p \times \{0\}}$ is generated by

$$t^{r_0 s^{\text{ord}_p X_0}} t^{r_1 s^{\text{ord}_p X_1}}, \ldots, t^{r_n s^{\text{ord}_p X_n}},$$

where $s^{\text{ord}_p X_i} = 0$ if $\text{ord}_p X_i = \infty$. Let us use the notation

$$\mathcal{I}(C)_{p \times \{0\}} = (t^{r_0 s^{\text{ord}_p X_0}} t^{r_1 s^{\text{ord}_p X_1}}, \ldots, t^{r_n s^{\text{ord}_p X_n}}).$$

Definition 2.5. In the situation above, suppose that there is an $i$ with $r_i = 0$ and $C \not\subset (X_i = 0)$. For each point $p$ in $\tilde{C}$, we define

$$e_F(\tilde{C})_p := \text{n.l.c. dim}_k(\mathcal{O}_{\tilde{C} \times \mathbb{A}^1, p \times \{0\}}/\mathcal{I}(C)_{p \times \{0\}}^m).$$

Remark 2.6. In the setting of Definition 2.5, the quotient sheaf $\mathcal{O}_{\tilde{C} \times \mathbb{A}^1}/\mathcal{I}(C)$ is supported at the points over $C \cap L_n$ because $r_n = 0$. Therefore

$$e_F(C) = \text{n.l.c. dim}_k(\mathcal{O}_{\tilde{C} \times \mathbb{A}^1}(m)/\mathcal{I}(C)^m \mathcal{O}_{\tilde{C} \times \mathbb{A}^1}(m))$$

$$= \sum_{\alpha(p) \in L_n} \text{dim}_k(\mathcal{O}_{\tilde{C} \times \mathbb{A}^1, p \times \{0\}}/\mathcal{I}(C)_{p \times \{0\}}^m)$$

$$= \sum_{\alpha(p) \in L_n} e_F(\tilde{C})_p.$$

Lemma 2.7 ([16], Lemma 1.4). In the situation of Definition 2.5, set $v_i := \text{ord}_p X_i$. If $v_i + r_i \geq a$ for all $i = 0, \ldots, n$, then $e_F(\tilde{C})_p \geq a^n$.

2.3. Canonical curves

Definition 2.8. We say that a curve $C$ is honestly hyperelliptic if there is a morphism $C \to \mathbb{P}^1$ of degree 2, and is honestly non-hyperelliptic if it is not honestly hyperelliptic.

A Gorenstein curve is a curve $C$ with $\omega_C \cong \mathcal{O}_C(K_C)$ for a Cartier divisor $K_C$. A generically Gorenstein curve is a curve $C$ such that $\omega_C$ is locally isomorphic to $\mathcal{O}_C$ outside a finite set.
Theorem 2.9 ([1], Theorem 3.6). Let \( C \) be a numerically 3-connected Gorenstein curve. That is, for any generically Gorenstein strict subcurve \( D \subset C \),
\[
\deg O_D(K_C) - \deg \omega_D \geq 3.
\]
Then either \( C \) is honestly hyperelliptic or \( K_C \) is very ample.

If \( C \) is a numerically 3-connected curve admitting nodal singularities only, then Theorem 2.9 implies that any irreducible component of \( C \) has at least three intersection points with the union of the other components.

Definition 2.10. A canonical curve is a numerically 3-connected honestly non-hyperelliptic Gorenstein genus \( g \) curve \( C \subset \mathbb{P}^{g-1} \) whose embedding is given by \(|\omega_C|\).

We remark that any canonical curve \( C \subset \mathbb{P}^{g-1} \) is a nondegenerate curve of degree \( 2g - 2 \).

3. Canonical curves of genus four

From now on, \( F \) is a weighted flag of \( \mathbb{P}^3 \) associated with coordinates \( X_0, \ldots, X_3 \) and weights \( r_0 \geq \cdots \geq r_3 = 0 \), and \( L_i \) is the linear subspace of \( \mathbb{P}^3 \) defined by \( X_i = \cdots = X_3 = 0 \).

Note that any canonical genus 4 curve in \( \mathbb{P}^3 \) has degree 6. Thus applying Theorem 2.2 we get that a canonical genus 4 curve \( C \subset \mathbb{P}^3 \) is Chow stable (resp. semistable) if and only if
\[
e_F(C) < (\text{resp.} \leq) 3 \sum r_i
\]
for any weighted flag \( F \).

3.1. Upper bounds of \( e_F(C) \)

In this subsection, we gather some preliminary results which will be used to give upper bounds of \( e_F(C) \) in the next subsection.

Lemma 3.1. Let \( C \subset \mathbb{P}^3 \) be a curve of degree \( d \), and let \( e_i \) be the same as that in Definition 2.3. Assume that each irreducible component of \( C \) does not contained in \( L_i \). Then
\[
e_F(C) \leq \min\{dr_0, e_1r_0 + dr_1, e_2r_0 + dr_2, e_1r_0 + e_2r_1 + (d - e_1)r_2\}.
\]

Proof. The lemma immediately comes by applying Proposition 2.4 to the sequences \( 0 < 3, 0 < 1 < 3, 0 < 2 < 3 \) and \( 0 < 1 < 2 < 3 \). \( \square \)

Lemma 3.2. Let \( R := k[s, t] \) and \( I \) an ideal of \( R \).

(1) If \( I = (t^a, s^b) \) for integers \( a, b \geq 1 \), then \( \dim_k R/I^m = ab \).

(2) If \( I = (t^n, t^ps^q, s^b) \) for integers \( a, b, p, q \geq 1 \), then
\[
\dim_k R/I^m \leq aq + bp.
\]
Lemma 3.4. 

Let 

\[
\{ f^{n_1} s^{n_2} | n_1 + n_2 \geq m, 0 \leq r_1 < a, 0 \leq r_2 < b \}.
\]

Thus the following set of monomials 

\[
\{ s^{k+j} | 0 \leq i \leq m-1, 0 \leq j \leq a(m-i) - 1, 0 \leq k \leq b-1 \}
\]

forms a basis of \( R/I^m \). Therefore  

\[
\dim_k R/I^m = \sum_{i=0}^{m-1} a(m-i)b = ab(m^2 + m)/2,
\]

which implies (1). Similarly, (2) can be proved by describing the set of the monomials spanning \( R/I^m \). □

Lemma 3.3. Let \( C \subset \mathbb{P}^3 \) be a curve of degree \( d \), and assume that each irreducible component of \( C \) does not lie in the hyperplane \( L_3 \). Then  

\[
e_F(C) \leq ( \sum_{\alpha(p) = L_1} \text{ord}_p X_3 )r_0 + ( \sum_{\alpha(p) \notin L_2} \text{ord}_p X_3 )r_1 + ( \sum_{\alpha(p) \notin L_2} \text{ord}_p X_3 )r_2.
\]

Proof. We may assume that \( C \) is irreducible and reduced. Let \( \alpha : \tilde{C} \rightarrow C \) be the normalization of \( C \). Take a point \( p \) in \( \tilde{C} \) and set \( \nu_i = \text{ord}_p X_i \). Then  

\[
\mathcal{I}(C)_{p \times \{0\}} = (t^{r_0}s^r, t^{r_1}s^s, t^{r_2}s^t, s^{s_2}).
\]

From this, it is induced that  

\[
\mathcal{I}(C)_{p \times \{0\}} > \begin{cases} 
(t^{r_0}, s^{s_2}), & \text{for all } p \\
(t^{r_1}, s^{s_2}), & \text{if } \alpha(p) \notin L_1 \\
(t^{r_2}, s^{s_2}), & \text{if } \alpha(p) \notin L_2.
\end{cases}
\]

Applying Lemma 3.1 to these inclusions, we obtain that  

\[
e_F(\tilde{C})_p \leq \begin{cases} 
r_0\nu_3, & \text{for all } p \\
r_1\nu_3, & \text{if } \alpha(p) \notin L_1 \\
r_2\nu_3, & \text{if } \alpha(p) \notin L_2.
\end{cases}
\]

Using the equality \( e_F(C) = \sum_{p \in \tilde{C}} e_F(\tilde{C})_p \), the desired inequality can be verified. □

Lemma 3.4. Let \( C \subset \mathbb{P}^3 \) be a reduced irreducible curve of degree \( d \) and assume that \( C \subset L_3 \) and \( C \neq L_2 \). Then  

\[
e_F(C) \leq ( \sum_{\alpha(p) = L_1} \text{ord}_p X_3 )r_0 + ( \sum_{\alpha(p) \neq L_1} \text{ord}_p X_i )r_1 + dr_2.
\]

Proof. Let \( \alpha : \tilde{C} \rightarrow C \) be the normalization of \( C \). Let \( F' \) be the weighted flag of \( L_3 \cong \mathbb{P}^2 \) associated with the coordinates \( X'_0 := X_0|_{L_3}, X'_1 := X_1|_{L_3}, X'_2 := X_2|_{L_3}, \)
$X'_2 := X_2|_{L_2}$ and the weights $r'_0 = r_0 - r_2 \geq r'_1 = r_1 - r_2 \geq r'_2 = 0$. From the proof of Theorem 2.9 in [14] it is induced that 

$$e_F(C) = e_F'(C) + 2dr_2.$$ 

Take a point $p \in \tilde{C} \cap (X'_2 = 0)$ and set $v_i := \text{ord}_p \alpha^* X_i$. Then 

$$e_{F'}(\tilde{C})_p \leq \begin{cases} r'_0 v_2, & \text{for all } p \\ r'_1 v_2, & \text{if } \alpha(p) \neq L_1. \end{cases}$$

The first inequality is given by applying from Lemma 3.2 to the inclusion 

$$I(C)_{p \times \{0\}} = (t^{r'_0} s^{v_0}, t^{r'_1} s^{v_1}, s^{v_2}) \supset (t^{r'_1}, s^{v_2}).$$

If $\alpha(p) \neq L_1$, then $v_1 = 0$, and hence we get the next inclusion 

$$I(C)_{p \times \{0\}} \supset (t^{r'_1}, s^{v_2})$$

which implies the second inequality by Lemma 3.2. From the equality $e_{F'}(C) = \sum_{p \in \tilde{C}} e_{F'}(\tilde{C})_p$, we get the lemma. \hfill \Box

Lemma 3.5. If $C \subset \mathbb{P}^3$ is equal to $L_2$, then $e_F(C) = r_0 + r_1$.

Proof. The coordinate ring of $C$ is $R = \mathbb{k}[X_0, X_1]$. Let $I = (X_0 t^{r_0}, X_1 t^{r_1})$. Applying Lemma 1.3 in [16], we get that 

$$e_F(C) = \text{n.l.c.dim}_k(R/I)_m.$$ 

Since $I^m$ is generated by 

$$\left\{ t^{r_0 i + r_1 j} X_0^i X_1^j \mid i + j = m \right\},$$

we get that 

$$\text{dim}_k(R/I^m)_m = \sum_{i+j=m} r_0 i + r_1 j = \sum_{i=0}^m r_0 i + r_1 (m - i)$$

$$= \sum_{i=0}^m i(r_0 - r_1) + mr_1$$

$$= \frac{m(m+1)}{2} (r_0 - r_1) + m(m+1)r_1$$

$$= \frac{r_0 + r_1}{2} (m^2 + m),$$

and thus the required equality is obtained. \hfill \Box

3.2. Main results

Next proposition says that Chow stable curves admit at worst double points.

Proposition 3.6. Let $C \subset \mathbb{P}^3$ be a curve of degree 6. If $C$ admits a singular point of multiplicity $\geq 3$, then it is not Chow stable. Furthermore, if $C$ has a point of multiplicity $\geq 4$, then it is not Chow semistable.
Proof. Let $p$ be a point of $C$ with multiplicity bigger than or equal to 3. Take coordinates $X_0, \ldots, X_3$ so that $X_1, X_2,$ and $X_3$ vanish at $p$, and let $r_0 = 1, r_1 = r_2 = r_3 = 0$. For the associated weighted flag $F$, it follows that
\[
I_{p \times \{0\}}(C) = (l, m_p)O_{C \times \mathbb{A}^1, p \times \{0\}}
\]
which is the maximal ideal of $O_{C \times \mathbb{A}^1, p \times \{0\}}$ where $m_p$ is the maximal ideal of $O_{C, p}$. Hence
\[
e_F(C) = e_F(C)_p = \text{mult}_{p \times \{0\}}(C \times \mathbb{A}^1) = \text{mult}_p C \geq 3 = 3 \sum r_i.
\]
Furthermore, the last inequality is strict if $\text{mult}_p C \geq 4$.

The next proposition will be used in the proof of the following theorems.

**Proposition 3.7.** Let $C \subset \mathbb{P}^3$ be an honestly non-hyperelliptic curve of degree 6 in the sense of Definition 2.8, and assume that each irreducible component of $C$ does not contained in $L_n$. Suppose that $e_1 \leq 2$ and $e_2 \leq 4$ where $e_i$ be the same as that in Definition 2.3. Then $C$ is Chow stable with respect to $F$.

**Proof.** From Lemma 3.1, it follows that
\[
e_F(C) \leq \min\{6r_0, 2r_0 + 6r_1, 4r_0 + 6r_2, 2r_0 + 4r_1 + 4r_2\}.\]
If the right hand side in the above inequality is greater than or equal to $3 \sum r_i$ simultaneously, then it should be satisfied that
\[
6r_0 = 2r_0 + 6r_1 = 4r_0 + 6r_2 = 2r_0 + 4r_1 + 4r_2 = 3 \sum r_i
\]
which implies that $e_F(C) \leq 3 \sum r_i$, and the equality $e_F(C) = 3 \sum r_i$ holds only when $r_0 = 3r, r_1 = 2r, r_2 = r$ for some $r \in \mathbb{Z}_{>0}$, and $e_2 = 2, e_4 = 4$.

We now assume that $r_0 = 3r, r_1 = 2r$ and $r_2 = r$ for some $r \in \mathbb{Z}_{>0}$, and $e_1 = 2$ and $e_4 = 4$. If $C$ meets $L_3$ at a point not equal to $L_1$, then
\[
e_F(C) \leq 5r_0 + r_1 = 17r < 3 \sum r_i
\]
by Lemma 3.3. On the other hand, if $C$ intersects $L_3$ only at $L_1$, then the restricted projection morphism $P_{L_3}\big|_{C \cap (\mathbb{P}^3 - L_2)}$ extends to a morphism $C \to \mathbb{P}^1$ of degree 2 because $e_2 = 4$ and the assumption that $C \cap L_3$ consists of only one point $L_1$. This gives a contradiction because $C$ is honestly non-hyperelliptic.

Our next result shows that any irreducible canonical curve admitting only mild singularities is Chow stable.

**Theorem 3.8.** Let $C \subset \mathbb{P}^3$ be an irreducible canonical curve of genus 4 admitting at worst $A_n, n \leq 4$, singularities. Then $C$ is Chow stable.

**Proof.** From the assumptions, it is induced that $C$ admits at most double points, and is nondegenerate. Thus it follows that $e_1 \leq 2$ and $e_2 \leq 5$. Via Proposition 3.7, it is enough to show that $e_2 \neq 5$. Suppose not. The composition of the partial normalization morphism $\tilde{C} \to C$ at the points in $C \cap L_2$
Thus the only nontrivial cases are each consisting of two smooth components meeting at nodes and having genus degree 3.

Moreover smooth elliptic curves.

Theorem 3.10. Let $C \subset \mathbb{P}^3$ be a general curve in $\delta_{1,1}$, then it is Chow semistable but not Chow stable. Furthermore, all Chow semistable curves in $\delta_{1,1}$ are identified in $\text{Chow}_{4,1}/\text{SL}_4$.

Proof. Without loss of generality, we may assume that $C$ is a union of two smooth elliptic curves $C_1$ and $C_2$ meeting at three nodes denoted by $p_1$, $p_2$ and $p_3$. Note that each $C_i$ is contained in a hyperplane denoted by $H_i$, and has degree 3.
If $L_2$ is not contained in any $H_i$, then $e_1 \leq 2$ and $e_2 \leq 4$ which implies that $e_F(C) \leq 3 \sum r_i$ by Proposition 3.7, and thus we may assume that $L_2 \subset H_2$.

If $H_2$ is not equal to $L_3$, then

$$e_F(C) = e_F(C_1) + e_F(C_2) \leq \begin{cases} (r_0 + 2r_1) + (r_0 + 2r_1), & \text{if } L_2 = H_1 \cap H_2 \\ (r_0 + 2r_2) + (r_0 + 3r_1), & \text{if } L_2 \neq H_1 \cap H_2 \end{cases}$$

which implies that $e_F(C) \leq 3 \sum r_i$.

Now assume that $H_2 = L_3$. Then it is easy to check that

$$e_F(C) = e_F(C_1) + e_F(C_2) \leq \begin{cases} (r_0 + 2r_1) + (r_0 + 2r_1 + 3r_2), & \text{if } L_2 = H_1 \cap H_2 \\ (r_0 + 2r_2) + (r_0 + 2r_1 + 3r_2), & \text{if } L_2 \neq H_1 \cap H_2 \end{cases}$$

which yields that $e_F(C) \leq 3 \sum r_i$. Finally we showed that $C$ is Chow semistable.

Choose coordinates $X_0, \ldots, X_3$ so that $H_1$ and $H_2$ are hyperplanes defined by $X_2 = 0$ and $X_3 = 0$ respectively. Set $r_0 = r_1 = r$ and $r_2 = 0$. Then for each $i$ it follows that

$$e_F(C_i) = e_F(C_i)_{p_1} + e_F(C_i)_{p_2} + e_F(C_i)_{p_3} = r + r + r = 3r$$

and thus

$$e_F(C) = e_F(C_1) + e_F(C_2) = 6r = 3 \sum r_i.$$  

This shows that $C$ is not Chow stable.

Now it remains to show the last statement of the theorem. Choose coordinates $X_0, \ldots, X_3$ of $\mathbb{P}^3$ so that $C$ is defined by

$$X_0X_2^2 + X_0X_3^2 - X_1(X_1 - aX_0)(X_1 - bX_0) = 0 \text{ and } X_2X_3 = 0,$$

where $1, a$ and $b$ are distinct where $X_0, \ldots, X_3$ is a homogeneous coordinates on $\mathbb{P}^3$. Note that general curve satisfying the assumptions in the proposition can be defined in this way if we choose suitable coordinates.

Consider the one parameter subgroup $\lambda : \mathbb{G}_m \to \text{GL}_4$ defined by

$$\lambda(t)X_0 = tX_0, \quad \lambda(t)X_1 = tX_1, \quad \lambda(t)X_2 = X_2, \quad \text{and } \lambda(t)X_3 = X_3.$$  

Let $\hat{C}$ be the limit of $C$ as $t \to \infty$ under the action $\lambda$. Applying the computation in [9], it follows that $\hat{C}$ is given by

$$X_1(X_1 - aX_0)(X_1 - bX_0) = 0 \text{ and } X_2X_3 = 0.$$  

We note that $\hat{C}$ is a union of $\hat{C}_1$ and $\hat{C}_2$ satisfying

- (a) each $\hat{C}_1$ is contained in $H_i$,
- (b) $\hat{C}_1 = L_{1,1} \cup L_{1,2} \cup L_{1,3}$ and $\hat{C}_2 = L_{2,1} \cup L_{2,2} \cup L_{2,3}$ where each $L_{i,j}$ is a line,
- (c) $L_{i,1}, L_{i,2}$ and $L_{i,3}$ intersect at one point $q_i$ for each $i = 1, 2$, and
- (d) $L_{1,j}$ and $L_{2,j}$ meet at a point $p_j$.

From Section 11.3 in [4], it is induced that $\hat{C}$ is Chow semistable. Note that any two curves satisfying (a)~(d) are projectively equivalent which yields the last statement in the theorem. \[ \square \]
Theorem 3.11. If \( C \subset \mathbb{P}^3 \) is a general curve in \( \delta_{2,0}, \delta_{1,0}, \) or \( \delta_{0,0}, \) then it is Chow stable.

Proof. Without loss of generality, we may assume that \( C \) is a canonical curve consisting of two smooth components \( C_1 \) and \( C_2 \) meeting at nodes. It is easy to check that \( e_1 \leq 2 \) and \( e_2 \leq 4 \) for any weighted flag \( F. \) Therefore by Proposition 3.7, we can also assume that \( C_2 \) is contained in \( L_3. \)

If \( C \) belongs to a class in \( \delta_{0,0}, \) then \( C_1 \) and \( C_2 \) are twisted cubic curves by Fig. 18 in p. 354 [5], and thus they are nondegenerate, a contradiction.

Assume that \( C \) belongs to a class in \( \delta_{1,0}. \) From Fig. 18 in p. 354 [5], we obtain that \( \deg C_1 = 4 \) and \( \deg C_2 = 2. \) We note that the intersection \( C_1 \cap C_2 \) consists of four distinct nodes of \( C \) and \( C_2 \subset L_3. \) Therefore the points in \( C_1 \cap L_3 \) are exactly the same as that in \( C_1 \cap C_2. \) Hence in \( C_1 \cap L_3, \) there exist at least two points not lying on \( L_2, \) and at least three points not equal to \( L_1, \) which implies that \( e_F(C_1) \leq r_0 + r_1 + 2r_2 \) by Lemma 3.3. Applying Lemma 3.4, it is induced that \( e_F(C_2) \leq 2r_0 + 2r_2. \) Therefore

\[
e_F(C) = e_F(C_1) + e_F(C_2) \leq 3r_0 + r_1 + 4r_2 \leq 3 \sum r_i.
\]

In the last inequality, the equality holds if and only if \( r_1 = r_2 = 0. \) In the case when \( r_1 = r_2 = 0, \) it is induced that

\[
e_F(C) = e_F(C_1) + e_F(C_2) = e_F(C_1)_p + e_F(C_2)_p \leq 2r < 3 \sum r_i,
\]

where \( p \) is the point on which \( X_1, X_2 \) and \( X_3 \) vanish.

The cases when \( C \) belongs to a class in \( \delta_{2,0} \) can be proved by similar arguments. \( \square\)

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References


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