REducing Subspaces For Toeplitz Operators On The Polydisk

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Abstract. In this note, we completely characterize the reducing subspaces of \( T_{z^N}T_{z^M} \) on \( A_2^\alpha(D^2) \) where \( \alpha > -1 \) and \( N, M \) are positive integers with \( N \neq M \), and show that the minimal reducing subspaces of \( T_{z^N}T_{z^M} \) on the unweighted Bergman space and on the weighted Bergman space are different.

1. Introduction

Let \( D \) denote the open unit disk in the complex plane. For \(-1 < \alpha < +\infty\), \( L_2^\alpha(D, dA_\alpha) \) is the space of functions on \( D \) which are square integrable with respect to the measure \( dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z) \), where \( dA \) denotes the normalized Lebesgue area measure on \( D \). \( L_2^\alpha(D, dA_\alpha) \) is a Hilbert space with the inner product \( \langle f, g \rangle_\alpha = \int_D f(z)g(z)dA_\alpha \). The weighted Bergman space \( A_\alpha^2 \) is the closed subspace of \( L_2^\alpha(D, dA_\alpha) \) consisting of analytic functions on \( D \). If \( \alpha = 0 \), \( A_0^2 \) is the Bergman space. We write \( A^2 = A_0^2 \). It is known that \( \{ \frac{z^n}{\|z^n\|_\alpha} \}_{n=0}^{+\infty} \) is an orthogonal basis of \( A_\alpha^2(D) \). Let \( \gamma_n = \|z^n\|_\alpha = \sqrt{\frac{n!\Gamma(2+\alpha)}{\Gamma(2+\alpha+n)}} \) for \( n = 0, 1, 2, \ldots \). Therefore,

\[
\|f\|_\alpha^2 = \sum_{n=0}^{+\infty} \gamma_n^2|a_n|^2 < \infty,
\]

with \( f(z) = \sum_{n=0}^{+\infty} a_nz^n \in A_\alpha^2(D) \).

Denote the unit polydisk by \( D^n \). The weighted Bergman space \( A_\alpha^2(D^n) \) is then the space of all holomorphic functions on \( L_2^\alpha(D^n, dv_\alpha) \), where \( dv_\alpha(z) = dA_\alpha(z_1)\cdots dA_\alpha(z_n) \). For multi-index \( \beta = (\beta_1, \ldots, \beta_n) \), \( \beta \geq 0 \) means that

\[\beta_1 + \cdots + \beta_n = \|\beta\|_\alpha = \frac{1}{\Gamma(2+\alpha)} \prod_{k=1}^n \Gamma(\alpha+\beta_k) \]

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\[ \beta_i \geq 0 \text{ for any } i \geq 0. \] Denote by \[ z^\beta = z_1^{\beta_1} z_2^{\beta_2} \cdots z_n^{\beta_n} \] and 
\[ e_\beta = \frac{z^\beta}{\gamma_{\beta_1} \cdots \gamma_{\beta_n}}, \]
then \( \{e_\beta\}_\beta \) is an orthogonal basis in \( A_2^0(D^n) \).

Let \( P \) be the Bergman basis from \( L^2(D^n) \) onto \( A_2^0(D^n) \).

For a bounded measurable function \( f \in L^\infty(D^n) \), the Toeplitz operator with symbol \( f \) is defined by \( T_f h = P(fh) \) for every \( h \in A_2^0(D^n) \).

Recall that in a Hilbert space \( \mathcal{H} \), a (closed) subspace \( \mathcal{M} \) is called a reducing subspace of the operator \( T \) if \( T(\mathcal{M}) \subseteq \mathcal{M} \) and \( T^*(\mathcal{M}) \subseteq \mathcal{M} \). A nontrivial reducing subspace \( \mathcal{M} \) is said to be minimal if the only reducing subspaces contained in \( \mathcal{M} \) are \( \mathcal{M} \) and \( \{0\} \). On the Bergman space \( A_2^0(D) \), the reducing subspaces of the Toeplitz operators with finite Blaschke products are well studied (see [1, 2, 8] for example). On \( A_2^0(D^2) \), Y. Lu and X. Zhou [4] characterized the reducing subspaces of Toeplitz operators \( T_{z_1^{N-1}z_2^{N-1}}, T_{z_1^N} \) and \( T_{z_2^N} \).

In this note, we consider the reducing subspaces of the Toeplitz operators \( T_{z_1^{N-1}z_2^{M}} \) on \( A_2^0(D^2) \) and \( T_{z_1^Nz_2^{M}} \) on \( A_2^0(D^n) \), where \( N, M \geq 1 \) are integers and \( 1 \leq i < j \leq n \). Usually, the Toeplitz operators on the unweighted Bergman space and the weighted Bergman space have similar properties (see [5, 6, 7, 9] for example). However, we obtain that the minimal reducing subspaces of \( T_{z_1^{N-1}z_2^{M}} \) with \( N \neq M \) on \( A_2^0(D^2) (\alpha \neq 0) \) are less than that on \( A^2(D^2) \) (see Theorem 2.4 and Theorem 3.2).

### 2. The results on the Bergman space

Let \( M, N \) be integers with \( M, N \geq 1 \) and \( M \neq N \). In this section, we consider the minimal reducing subspace of \( T_{z_1^{N-1}z_2^{M}} \) on \( A^2(D^2) \). Here \( \gamma_k = \|z^k\|_0 = \sqrt{\frac{1}{k+1}} \). Let \( \rho_1(k) = \frac{(k+1)N}{M} - 1 \) and \( \rho_2(k) = \frac{(k+1)M}{N} - 1 \). Let \( \mathcal{H}_{nm} = \text{Span}\{z_1^{n-m}z_2^{m}, z_1^{n(m)}z_2^{(n)}\} \) and \( P_{nm} \) be the orthogonal projection from \( A_2^0(D^2) \) onto \( \mathcal{H}_{nm} \).

**Lemma 2.1.** Let \( n, m, h \) be nonnegative integers. Then the following statements hold:

- (a) if \( \rho_1(m) \) is an integer, then \( \rho_1(m + hM) = \rho_1(m) + hN \) is an integer for every \( h \geq 0 \);
- (b) if \( \rho_2(n) \) is an integer, then \( \rho_2(n + hN) = \rho_2(n) + hM \) is an integer for every \( h \geq 0 \);
- (c) if \( \rho_1(m) \) and \( \rho_2(n) \) are positive integers, then \( \gamma_{\rho_1(m)}\gamma_{\rho_2(n)} = \gamma_m\gamma_n \);
- (d) \( \rho_1(\rho_2(n)) = n \) and \( \rho_2(\rho_1(m)) = m \).

**Proof.** Notice that if \( \rho_1(m) \) and \( \rho_2(n) \) are positive integers, then \( \gamma_{\rho_1(m)} = \sqrt{\frac{M}{N}}\gamma_m \) and \( \gamma_{\rho_2(n)} = \sqrt{\frac{N}{M}}\gamma_n \). So (c) holds. By the direct calculation, (a), (b) and (d) are obvious. \( \square \)
Theorem 2.2. Let \( n, m \) be integers such that \( 0 \leq n \leq N - 1 \) or \( 0 \leq m \leq M - 1 \), and both of \( \rho_1(m) \) and \( \rho_2(n) \) are integers. Then for \( a, b \in \mathbb{C} \),

\[
\mathcal{M} = \text{Span}\{a z_1^{m+nH} z_2^m + b z_1^{\rho_1(m)+hM} z_2^{\rho_2(n)+hN}; h = 0, 1, 2, \ldots\}
\]

is a minimal reducing subspace of \( T_{z_1^{m+nH} z_2^m} \) on the polydisk.

Proof. By Lemma 2.1(a) and (b), it is easy to check that \( T_{z_1^{m+nH} z_2^m}(\mathcal{M}) \subseteq \mathcal{M} \).

On the other hand,

\[
T_{z_1^{m+nH} z_2^m}(z_1^{k-1} z_2^l) = \sum_{\beta \geq 0} \langle T_{z_1^{m+nH} z_2^m}^k z_1^{k-1} z_2^l, e^\beta \rangle e^\beta
\]

\[
= \begin{cases} 
\gamma_2^{k-1} z_1^{k-N} z_2^{-M}, & \text{if } k \geq N, l \geq M, \\
0, & \text{if others.}
\end{cases}
\]

For each \( h \geq 1 \),

\[
T_{z_1^{m+nH} z_2^m}(z_1^{n+hN} z_2^m + hM)
\]

\[
= \frac{\gamma_{n+hN}^{2} z_1^{n+hN} z_2^m + hM}{\gamma_{n+hN}^{2} z_1^{n+hN} z_2^m + hM},
\]

\[
T_{z_1^{m+nH} z_2^m}(z_1^{n+hN} z_2^m + hM)
\]

\[
= \frac{\gamma_{\rho_1(m)+hM}^{2} z_1^{\rho_1(m)+hM} z_2^{\rho_2(n+hN)-M}}{\gamma_{\rho_1(m)+hM}^{2} z_1^{\rho_1(m)+hM} z_2^{\rho_2(n+hN)-M}}.
\]

Combining this with Lemma 2.1(c), it is easy to check that

\[
T_{z_1^{m+nH} z_2^m}(a z_1^{n+hN} z_2^m + b z_1^{\rho_1(m)+hM} z_2^{\rho_2(n)+hN})
\]

\[
= \mu(a z_1^{n+hN-\rho_1(m)+hM} + b z_1^{\rho_1(m)+hM-M} z_2^{\rho_2(n)+hN-N} \in \mathcal{M},
\]

where \( \mu = \frac{\gamma_{n+hN+M}^{2} z_1^{n+hN+M} + \gamma_{\rho_1(m)+hM-M}^{2} z_1^{\rho_1(m)+hM-M}}{\gamma_{n+hN+M}^{2} z_1^{n+hN+M} + \gamma_{\rho_1(m)+hM-M}^{2} z_1^{\rho_1(m)+hM-M}}. \)

Since \( 0 \leq n \leq N - 1 \) and \( 0 \leq m \leq M - 1 \), we get \( p_2(n) < M \) (or \( \rho_1(m) < N \), respectively). Therefore, \( T_{z_1^{m+nH} z_2^m}(a z_1^{n+hN} z_2^m + b z_1^{\rho_1(m)+hM} z_2^{\rho_2(n)+hN}) = 0 \in \mathcal{M} \). So \( T_{z_1^{m+nH} z_2^m}(\mathcal{M}) \in \mathcal{M} \), which finishes the proof. \( \Box \)

Lemma 2.3. Suppose \( \mathcal{M} \neq 0 \) is a reducing subspace of \( T_{z_1^{m+nH} z_2^m} \) in \( A^2(D^2) \). Let

\[
f = \sum_{(k,l) \geq 0} a_{k,l} z_1^k z_2^l \in \mathcal{M}.
\]

For each nonnegative integers \( n, m \) with \( a_{nm} \neq 0 \), the following statements hold:

(I) if \( \rho_1(m), \rho_2(n) \) are integers and \( a_{\rho_1(m)\rho_2(n)} \neq 0 \), then

\[
a_{nm} z_1^n z_2^m + a_{\rho_1(m)\rho_2(n)} z_1^{\rho_1(m)} z_2^{\rho_2(n)} \in \mathcal{M}.
\]

(II) if at least one of \( \rho_1(m), \rho_2(n) \) is not an integer, or \( a_{\rho_1(m)\rho_2(n)} = 0 \), then \( z_1^n z_2^m \in \mathcal{M} \).
Proof. For every integer \( h \geq 0 \), denote by \( T_h = T_{1_{hH_2 M}} \). Notice that

\[
T_h^* T_h(z_1^n, z_2^m) = \frac{\gamma_{2hN + n}^2 \gamma_{2hM + m}^2}{\gamma_{hN + n}^2 \gamma_{hM + m}^2} z_1^n z_2^m \in M, \forall n, m \geq 0.
\]

Let \( P_M \) be the orthogonal projection from \( A^2_{1_{hH_2 M}} \) onto \( M \), then for nonnegative integers \( n, m, k, l \),

\[
\langle P_M T_h^* T_h z_1^n, z_2^m : z_1^{k}, z_2^{l} \rangle = \langle T_h^* T_h P_M z_1^n, z_2^m : z_1^{k}, z_2^{l} \rangle = \langle P_M z_1^n, z_2^m : T_h^* T_h z_1^{k}, z_2^{l} \rangle.
\]

Thus \( \frac{\gamma_{2hN + n}^2 \gamma_{2hM + m}^2}{\gamma_{hN + n}^2 \gamma_{hM + m}^2} = \frac{\gamma_{2hN + n}^2 \gamma_{2hM + m}^2}{\gamma_{hN + n}^2 \gamma_{hM + m}^2} \). Equivalently,

\[
\frac{(k + 1)(l + 1)}{(n + 1)(m + 1)} = \frac{(k + hN + 1)(l + hM + 1)}{(n + hN + 1)(m + hM + 1)} \quad h \geq 0.
\]

We claim that \( (k, l) = (n, m) \) or \( (k, l) = (\rho_1(m), \rho_2(n)) \). In fact, let \( h \to +\infty \), then

\[
(k + 1)(l + 1) = (n + 1)(m + 1).
\]

It follows that \( (k + hN + 1)(l + hM + 1) = (n + hN + 1)(m + hM + 1) \). Since \( g(\lambda) = (k + \lambda N + 1)(l + \lambda M + 1) - (n + \lambda N + 1)(m + \lambda M + 1) \) is an analytic polynomial on \( \mathbb{C} \), \( g(\lambda) = 0 \) for any \( \lambda \in \mathbb{C} \). The coefficient of \( \lambda \) must be zero. We get

\[
M(n - k) = N(l - m).
\]

This together with (2.3) implies the claim.

Therefore, \( P_M(z_1^n, z_2^m) \in H_{nm} \). Hence,

\[
P_{nm} P_M(z_1^n, z_2^m) = P_M(z_1^n, z_2^m).
\]

Since \( P_M f = f \) for every \( f \in M \), we arrive to

\[
\langle P_M P_{nm} f, z_1^n, z_2^m \rangle = \langle f, P_M z_1^n, z_2^m \rangle = \langle P_M f, z_1^n, z_2^m \rangle.
\]

Notice that \( \rho_2(\rho_1(m)) = m, \rho_1(\rho_2(n)) = n \) and \( H_{\rho_1(m)\rho_2(n)} = H_{nm} \). Replacing \( n, m \) by \( \rho_1(m) \) and \( \rho_2(n) \), respectively, it is easy to get that

\[
\langle P_M P_{nm} f, z_1^{\rho_1(m)}, z_2^{\rho_2(n)} \rangle = \langle P_{nm} f, z_1^{\rho_1(m)}, z_2^{\rho_2(n)} \rangle.
\]

Moreover, \( (P_M P_{nm} f, z_1^{k}, z_2^{l}) = (P_{nm} f, z_1^{k}, z_2^{l}) = 0 \) for any \( (k, l) \neq (\rho_1(m), \rho_2(n)) \) and \( (k, l) \neq (n, m) \). Hence \( P_{nm} f = P_M P_{nm} f \in M \). So we get the result. \( \square \)

**Theorem 2.4.** Suppose \( M \neq \{0\} \) is a reducing subspace of \( T_{z_1^n, z_2^m} \) in the Bergman space \( A^2(D^2) \). Then there exist \( a, b \in \mathbb{C} \) and nonnegative integers \( m, n \) with \( 0 \leq n \leq N - 1 \) or \( 0 \leq m \leq M - 1 \), such that \( M \) contains a reducing subspace as follows

\[
M_{n, m, a, b} = \text{Span}\{az_1^{hN} z_2^{hM} + bz_1^{\rho_1(m + hN)} z_2^{\rho_2(n + hM)} : h = 0, 1, 2, \ldots \},
\]

where \( \rho_1(m + hN) = \frac{(m + hN + 1)M}{M} - 1 \) and \( \rho_2(n + hM) = \frac{(n + hM + 1)N}{N} - 1 \). In particular, if \( \rho_1(m) \) (or \( \rho_2(n) \)) is not a positive integer, then \( b = 0 \). Moreover, \( M \) is minimal if and only if \( M = M_{n, m, a, b} \).
Proof. (I) If \( \mathcal{M} \neq 0 \), there exist nonzero function \( f \in \mathcal{M} \) and \( k, l \), such that \( P_M f \neq 0 \). Lemma 2.3 implies that

\[
g_{kl} = P_{kl} f = az_1^k + bz_2^l \in \mathcal{M}.
\]

Observe that there is a positive integer \( h_0 \) such that \( az_1^n z_2^m + bz_1^{\rho_1(m)} z_2^{\rho_2(n)} = (T_{z_2}^{n + h_0})^h (g_{kl}) \neq 0, (T_{z_2}^{n + h_0})^{h_0 + 1} (g_{kl}) = 0 \), where \( n = k - h_0 N, m = l - h_0 M \).

Clearly, \( 0 \leq n \leq N - 1 \) or \( 0 \leq m \leq M - 1 \). So Theorem 2.2 shows that \( az_1^n z_2^m + bz_1^{\rho_1(m)} z_2^{\rho_2(n)} \in \mathcal{M}_{n, m, a, b} \subseteq \mathcal{M} \).

(II) Suppose \( \mathcal{M} \) is minimal. As in (I), there is a nonzero function \( az_1^n z_2^m + bz_1^{\rho_1(m)} z_2^{\rho_2(n)} \in \mathcal{M} \). Then the following statements hold:

(a) if \( z_1^n z_2^m \in \mathcal{M} \), then \( \mathcal{M} = \text{Span}\{z_1^{n + h_0 N} z_2^{m + h_0 M}, h \geq 0\} \); 
(b) if \( \rho_1(m), \rho_2(n) \) are integers, and \( z_1^{\rho_1(m)} z_2^{\rho_2(n)} \in \mathcal{M} \), then \( \mathcal{M} = \text{Span}\{z_1^{\rho_1(m) + h_0 N} z_2^{\rho_2(n) + h_0 M}, h \geq 0\} \);  
(c) if none of \( z_1^n z_2^m \) and \( z_1^{\rho_1(m)} z_2^{\rho_2(n)} \) is in \( \mathcal{M} \), then \( \mathcal{M} = \mathcal{M}_{n, m, a, b} \) with \( ab \neq 0 \).

So we finish the proof. \( \square \)

Remark 2.5. Note that

\[
z_1^\rho_1(m + h_0 N) z_2^\rho_2(n + h_0 M) = z_1^{m + (M - N)(m + 1)} z_2^{n + (N - M)(n + 1)} z_1^h M z_2^N.
\]

If \( N = M \), then \( \rho_1(n) = m \) and \( \rho_2(m) = n \). Y. Lu and X. Zhou [4] showed that \( \text{Span}\{(z_1^n z_2^m + z_1^{-n} z_2^{-m})(z_1 z_2)^{h N} : h = 0, 1, 2, \ldots\} \) and \( \text{Span}\{z_1^n z_2^m (z_1 z_2)^{h N} : h = 0, 1, 2, \ldots\} \) are the only minimal reducing subspaces of \( T_{z_2}^{N} z_2^{N} \). Let \( ab \neq 0 \) with \( a \neq b \). Then \( \mathcal{M}_{n, m, a, b} \) is a reducing subspace of \( T_{z_2}^{N} z_2^{N} \) when \( N \neq M \), but is not a reducing subspace of \( T_{z_2}^{N} z_2^{N} \).

3. The results on the weighted Bergman space

Let \(-1 < \alpha < +\infty\) with \( \alpha \neq 0 \). In this section, we consider the reducing subspace of \( T_{z_2}^{N} z_2^{N} \) on the weighted Bergman Space \( A_\alpha^2(D) \). Here \( \gamma_n = \|z^n\|_\alpha = \sqrt{n/1} \). We begin with a useful lemma.

Lemma 3.1. Let \( M, N, n, m, k, l \) be nonnegative integers with \( l > m, n > k \) and \( M, N \geq 1 \). If

\[
\gamma_{N + k}^2 \gamma_{M + l}^2 = \gamma_{N + n}^2 \gamma_{M + m}^2, \quad h \geq 0,
\]

then \( N = M, l = n \) and \( m = k \).

Proof. First, note that the equality (3.1) holds if and only if for any \( \lambda \in \mathbb{C} \) the following equality holds:

\[
\prod_{j=1}^{n-k} (\lambda N + j + k) \prod_{j=1}^{l-m} (\lambda M + 2 + \alpha + l - j)
\]
\[ M \alpha + \sum_{j=1}^{n-k+1} (2+\alpha+l-j) \prod_{j=1}^{l-m} (\lambda M + j + m). \]

By computing the coefficient of \( \lambda^{n-k+l-m-1} \) in the equality (3.2), we obtain
\[ M \sum_{j=1}^{n-k} (j+k)+N \sum_{j=1}^{l-m} (2+\alpha+l-j) = M \sum_{j=1}^{n-k} (2+\alpha+n-j) + N \sum_{j=1}^{l-m} (j + m). \]
It follows that \( M(n-k) = N(l-m) \).

Second, we prove that if \( \alpha \) is not an integer, then the following statements hold:

(3.3) \( (m+1)N = (k+1)M \) and \( (l+1+\alpha)N = (n+1+\alpha)M \).

(a) Let \( \lambda_1 = -\frac{k+1}{M} \). Then \( \lambda_1 N + k + 1 = 0 \) and \( \lambda_1 N + 2 + \alpha + n - j \neq 0 \) for any \( 1 \leq j \leq n-k \), because \( \lambda_1 N + 2 + \alpha + n - j \) is not an integer. Therefore, the equality (3.2) implies that \( \prod_{j=1}^{l-m} (\lambda_1 M + j + m) = 0 \). That is, there exists \( 1 \leq h_1 \leq l - m \) such that \( \lambda_1 M + m + h_1 = 0 \). So, \( h_1 = \frac{k+1}{M}M - m \geq 1 \). It follows that \( (m+1)N \leq (k+1)M \).

(b) Let \( \lambda_2 = -\frac{m+1}{N} \). Then \( \lambda_2 M + m + 1 = 0 \). Similarly, we can get an integer \( h_2 \) such that \( 1 \leq h_2 \leq l - m \) and \( \lambda_2 N + k + h_2 = 0 \), which implies that \( h_2 = \frac{m+1}{N}N - k \geq 1 \). Thus \( (m+1)N \geq (k+1)M \).

Comparing (a) with (b), we arrive at \( (m+1)N = (k+1)M \).

(c) Let \( \mu_1 = -\frac{n+1+\alpha}{M} \). Then \( \mu_1 N + n + 1 + \alpha = 0 \), \( \mu_1 N + k + j \neq 0 \) for any \( 1 \leq j \leq n-k \). Therefore, \( \prod_{j=1}^{n-k} (\mu_1 M + 2 + \alpha + l - j) = 0 \). That is, there exists \( 1 \leq h_3 \leq l - m \) such that \( \mu_1 M + 2 + \alpha + l - h_3 = 0 \). So, \( h_3 = -\frac{n+1+\alpha}{M}M + (2 + \alpha + l) \geq 1 \), i.e., \( (l+1+\alpha)N \geq (n+1+\alpha)M \).

(d) Let \( \mu_2 = -\frac{l+1+\alpha}{N} \). Then \( \mu_2 M + l + 1 + \alpha = 0 \). As in (c), there exists \( 1 \leq h_4 \leq n - k \) such that \( \mu_2 N + \alpha + 2 + n - h_4 = 0 \). So, \( 1 \leq h_4 = -\frac{l+1+\alpha}{N}N + (2 + \alpha + n) \leq n - k \) and \( (l+1+\alpha)N \leq (n+1+\alpha)M \).

Comparing (c) with (d), we arrive at \( (l+1+\alpha)N = (n+1+\alpha)M \).

Third, we prove that if \( \alpha \) is an positive integer, then (3.3) holds. In fact, if \( 1 + \alpha \geq 2 \) is an integer, then (3.2) can be simplified into
\[ \prod_{j=1}^{k_1} (\lambda N + j + k) \prod_{j=1}^{m_1} (\lambda M + 2 + \alpha + l - j) \]
\[ = \prod_{j=1}^{k_1} (\lambda N + 2 + \alpha + n - j) \prod_{j=1}^{m_1} (\lambda M + j + m), \forall \lambda \in \mathbb{C}, \]
where \( 2 \leq k_1 \leq n - k \), \( 2 \leq m_1 \leq l - m \), \( 2 + \alpha + n - k_1 > k_1 + k \) and \( 2 + \alpha + l - m_1 > m_1 + m \). By the same technique as in second part of the proof, we can get the equalities in (3.3).

Finally, combining the equalities (3.3) with \( M(n-k) = N(l-m) \), it is easy to get \( n = \alpha M \). Since \( \alpha \neq 0 \), we have \( N = M \), \( l = n \), \( k = m \). \( \square \)

**Theorem 3.2.** Let \( \alpha \neq 0 \), \( M, N \geq 1 \) with \( M \neq N \). Suppose \( M \neq \{0\} \) is a reducing subspace of \( T^2_{<N,\alpha} \) in the weighted Bergman space \( A^2_{\alpha}(D^2) \). Then
there exist nonnegative integers \(n, m\) with \(0 \leq n \leq N - 1\) or \(0 \leq m \leq M - 1\) such that

\[
\mathcal{M}_{nm} = \text{Span}\{z_1^{hN+n}z_2^{hM+m} : h = 0, 1, 2, \ldots\} \subseteq \mathcal{M}.
\]

In particular, \(\mathcal{M}\) is minimal if and only if there exist \(n, m\) as in assumption such that \(\mathcal{M} = \mathcal{M}_{nm}\).

**Proof.** Suppose \(\mathcal{M} \neq \{0\}\) is a reducing subspace. As in the proof of Lemma 2.3, there exist integers \(n, m\) such that \(P_M(z_1^n z_2^m) \neq 0\) and

\[
\frac{\gamma_{hN+k}^2 \gamma_{M+l}^2}{\gamma_k^2 \gamma_l^2} = \frac{2}{\gamma_{hN+n}^2 \gamma_{M+m}^2},\ h \geq 0,
\]

whenever \(\langle P_M(z_1^n z_2^m), z_1 z_2 \rangle \neq 0\). Considering that \(\{\gamma_j\}_{j=0}^{\infty}\) is strictly decreasing and \(\gamma_{hN+k}^2 \gamma_{M+l}^2 \to 1\) as \(h \to +\infty\), we obtain that \(\gamma_k^2 \gamma_l^2 = \gamma_{hN+n}^2 \gamma_{M+m}^2\), \(h \geq 0\). This means that one of the following statements holds:

1. \(l = m, n = k\);
2. \(l > m\) and \(n > k\);
3. \(l < m\) and \(n < k\).

Since \(N \neq M\), Lemma 3.1 implies that (2) does not hold. By the same technique, (3) does not hold. So, (1) holds, that is, there exists \(c_{nm} \in \mathbb{C}\) such that \(P_M(z_1^n z_2^m) = c_{nm} z_1^n z_2^m\). For \(f = \sum (k,l) \geq 0 a_{kl} z_1^k z_2^l \in \mathcal{M}\), we claim that if \(a_{nm} \neq 0\), then \(c_{nm} \neq 0\). In fact,

\[
Q_{nm} f = Q_{nm} P_M(f) = Q_{nm} \sum (k,l) \geq 0 P_M(a_{kl} z_1^k z_2^l)) = c_{nm} a_{nm} z_1^n z_2^m = c_{nm} Q_{nm} f,
\]

where \(Q_{nm}\) is the orthogonal projection from \(A^2_\alpha(D^2)\) onto \(\text{Span}\{z_1^n z_2^m\}\). Therefore, \(c_{nm} = 1 \neq 0\).

Hence \(z_1^n z_2^m \in \mathcal{M}\). Choose an integer \(h_0\) such that \(0 \leq n - h_0 N \leq N - 1\), \(m - h_0 M \geq 0\) or \(0 \leq m - h_0 M \leq M - 1\), \(n - h_0 N \geq 0\). As in the proof of Theorem 2.4, \(\text{Span}\{z_1^{n+(h-h_0)N} z_2^{m+(h-h_0)M} : h = 0, 1, 2, \ldots\} \subseteq \mathcal{M}\) is a minimal reducing subspace of \(T_{z_1^{N} z_2^{M}}\). The proof is complete. \(\square\)

**Remark 3.3.** By the proof of above theorem, we know that on the weighted Bergman space, either \(\text{Span}\{z_1^n z_2^m\} \subseteq \mathcal{M}\) or \(\text{Span}\{z_1^n z_2^m\} \subseteq \mathcal{M}^4\) holds.

**Theorem 3.4.** Let \(N, M \geq 1\) and \(N \neq M\). Every nonzero reducing subspace \(\mathcal{M}\) of \(T_{z_1^{N} z_2^{M}}\) in \(A^2_\alpha(D^2)\) for every \(\alpha > -1\) is a direct (orthogonal) sum of some minimal reducing subspaces.

**Proof.** We prove the theorem in two cases.

Case one: \(\alpha \neq 0\). Let us denote

\[
\mathcal{M}_{nm} = \text{Span}\{z_1^{hN+n} z_2^{hM+m} : h = 0, 1, 2, \ldots\},
\]

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where $0 \leq n \leq N - 1$ or $0 \leq m \leq M - 1$. By Lemma 3.1, we have $M_{nm} \subseteq M$ if and only if there exist some $f \in M$ with $\langle f, z_{n}^{m}z_{2}^{m} \rangle \neq 0$. Let $E_{1} = \{(n, m) \geq 0; n \leq N - 1$ or $m \leq M - 1, \langle f, z_{n}^{m}z_{2}^{m} \rangle \neq 0$ for some $f \in M\}$. Then $M = \bigoplus_{(n, m) \in E_{1}} M_{nm}$.

Case two: $\alpha = 0$. For $n, m \geq 0$, there exist $a, b \in \mathbb{C}$ such that $M$ contains the minimal reducing subspace of $T_{z_{n}^{m}z_{2}^{m}}$ defined by

$$M_{n, m, a, b} = \text{Span}\{az_{1}^{hN+n}z_{2}^{hM+m} + b_{2}^{\rho_{1}(m+hN)}z_{2}^{\rho_{2}(m+hM)} : h = 0, 1, 2, \ldots\}.$$  

In fact,

1. If $z_{1}^{m}z_{2}^{m} \in M$, then $M_{n, m, 1, 0} = M_{nm}$.
2. If $z_{1}^{m}z_{2}^{m} \notin M$, then $M_{n, m, 0, 1} = M_{\rho_{1}(m)\rho_{2}(n)}$.
3. If neither $z_{1}^{m}z_{2}^{m}$ nor $z_{2}^{\rho_{1}(m)}z_{2}^{\rho_{2}(n)}$ are in $M$, and there exists $f \in M$ such that $P_{nm}f \neq 0$, then Theorem 2.4 implies that $M_{n, m, a, b} \subseteq M$ is a minimal reducing subspace of $T_{z_{n}^{m}z_{2}^{m}}$, where $P_{nm}f = az_{1}^{m}z_{2}^{m} + b_{2}^{\rho_{1}(m)}z_{2}^{\rho_{2}(n)}$. It follows that $P_{nm}g = \lambda (az_{1}^{m}z_{2}^{m} + b_{2}^{\rho_{1}(m)}z_{2}^{\rho_{2}(n)})$ for every $g \in M$ with $P_{nm}g \neq 0$.
4. If $P_{nm}f = 0$ for any $f \in M$, then $M_{n, m, a, b} \subseteq M$ if and only if $a = 0, b = 0$, i.e., $M_{n, m, 0, 0} = \{0\}$.

Let $M' = M \ominus M_{n, m, a, b}$. Then $M'$ is a reducing subspace. Continuing this process, since $A^{2}(D^{2}) = \bigoplus_{n, m \geq 0} z_{n}^{m}z_{2}^{m}$, it is not difficult to prove that $M$ is the direct (orthogonal) sum of some minimal reducing subspaces as $M_{n, m, a, b}$.

In [8], Kehe Zhu shows that a reducing subspace of $T_{z_{n}^{m}z_{2}^{m}}$ on $A^{2}(D)$ is the direct (orthogonal) sum of at most $N$ minimal reducing subspaces. However, the reducing subspace of $T_{z_{n}^{m}z_{2}^{m}}$ on $A^{2}(D^{2})$ may be the direct (orthogonal) sum of infinity numbers of minimal reducing subspaces. For example, $M = \text{Span}\{z_{1}^{1+2h}f(z_{2}); f \in A^{2}_{h}(D), h = 0, 1, 2, \ldots\}$ is a reducing subspace of $T_{z_{n}^{m}z_{2}}$ and $M = \bigoplus_{n=0}^{+\infty} M_{n}$, where $M_{n} = \text{Span}\{z_{1}^{1+2h}z_{2}^{n+3h}; h = 0, 1, 2, \ldots\}$.

4. The results on the polydisk $A^{2}_{h}(D^{n})$

In this section, we consider the reducing subspace of $T_{z_{n}^{m}z_{2}^{m}}$ in the weighted Bergman space $A^{2}_{h}(D^{n})$ with $N \neq M$.

**Theorem 4.1.** Suppose $M \neq \{0\}$ is a reducing subspace of $T_{z_{n}^{m}z_{2}^{m}}$ (N, $M \geq 1, N \neq M, i \neq j$) in the weighted Bergman space $A^{2}_{h}(D^{n})$. Then the following statements hold:

(a) if $\alpha = 0$, then there exist functions $g_{1}, g_{2} \in A^{2}_{h}(D^{n-2})$ and integers $l, m$ with $0 \leq l \leq N - 1$ or $0 \leq m \leq M - 1$, such that $M$ contains the reducing subspace

$$M' = \text{Span}\{(g_{1}(z'))z_{1}^{hN+l}z_{2}^{hM+m} + g_{2}(z')z_{1}^{\rho_{1}(l+hN)}z_{2}^{\rho_{2}(m+hM)}; h \geq 0\};$$
(b) if $\alpha \neq 0$, then there exist a function $g \in A^2_{\alpha}(D^{n-2})$ and integers $l, m$ with $0 \leq l \leq N - 1$ or $0 \leq m \leq M - 1$ such that $\mathcal{M}$ contains the reducing subspace

$$
\mathcal{M}_{lmg} = \text{Span}\{z_i^{hN+l}z_j^{hM+m}g(z') : h = 0, 1, 2, \ldots\},
$$

where $z' = (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n)$. Moreover, $\mathcal{M}'$ is the only minimal reducing subspace of $T_{z_i^{N}z_j^{M}}$ on $A^2(D^n)$ and $\mathcal{M}_{lmg}$ is the only minimal reducing subspace of $T_{z_i^{N}z_j^{M}}$ on $A^2_{\alpha}(D^n)$ with $\alpha \neq 0$.

Proof. Without loss of generality, let $i = 1$ and $j = 2$. Denote by $P_M$ the orthogonal projection from $A^2_{\alpha}(D^n)$ onto $\mathcal{M}$. Let $z^K = z_1^{k_1}z_2^{k_2} \cdots z_n^{k_n}$ with $P_M(z^K) \neq 0$. Let $T_h = T_{z_i^{N}z_j^{M}}$. Then $(T_h^*T_h P_Mz^K, z^L) = (P_M T_h^*T_h z^K, z^L)$ for any $z^L = z_1^{l_1}z_2^{l_2} \cdots z_n^{l_n}$. Observe that

$$
\langle P_Mz^K, T_h^*T_h z^L \rangle = \frac{\gamma_1^{hN+k_1} \gamma_2^{hM+k_2}}{\gamma_1^h \gamma_2^l} \langle P_Mz^K, z^L \rangle,
$$

and

$$
\langle T_h^*T_h z^K, P_Mz^L \rangle = \frac{\gamma_1^{hN+k_1} \gamma_2^{hM+k_2}}{\gamma_1^h \gamma_2^l} \langle z^K, P_Mz^L \rangle.
$$

Therefore,

$$
\frac{\gamma_1^{hN+k_1} \gamma_2^{hM+k_2}}{\gamma_1^h \gamma_2^l} = \frac{\gamma_1^{hN+l_1} \gamma_2^{hM+l_2}}{\gamma_1^h \gamma_2^l}, \forall h \geq 0,
$$

whenever $\langle P_Mz^K, z^L \rangle \neq 0$.

If $\alpha = 0$, then as in Lemma 2.3 we have $(l_1, l_2) = (k_1, k_2)$ or $(l_1, l_2) = (\rho_1(k_2), \rho_2(k_1))$ where $\rho_1(k_2), \rho_2(k_1)$ are integers. Thus $P_Mz^K$ and $P_Mz^L$ are in $z_1^{k_1}z_2^{k_2} A^2(D^{n-2}) + z_1^{\rho_1(k_2)}z_2^{\rho_2(k_1)} A^2(D^{n-2})$, where $z' = (z_3, \ldots, z_n)$ and $K' = (k_3, \ldots, k_n)$. Let $P_{k_1k_2}$ be the orthogonal projection from $A^2(D^n)$ onto

$$
\text{Span}\{z_1^{k_1}z_2^{k_2} A^2(D^{n-2}) + z_1^{\rho_1(k_2)}z_2^{\rho_2(k_1)} A^2(D^{n-2}); h = 0, 1, 2, \ldots\}.
$$

Then $P_{k_1k_2}P_Mz^K = P_M P_{k_1k_2}z^K$. For each $f \in \mathcal{M}$ with $f \neq 0$, there are integers $l, m \geq 0$ such that $P_M f \neq 0$. By the similar technique, we can prove that $\langle P_M P_M f, z^K \rangle = (P_M f, z^K)$ for any $K \geq 0$, i.e., $P_M P_M f = P_M f$. So, there exist $f_1(z')$ and $g_2(z') \in A^2(D^{n-2})$ such that $P_M f = g_1(z') z_1^{l_1}z_2^{l_2} + g_2(z') z_1^{\rho_1(l)} z_2^{\rho_2(m)} \in \mathcal{M}$, which implies that (a) holds.

If $\alpha \neq 0$, then we arrive at $P_M z^K \in z_1^{l_1}z_2^{l_2} A^2(D^{n-2})$. Denote by $P'_{k_1k_2}$ the orthogonal projection from $A^2_{\alpha}(D^n)$ onto

$$
\text{Span}\{z_1^{hN}z_2^{hM} A^2(D^{n-2}); h = 0, 1, 2, \ldots\}.
$$

Then $P'_{k_1k_2}P_M(f) = P_{k_1k_2}P_M(f) = P_M P'_{k_1k_2}(f) \in \mathcal{M}$ for each $f \in \mathcal{M}$. Hence (b) holds. The rest of the proof is obvious. \(\square\)
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