MAXIMAL PROPERTIES OF SOME SUBSEMMIBANDS OF ORDER-PRESERVING FULL TRANSFORMATIONS

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Abstract. Let \([n] = \{1, 2, \ldots, n\}\) be ordered in the standard way. The order-preserving full transformation semigroup \(O_n\) is the set of all order-preserving singular full transformations on \([n]\) under composition. For this semigroup we describe maximal subsemibands, maximal regular subsemibands, locally maximal regular subsemibands, and completely obtain their classification.

1. Introduction

A semigroup is called idempotent-generated or semiband if it is generated by its idempotents. The latter term was introduced by F. Pastijn [7].

Let \([n] = \{1, 2, \ldots, n\}\) ordered in the standard way. We denote by \(\text{Sing}_n\) the semigroup (under composition) of all singular full transformations on \([n]\). We say that a full transformation \(\alpha\) in \(\text{Sing}_n\) is order-preserving if, for all \(x, y \in [n]\), \(x \leq y\) implies \(x\alpha \leq y\alpha\). We denote by \(O_n\) the subsemigroup of \(\text{Sing}_n\) of all order-preserving singular full transformations.

The semigroup \(O_n\) was studied first by Aizenstat [1] and subsequently by many authors (see, for example [2-6], [8-13]). In particular, Howie [5] proved that \(O_n\) is a regular semiband. Garba [2] further proved that each one of the ideals of \(O_n\) is also a regular semiband. Yang [12] classified completely maximal subsemibands and maximal regular subsemibands of \(O_n\). Recently, Xu, Zhao and Li [9] obtained a complete classification of locally maximal subsemibands of \(O_n\). Further, Zhao, Xu and Yang [13] obtained a simpler form of the classification of maximal (regular) subsemibands of \(O_n\), using results of Xu, Zhao and Li [9].

In view of the above work, it is natural to seek a description of the locally maximal regular subsemibands of \(O_n\). In Section 2 we obtain a same simpler form of the classification of maximal (regular) subsemibands of \(O_n\), using a
different approach from Zhao, Xu and Yang [13]. In Section 3 we obtain a classification of locally maximal regular subsemibands of $O_n$.

From Gomes and Howie [3], Green’s equivalences on $O_n$ are characterized as:

\[
\begin{align*}
\alpha L \beta &\iff \text{im}(\alpha) = \text{im}(\beta), \\
\alpha R \beta &\iff \text{ker}(\alpha) = \text{ker}(\beta), \\
\alpha J \beta &\iff |\text{im}(\alpha)| = |\text{im}(\beta)|.
\end{align*}
\]

Thus $O_n$ has $n-1$ $J$-classes: $J_1, J_2, \ldots, J_{n-1}$, where $J_r = \{\alpha \in O_n : |\text{im}(\alpha)| = r\}$.

Gomes and Howie [3] used the notation $[i \rightarrow i + 1]$ for the increasing idempotent $e$ defined by $ie = i+1, xe = x$ ($x \neq i$), and $[i \rightarrow i - 1]$ for the decreasing idempotent $f$ defined by $if = i-1, xf = x$ ($x \neq i$). As usual, we denote by $E(S)$ the set of all idempotents of a subset $S$ of $O_n$. Let $E_{n-1}^+ = \{[i \rightarrow i + 1] : 1 \leq i \leq n-1\}$ and $E_{n-1}^- = \{[i \rightarrow i - 1] : 2 \leq i \leq n\}$ be the increasing and decreasing idempotent sets, respectively. Then $E(J_{n-1}) = E_{n-1}^+ \cup E_{n-1}^-$. 

2. Maximal (regular) subsemibands of $O_n$

Both maximal subsemibands and maximal regular subsemibands of $O_n$ were studied by [12]. Zhao, Xu and Yang [13] obtained a simpler form of the classification of maximal (regular) subsemibands of $O_n$, using results of Xu, Zhao and Li [9]. In this section, we obtain a same simpler form of the classification of maximal (regular) subsemibands of $O_n$, using a different approach from Zhao, Xu and Yang [13]. For convenience, we introduce the following notation from [8].

Let

\[
\begin{align*}
C_n^- &= \{\alpha \in O_n : (\forall x \in [n]) \ x \alpha \leq x\}, \\
C_n^+ &= \{\alpha \in O_n : (\forall x \in [n]) \ x \alpha \geq x\},
\end{align*}
\]

be the semigroups of all singular order-preserving and decreasing full transformations and order-preserving and increasing full transformations on $[n]$, respectively.

As in [4], for any $\alpha \in O_n$, let

\[
\begin{align*}
x \alpha^- &= \left\{ \begin{array}{ll} 
 x \alpha, & x \in [n]^-_{\alpha} \\
 x, & \text{otherwise}, \end{array} \right.
\end{align*}
\]

\[
\begin{align*}
x \alpha^+ &= \left\{ \begin{array}{ll} 
 x \alpha, & x \in [n]^+_{\alpha} \\
 x, & \text{otherwise}, \end{array} \right.
\end{align*}
\]

where $[n]^-_{\alpha} = \{x \in [n] : x \alpha \leq x\}$, and $[n]^+_{\alpha} = \{x \in [n] : x \alpha \geq x\}$. It is obvious that $\alpha^- \in C_n^-$ and $\alpha^+ \in C_n^+$. The following lemma was proved by Higgins [4, page 1053].
Lemma 2.1. Let $\alpha \in \mathcal{O}_n$. Then
$$\alpha = \alpha^+ \alpha^- = \alpha^- \alpha^+,$$
with $\alpha^- \in \mathcal{C}_n^-$, $\alpha^+ \in \mathcal{C}_n^+$. 

For convenience, we use $[n \to n + 1]$ or $[1 \to 0]$ to denote $\emptyset$ (the empty mapping). With this notation, we have:

Lemma 2.2. Let $\alpha \in \mathcal{C}_n^+$. If $k \alpha = k$ for some $1 \leq k \leq n$, then
$$\alpha \in (E_{n-1}^+ \backslash \{[k \to k + 1]\}).$$

Proof. Since $\alpha \in \mathcal{C}_n^+ \subseteq \mathcal{O}_n$, we have that the $\ker(\alpha)$-classes are convex subsets $C$ of $[n]$, in the sense that
$$x, y \in C \text{ and } x \leq z \leq y \implies z \in C.$$ 

Then $\alpha$ can be expressed as
$$\alpha = \left( \begin{array}{ccc} A_1 & A_2 & \cdots & A_r \\ b_1 & b_2 & \cdots & b_r \end{array} \right),$$
where $A_i = \{a_i, a_i + 1, \ldots, a_{i+1} - 1\} \ (1 \leq i \leq r - 1)$, $A_r = \{a_r, a_r + 1, \ldots, n\}$, $1 = a_1 < a_2 < \cdots < a_r$ and $b_1 < b_2 < \cdots < b_r$. Since $\alpha \in \mathcal{C}_n^+$, we have
$$a_i = \min A_i \leq \max A_i \leq (\max A_i) \alpha = b_i, \ 1 \leq i \leq r - 1,$$
$$a_r = \min A_r \leq \max A_r \lim (n) \leq (\max A_r) \alpha = b_r.$$ 

Thus
$$b_r = n \text{ and } a_i \leq b_i, \ i \in [n].$$

Let $e_0$ be the identity mapping on $[n]$, and let
$$E^+(i, j) = [i \to i + 1] \cdot [i + 1 \to i + 2] \cdots [j - 1 \to j], \ 1 \leq i < j \leq n,$$
$$E^+(i, i) = e_0, \ i \in [n].$$

Further, let
$$\beta = E^+(a_r, b_r)E^+(a_{r-1}, b_{r-1}) \cdots E^+(a_1, b_1).$$

We claim that $\alpha = \beta$. To prove that $\alpha = \beta$, take any $x \in [n]$. Suppose that $x \in A_s \ (1 \leq s \leq r)$. Then
$$x \beta = xE^+(a_s, b_s)E^+(a_{s+1}, b_{s+1}) \cdots E^+(a_1, b_1) = b_s = x \alpha.$$ 

Note that $[n \to n + 1] = \emptyset$ (the empty mapping). If $k = n$, then $\alpha = \beta \in (E_{n-1}^+ \cup \{e_0\}) = (E_{n-1}^+ \backslash \{[n \to n + 1]\}) \cup \{e_0\}$. Since $\alpha \in \mathcal{C}_n^+ \subseteq \mathcal{O}_n \subseteq \operatorname{Sing}_n$, we have $\alpha \in (E_{n-1}^+ \backslash \{[n \to n + 1]\})$. If $1 \leq k \leq n - 1$. Note that $b_r = n$. Since $ka = k$, there exists $1 \leq j \leq r - 1$ such that $k \in A_j$ and $b_j = k$. Since $\alpha \in \mathcal{C}_n^+$, we have $b_j = \max A_j = k$ and so $a_{j+1} = \min A_{j+1} = k + 1$. Thus

(2.1) $[k \to k + 1] \notin \{[a_i \to a_i + 1], [a_i + 1 \to a_i + 2], \ldots, [b_i - 1 \to b_i]\}, \ a_i < b_i.$

Note that $E^+(a_i, b_i) = [a_i \to a_i + 1] [a_i + 1 \to a_i + 2] \cdots [b_i - 1 \to b_i]$ if $a_i < b_i$; $E^+(a_i, b_i) = e_0$ if $a_i = b_i$. It follows immediately from (2.1) that
$$E^+(a_i, b_i) \in (E_{n-1}^+ \backslash \{[k \to k + 1]\}) \cup \{e_0\}, \ 1 \leq i \leq r.$$
Lemma 2.4. For any $k \geq 1$.

Let $s\alpha = k$.

Proof. We have

$$\alpha \in \langle E_{n-1}^+ \{[k \rightarrow k + 1]\} \cup \{e_0\} \rangle.$$ 

Since $\alpha \in C_n^+ \subseteq O_n \subseteq Sing_n$, we have

$$\alpha \in \langle E_{n-1}^+ \{[k \rightarrow k + 1]\} \rangle.$$ 

$\square$

Similarly, we can prove:

Lemma 2.3. Let $\alpha \in C_n^-$. If $k\alpha = k$ for some $1 \leq k \leq n$. Then

$$\alpha \in \langle E_{n-1}^+ \{[k \rightarrow k - 1]\} \rangle.$$ 

The following lemma is immediate by definition of $\alpha^+, \alpha^-$:

Lemma 2.4. For any $\alpha \in O_n$, we have

(i) If $k\alpha \leq k$ for some $1 \leq k \leq n$, then $k\alpha^+ = k$.

(ii) If $k\alpha \geq k$ for some $1 \leq k \leq n$, then $k\alpha^- = k$.

For any $s, t \in [n]$, let

(2.2) $M_{st} = \{ \alpha \in O_n : s\alpha \leq s, t\alpha \geq t \}$.

With above notation, we have:

Lemma 2.5. Let $n \geq 3$. Then

$$M_{st} = \langle E(J_{n-1}) \{[s \rightarrow s + 1], [t \rightarrow t - 1]\} \rangle, \ s, t \in [n].$$

Proof. Let $P_{st} = \langle E(J_{n-1}) \{[s \rightarrow s + 1], [t \rightarrow t - 1]\} \rangle$. It is easy to prove that $M_{st}$ is a subsemigroup of $O_n$. It is obvious that $E(J_{n-1}) \{[s \rightarrow s + 1], [t \rightarrow t - 1]\} \subseteq M_{st}$. Then $P_{st} = \langle E(J_{n-1}) \{[s \rightarrow s + 1], [t \rightarrow t - 1]\} \rangle \subseteq M_{st}$.

It remains to prove that $M_{st} \subseteq P_{st}$. Let $\alpha \in M_{st} \subseteq O_n$. By Lemmas 2.1 and 2.4, we have

$$\alpha = \alpha^+ \alpha^- = \alpha^- \alpha^+, \ \alpha^- \in C_n^-, \ \alpha^+ \in C_n^+,$$

and $s\alpha^+ = s, t\alpha^- = t$. Note that $E(J_{n-1}) = E_{n-1}^+ \cup E_{n-1}^-$. Thus, by Lemmas 2.2 and 2.3,

$$\alpha = \alpha^+ \alpha^- \in \langle E_{n-1}^+ \{[s \rightarrow s + 1]\} \rangle \cdot \langle E_{n-1}^- \{[t \rightarrow t - 1]\} \rangle$$

$$\subseteq \langle E(J_{n-1}) \{[s \rightarrow s + 1], [t \rightarrow t - 1]\} \rangle = P_{st}.$$

A subsemiband $S$ of $O_n$ is called maximal subsemiband if for an arbitrary subsemiband $T$ of $O_n$ such that $S \subset T$, then $T = O_n$. Combining [12, Theorem 2.1 and Lemma 2.3], we obtain the following.

Lemma 2.6. Let $n \geq 3$. Let $I_{n-2} = \{ \alpha \in O_n : |im(\alpha)| \leq n - 2 \}$. Then each maximal subsemiband of $O_n$ must be one of the following forms:

(C) $C_s = I_{n-2} \cup \langle E(J_{n-1}) \{[s \rightarrow s + 1]\} \rangle, \ s = 1, 2, \ldots, n - 1$.

(D) $D_s = I_{n-2} \cup \langle E(J_{n-1}) \{[s \rightarrow s - 1]\} \rangle, \ s = 2, 3, \ldots, n$.

Now, it is easy to prove one of the main results of this section:
Theorem 2.7. Let $n \geq 3$. Let $I_{n-2} = \{ \alpha \in \mathcal{O}_n : |\text{im}(\alpha)| \leq n - 2 \}$. Then each maximal subsemiband of $\mathcal{O}_n$ must be one of the following forms:

(A) $A_s = I_{n-2} \cup \{ \alpha \in \mathcal{O}_n : s \alpha \leq s \}$, $s = 1, 2, \ldots, n - 1$.

(B) $B_s = I_{n-2} \cup \{ \alpha \in \mathcal{O}_n : s \alpha \geq s \}$, $s = 2, 3, \ldots, n$.

Proof. Let $M_{ns}$ be as defined in (2.2). Then $M_{s1} = \{ \alpha \in \mathcal{O}_n : s \alpha \leq s \}$ and $M_{ns} = \{ \alpha \in \mathcal{O}_n : s \alpha \geq s \}$. Note that $[1 \to 0] = [n \to n + 1] = \emptyset$ (the empty mapping). Thus, by Lemma 2.5,

$$A_s = I_{n-2} \cup \{ \alpha \in \mathcal{O}_n : s \alpha \leq s \} = I_{n-2} \cup M_{s1}$$

$$= I_{n-2} \cup (E(J_{n-1})\{[s \to s + 1]\}) = C_s,$$

$$B_s = I_{n-2} \cup \{ \alpha \in \mathcal{O}_n : s \alpha \geq s \} = I_{n-2} \cup M_{ns}$$

$$= I_{n-2} \cup (E(J_{n-1})\{[s \to s - 1]\}) = D_s.$$

Hence Theorem 2.7 holds by Lemma 2.6.

A regular subsemiband $S$ of $\mathcal{O}_n$ is called maximal regular subsemiband if for an arbitrary regular subsemiband $T$ of $\mathcal{O}_n$ such that $S \subset T$, then $T = \mathcal{O}_n$. Note that $[1 \to 0] = [n \to n + 1] = \emptyset$ (the empty mapping). Combining [12, Lemma 2.3 and Theorem 4.1], we obtain the following.

Lemma 2.8. Let $n \geq 4$. Let $I_{n-2} = \{ \alpha \in \mathcal{O}_n : |\text{im}(\alpha)| \leq n - 2 \}$. Then each maximal regular subsemiband of $\mathcal{O}_n$ must be the following forms:

(F) $F_s = I_{n-2} \cup (E(J_{n-1})\{[s \to s + 1], [s \to s - 1]\})$, $s = 1, 2, \ldots, n$.

(G) $G_s = I_{n-2} \cup (E(J_{n-1})\{[s \to s + 1], [s + 1 \to s]\})$, $s = 2, 3, \ldots, n - 2$.

Using Lemma 2.5 and Lemma 2.8, the other main result of this section is now established:

Theorem 2.9. Let $n \geq 4$. Let $I_{n-2} = \{ \alpha \in \mathcal{O}_n : |\text{im}(\alpha)| \leq n - 2 \}$. Then each maximal regular subsemiband of $\mathcal{O}_n$ must be the following forms:

(A) $A_s = I_{n-2} \cup \{ \alpha \in \mathcal{O}_n : s \alpha = s \}$, $s = 1, 2, \ldots, n$.

(B) $B_s = I_{n-2} \cup \{ \alpha \in \mathcal{O}_n : s \alpha \leq s, (s + 1)\alpha \geq s + 1 \}$, $s = 2, 3, \ldots, n - 2$.

Proof. Let $M_{ss}$ be as defined in (2.2). Then $M_{s1} = \{ \alpha \in \mathcal{O}_n : s \alpha = s \}$ and $M_{ss(s+1)} = \{ \alpha \in \mathcal{O}_n : s \alpha \leq s, (s + 1)\alpha \geq s + 1 \}$. Note that $[1 \to 0] = [n \to n + 1] = \emptyset$ (the empty mapping). Thus, by Lemma 2.5,

$$A_s = I_{n-2} \cup \{ \alpha \in \mathcal{O}_n : s \alpha = s \} = I_{n-2} \cup M_{ss}$$

$$= I_{n-2} \cup (E(J_{n-1})\{[s \to s + 1], [s \to s - 1]\}) = F_s,$$

$$B_s = I_{n-2} \cup \{ \alpha \in \mathcal{O}_n : s \alpha \leq s, (s + 1)\alpha \geq s + 1 \} = I_{n-2} \cup M_{ss(s+1)}$$

$$= I_{n-2} \cup (E(J_{n-1})\{[s \to s + 1], [s + 1 \to s]\}) = G_s.$$

Hence Theorem 2.9 holds by Lemma 2.8.
3. Locally maximal regular subsemibands of $O_n$

Let $I$ be a subset of $E(J_{n-1})$. A subsemiband $\langle I \rangle$ of $O_n$ is called a locally maximal regular subsemiband if $\langle I \rangle$ is regular, and any regular subsemiband $\langle J \rangle$ ($J \subseteq E(J_{n-1})$) of $O_n$ properly containing $\langle I \rangle$ must be $O_n$. In this section, we obtain a classification of locally maximal regular subsemibands of $O_n$.

The main result of this section is:

**Theorem 3.1.** Let $n \geq 4$. Let $I_{n-2} = \{ \alpha \in O_n : |\text{im}(\alpha)| \leq n-2 \}$. Then each locally maximal regular subsemiband of $O_n$ must be the following forms:

(A) $A_s = \{ \alpha \in O_n : s \alpha = s \}, \ s = 1, 2, \ldots, n$.
(B) $B_s = \{ \alpha \in O_n : s \leq \alpha, (s+1) \alpha \geq s+1 \}, \ s = 2, 3, \ldots, n-2$.

To prove Theorem 3.1, we begin by establishing a series of lemmas. Combining [5, Lemmas 1.2 and 1.3], we know that $O_n$ is generated by $E(J_{n-1})$. Note that $|E(J_{n-1})| = 2n-2$. From the result [3, Theorem 2.8] that the rank of $O_n$ is $2n-2$, we immediately deduce:

**Lemma 3.2.** Let $n \geq 4$. Then

$O_n = \langle E(J_{n-1}) \rangle$ and no proper subset of $E(J_{n-1})$ can generate $O_n$.

It is well known that the characterized forms of the Green’s relations in $\text{Sing}_n$ are the same as in $O_n$ (see Section 1). $\text{Sing}_n$ has $n-1$ $\mathcal{J}$-classes: $SJ_r = \{ \alpha \in \text{Sing}_n : |\text{im}(\alpha)| = r \}, \ r = 1, 2, \ldots, n-1$. Let $SI_r = \{ \alpha \in \text{Sing}_n : |\text{im}(\alpha)| \leq r \}, \ r = 1, 2, \ldots, n-1$.

Then the sets $SI_r$ are two-sided ideal of $\text{Sing}_n$. As usual, we denote by $E(S)$ the set of all idempotents of a subset $S$ of $\text{Sing}_n$. Let $I$ be nonempty subsets of $E(SJ_{n-1})$. It is obvious that $I \subseteq E(I) \cap SJ_{n-1}$. In general, $E(I) \cap SJ_{n-1} \subseteq I$ is false. For example, let

$f = \begin{pmatrix} 2 & 3 & \cdots & n-1 & \{n, 1\} \\ 2 & 3 & \cdots & n-1 & 1 \end{pmatrix}, \ g = \begin{pmatrix} 2 & 3 & \cdots & n-1 & \{n, 1\} \\ 2 & 3 & \cdots & n-1 & n \end{pmatrix},$

then $f, g \in E(SJ_{n-1})$. Let $\eta = f \cdot [n-1 \rightarrow n] \cdot [n-2 \rightarrow n-1] \cdots [1 \rightarrow 2]$, then

$\eta = \begin{pmatrix} 2 & 3 & \cdots & n-1 & \{n, 1\} \\ 3 & 4 & \cdots & n & 2 \end{pmatrix}$

and so $\eta^{n-1} = g$. Clearly, $E_{n-1}^+ \subseteq E(SJ_{n-1})$. Let $I = E_{n-1}^+ \cup \{ f \}$. Then $g = \eta^{n-1} \in \langle I \rangle$ and so $g \in E(I) \cap SJ_{n-1}$. Clearly, $g \notin I$. Thus $E(I) \cap SJ_{n-1} \nsubseteq I$.

However, using Lemma 3.2, we have the following.

**Lemma 3.3.** Let $I$ be a subset of $E(J_{n-1})$. Then

$E(I) \cap J_{n-1} = I$.

**Proof.** Clearly, $I \subseteq E(I) \cap J_{n-1}$. Now, we need to prove that $E(I) \cap J_{n-1} \subseteq I$. Note that $I \subseteq E(I) \cap J_{n-1} \subseteq \langle I \rangle$. Then $\langle I \rangle \subseteq \langle E(I) \cap J_{n-1} \rangle \subseteq \langle I \rangle = \langle I \rangle.$
Lemma 3.4. Let \( E(I \cap J_{n-1}) = \{I\} \). Let \( I^* = E((I \cap J_{n-1}) \setminus I) \). Then \( I \subseteq E(J_{n-1}) \setminus I^* \) and so \( \langle I^* \rangle \subseteq E((I \cap J_{n-1}) \setminus I) \). Thus

\[
E(J_{n-1}) = I^* \cup (E(J_{n-1}) \setminus I^*) \subseteq \langle I^* \rangle \cup (E(J_{n-1}) \setminus I^*) = \langle E(J_{n-1}) \setminus I^* \rangle
\]

and so \( \langle E(J_{n-1}) \setminus I^* \rangle \subseteq \langle E(J_{n-1}) \setminus I^* \rangle \subseteq O_n \). It follows immediately form Lemma 3.2 that

\[
E(J_{n-1}) \setminus I^* = E(J_{n-1})
\]

Then \( I^* = \emptyset \) (the empty set) and so \( E((I \cap J_{n-1}) \subseteq I \). □

Further, we have:

**Lemma 3.5.** Let \( I_{n-2} = \{\alpha \in O_n : |\text{im}(\alpha)| \leq n-2\} \). Let \( I \) and \( J \) be nonempty subsets of \( E(J_{n-1}) \). Then

(i) \( I \subseteq J \Leftrightarrow (I) \subseteq (J) \Leftrightarrow I_{n-2} \cup \langle I \rangle \subseteq I_{n-2} \cup \langle J \rangle \).

(ii) \( I \subset J \Leftrightarrow \langle I \rangle \subseteq \langle J \rangle \Leftrightarrow I_{n-2} \cup \langle I \rangle \subseteq I_{n-2} \cup \langle J \rangle \).

**Proof.** (i) Clearly,

\[
I \subseteq J \Rightarrow (I) \subseteq (J) \Rightarrow I_{n-2} \cup \langle I \rangle \subseteq I_{n-2} \cup \langle J \rangle.
\]

To prove that

\[
I \subseteq J \Rightarrow (I) \subseteq (J) \Rightarrow I_{n-2} \cup \langle I \rangle \subseteq I_{n-2} \cup \langle J \rangle.
\]

It suffices to prove that

\[
I_{n-2} \cup \langle I \rangle \subseteq I_{n-2} \cup \langle J \rangle \Rightarrow I \subseteq J.
\]

Suppose that \( I_{n-2} \cup \langle I \rangle \subseteq I_{n-2} \cup \langle J \rangle \). Then \( (I) \cap J_{n-1} = (I_{n-2} \cup \langle I \rangle) \cap J_{n-1} = (I_{n-2} \cup \langle J \rangle) \cap J_{n-1} \). Thus, by Lemma 3.3,

\[
I = E((I) \cap J_{n-1}) \subseteq E((J) \cap J_{n-1}) = J.
\]

(ii) By (i), we easily deduce that

\[
I = J \Leftrightarrow (I) = (J) \Leftrightarrow I_{n-2} \cup \langle I \rangle = I_{n-2} \cup \langle J \rangle.
\]

It follows immediately that

\[
I \subset J \Leftrightarrow (I) \subset (J) \Leftrightarrow I_{n-2} \cup \langle I \rangle \subset I_{n-2} \cup \langle J \rangle.
\]

□

Now, we can use Lemmas 2.5, 2.8, 3.2 and 3.4 to obtain the following.

**Lemma 3.5.** For \( n \geq 4 \) and \( s \in [n] \), let \( M_{ss} \) be as defined in (2.2). Then \( M_{ss} \) is a locally maximal regular subsemiband of \( O_n \).

**Proof.** Recall that \( M_{ss} = \{\alpha \in O_n : s\alpha = s\} \). Let \( \alpha \in M_{ss} \). If \( |\text{im}(\alpha)| = 1 \), then clearly \( \alpha = \alpha \alpha \) and so \( \alpha \) is regular. If \( |\text{im}(\alpha)| \geq 2 \), suppose that

\[
\alpha = \begin{pmatrix}
A_1 & A_2 & \cdots & A_r \\
A_1 & A_2 & \cdots & A_r
\end{pmatrix} \in M_{ss}.
\]
where \(a_1 < a_2 < \cdots < a_r\), \(\min A_i = \max A_{i-1}, i = 2, 3, \ldots, r\). Since \(\alpha \in M_{ss}\), there exists \(k \in \{1, 2, \ldots, r\}\) such that \(s \in A_k\) and \(a_k = s\). Let

\[
\beta = \begin{pmatrix} B_1 & B_2 & \cdots & B_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix},
\]

where \(b_k = s \in A_k\), \(b_1 = \min A_1\), \(B_1 = \{1, 2, \ldots, a_1\}\), \(a_i < a_{i+1}, s = 2, 3, \ldots, r-1\), and \(B_r = \{a_r+1, \ldots, n\}\). Then \(\alpha = \alpha \beta \alpha\) and \(\beta \in M_{ss}\) (since \(s = a_k \in B_k\) and \(b_k = s\)). Then \(\alpha\) is regular and by Lemma 2.5, we have

\[
(3.1) \quad M_{ss} = \langle E(J_{n-1}) \setminus \{[s \to s + 1], [s \to s - 1]\} \rangle.
\]

Thus \(M_{ss}\) is a regular subsemiband.

For some \(J \subseteq E(J_{n-1})\), let \(\langle J \rangle\) be a regular subsemiband of \(\mathcal{O}_n\) properly containing \(M_{ss}\), see (3.1). Then, by Lemma 3.4(ii),

\[
(3.2) \quad E(J_{n-1}) \setminus \{[s \to s + 1], [s \to s - 1]\} \subseteq J.
\]

Let \(T = I_{n-2} \cup \langle J \rangle\) and let \(F_s\) be as defined in Lemma 2.8, i.e., \(F_s = I_{n-2} \cup \langle E(J_{n-1}) \setminus \{[s \to s + 1], [s \to s - 1]\} \rangle\), see (3.2). Then, by Lemma 3.4(ii), \(F_s \subseteq T\), and since \(\langle J \rangle\) is regular, \(I_{n-2}\) is a regular semiband (see [2]) and also an ideal of \(\mathcal{O}_n\), we deduce that \(T\) is a regular subsemiband of \(\mathcal{O}_n\). Thus, by maximality of \(F_s\) (by Lemma 2.8) and \(F_s \subseteq T\), \(T = I_{n-2} \cup \langle J \rangle = \mathcal{O}_n\). It now follows immediately that \(E(J_{n-1}) \subseteq \langle J \rangle\) and so \(\langle E(J_{n-1}) \rangle \subseteq \langle J \rangle\). Thus, by Lemma 3.2, \(\langle J \rangle = \mathcal{O}_n\).

Also, using Lemmas 2.5, 2.8, 3.2 and 3.4, we have:

**Lemma 3.6.** For \(n \geq 4\) and \(2 \leq s \leq n-2\), let \(M_{s(s+1)}\) be as defined in (2.2). Then \(M_{s(s+1)}\) is a locally maximal regular subsemiband of \(\mathcal{O}_n\).

**Proof.** Recall that \(M_{s(s+1)} = \{\alpha \in \mathcal{O}_n : s \leq s, (s+1)\alpha \geq s + 1\}\). Note that for any \(\alpha \in M_{s(s+1)}\), \(|\text{im}(\alpha)| \geq 2\) (since \(s \leq s\) and \(s+1)\alpha \geq s + 1\)). Consider a typical element

\[
\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_r \\ a_1 & a_2 & \cdots & a_r \end{pmatrix} \in M_{s(s+1)},
\]

where \(a_1 < a_2 < \cdots < a_r\), \(\min A_i = \max A_{i-1}, i = 2, 3, \ldots, r\). Since \(\alpha \in M_{s(s+1)}\), there exist \(k \in \{1, 2, \ldots, r-1\}\) such that \(s \in A_k\), \(s + 1 \in A_{k+1}\) and \(a_k \leq s < s + 1 \leq a_{k+1}\). Let \(c_i = a_i (i \neq k)\) and \(c_k = s\), then \(c_1 < c_2 < \cdots < c_r\). Let

\[
\beta = \begin{pmatrix} B_1 & B_2 & \cdots & B_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix},
\]

where \(b_{k+1} = s + 1 \in A_{k+1}, b_1 = \min A_1\), \(B_1 = \{1, 2, \ldots, c_1\}\), \(B_i = \{x \in [n] : c_{i-1} < x \leq c_i\}, i = 2, 3, \ldots, r-1\) and \(B_r = \{c_{r-1}+1, \ldots, n\}\). Clearly, \(\beta \in \mathcal{O}_n\). Note that \(s = c_k \in B_k\) and \(s + 1 \in B_{k+1}\) (since \(c_k = s < s + 1 \leq a_{k+1} = c_{k+1}\)). It follows that \(s \beta = B_k \beta = b_k = \min A_k\) \((s \in A_k)\) and \((s + 1)\beta = B_{k+1} \beta = b_{k+1} = s + 1\). Thus \(\beta \in M_{s(s+1)}\). Note that \(c_i = a_i (i \neq k)\) and \(a_k \leq s = c_k < s + 1 \leq a_{k+1}\). It follows that \(a_i \in B_i\)
(i = 1, 2, . . . , r) and so α = αβα. Thus α is regular and by Lemma 2.5, we have
\begin{equation}
M_{s(s+1)} = \langle E(J_{n-1}) \setminus \{[s \rightarrow s+1], [s+1 \rightarrow s]\} \rangle.
\end{equation}
Thus \( M_{s(s+1)} \) is a regular subsemiband.

For some \( J \subseteq E(J_{n-1}) \), let \( \langle J \rangle \) be a regular subsemiband of \( \mathcal{O}_{n} \) properly containing \( M_{s(s+1)} \), see (3.3). Then, by Lemma 3.4(ii),
\begin{equation}
E(J_{n-1}) \setminus \{[s \rightarrow s+1], [s+1 \rightarrow s]\} \subseteq J.
\end{equation}
Let \( T = I_{n-2} \cup \langle J \rangle \) and let \( G_{s} \) be as defined in Lemma 2.8, i.e., \( G_{s} = I_{n-2} \cup (E(J_{n-1}) \setminus \{[s \rightarrow s+1], [s+1 \rightarrow s]\}) \). Then, by Lemma 3.4(ii), \( G_{s} \subseteq T \). Since \( \langle J \rangle \) is regular, \( I_{n-2} \) is a regular semiband (see [2]) and also an ideal of \( \mathcal{O}_{n} \), we deduce that \( T \) is a regular subsemiband of \( \mathcal{O}_{n} \). Thus, by maximality of \( G_{s} \) (by Lemma 2.8) and \( G_{s} \subseteq T \), \( T = I_{n-2} \cup \langle J \rangle = \mathcal{O}_{n} \). It follows immediately that \( E(J_{n-1}) \subseteq \langle J \rangle \) and so \( (E(J_{n-1})) \subseteq \langle J \rangle \). Thus, by Lemma 3.2, \( \langle J \rangle = \mathcal{O}_{n} \). □

The following lemma gives a necessary condition for a locally regular subsemiband of \( \mathcal{O}_{n} \) to be maximal.

**Lemma 3.7.** Let \( I \) be a nonempty set of \( E(J_{n-1}) \). If \( \langle I \rangle \) is a locally maximal regular subsemiband of \( \mathcal{O}_{n} \), then \( T = I_{n-2} \cup \langle I \rangle \) is a maximal regular subsemiband of \( \mathcal{O}_{n} \).

**Proof.** Suppose that \( \langle I \rangle \) is a locally maximal regular subsemiband of \( \mathcal{O}_{n} \). Let \( M \) be a regular subsemiband of \( \mathcal{O}_{n} \) properly containing \( T \). Since \( M = \langle E(M) \rangle \) and \( I_{n-2} \subseteq M \) (since \( T \subseteq M \)), we have \( M = I_{n-2} \cup M = I_{n-2} \cup (E(M \cap J_{n-1})) \) and so
\[
I_{n-2} \cup \langle I \rangle = T \subseteq M = I_{n-2} \cup (E(M \cap J_{n-1}))).
\]
Note that \( E(M \cap J_{n-1}) \subseteq E(J_{n-1}) \). Then, by Lemma 3.4(ii), \( \langle I \rangle \subseteq \langle E(M \cap J_{n-1}) \rangle \) and so, by the locally maximality of \( \langle I \rangle \), \( (E(M \cap J_{n-1})) = \mathcal{O}_{n} \). Thus \( M = \mathcal{O}_{n} \) and so \( T = I_{n-2} \cup \langle I \rangle \) is a maximal regular subsemiband of \( \mathcal{O}_{n} \). □

Now, we can prove Theorem 3.1.

**Proof Theorem 3.1.** Let \( M_{s} \) and \( M_{s(s+1)} \) be defined earlier. It is obvious that
\begin{equation}
A_{s} = \{ \alpha \in \mathcal{O}_{n} : s \alpha = s \} = M_{s},
\end{equation}
\begin{equation}
B_{s} = \{ \alpha \in \mathcal{O}_{n} : s \alpha \leq s, (s+1) \alpha \geq s+1 \} = M_{s(s+1)}.
\end{equation}
Thus, by Lemmas 3.5 and 3.6, \( A_{s} \) and \( B_{s} \) are locally maximal regular subsemibands of \( \mathcal{O}_{n} \).

Conversely, we shall prove that each locally maximal regular subsemiband of \( \mathcal{O}_{n} \) must be of the form \( A_{s} \) or \( B_{s} \). For some \( I \subseteq E(J_{n-1}) \), let \( \langle I \rangle \) is a locally maximal regular subsemiband of \( \mathcal{O}_{n} \). Then, by Lemma 3.7, \( T = I_{n-2} \cup \langle I \rangle \) is a maximal regular subsemiband of \( \mathcal{O}_{n} \). Thus, by Lemma 2.8, there exists \( s \in [n] \) such that \( T = F_{s} = I_{n-2} \cup (E(J_{n-1}) \setminus \{[s \rightarrow s+1], [s+1 \rightarrow s]\}) \) or there exists...
$s \in \{2, 3, \ldots, n - 2\}$ such that $T = G_s = I_{n-2} \cup \langle E(J_{n-1})\backslash\{[s \to s + 1], [s + 1 \to s]\}\rangle$. It follows immediately from Lemmas 3.4 that

\[
\langle I \rangle = \langle E(J_{n-1})\backslash\{[s \to s + 1], [s \to s - 1]\}\rangle \text{ or } \langle I \rangle = \langle E(J_{n-1})\backslash\{[s \to s + 1], [s + 1 \to s]\}\rangle.
\]

Thus, by Lemma 2.5 and (3.4), (3.5),

\[
\langle I \rangle = \langle E(J_{n-1})\backslash\{[s \to s + 1], [s \to s - 1]\}\rangle = M_{ss} = A_s \text{ or } \\
\langle I \rangle = \langle E(J_{n-1})\backslash\{[s \to s + 1], [s + 1 \to s]\}\rangle = M_{s(s+1)} = B_s.
\]

\[\square\]

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