A NOTE ON THE $q$-ANALOGUE OF KIM’S $p$-ADIC log GAMMA TYPE FUNCTIONS ASSOCIATED WITH $q$-EXTENSION OF GENOCCHI AND EULER NUMBERS WITH WEIGHT $\alpha$

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Abstract. In this paper, we introduce the $q$-analogue of $p$-adic log gamma functions with weight alpha. Moreover, we give a relationship between weighted $p$-adic $q$-log gamma functions and $q$-extension of Genocchi and Euler numbers with weight alpha.

1. Introduction

Assume that $p$ is a fixed odd prime number. Throughout this paper $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of integers, the field of $p$-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Also we denote $\mathbb{N}^* = \mathbb{N} \cup \{0\}$ and $\exp(x) = e^x$. Let $v_p: \mathbb{C}_p \to \mathbb{Q} \cup \{\infty\}$ ($\mathbb{Q}$ is the field of rational numbers) denote the $p$-adic valuation of $\mathbb{C}_p$ normalized so that $v_p(p) = 1$. The absolute value on $\mathbb{C}_p$ will be denoted as $|\cdot|$, and $|x|_p = p^{-v_p(x)}$ for $x \in \mathbb{C}_p$. When one talks of $q$-extensions, $q$ is considered in many ways, e.g. as an indeterminate, a complex number $q \in \mathbb{C}$, or a $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, we assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we assume $|1 - q|_p < p^{-\frac{1}{p-1}}$, so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. We use the following notation

\[ (1.1) \quad [x]_q = \frac{1 - q^x}{1 - q}, \quad [x]_{-q} = \frac{1 - (-q)^x}{1 + q}, \]

where $\lim_{q \to 1} [x]_q = x$; cf. [1-21].

For a fixed positive integer $d$, we set

\[ X = X_d = \lim_{\mathcal{N}} \mathbb{Z}/dp^{\mathcal{N}}\mathbb{Z}, \quad X^* = \bigcup_{0 < a < dp \atop (a, p) = 1} a + dp\mathbb{Z}_p \]

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and 
\[ a + dp^N \mathbb{Z}_p = \left\{ x \in X \mid x \equiv a \pmod{dp^N} \right\}, \]
where \( a \in \mathbb{Z} \) satisfies the condition \( 0 \leq a < dp^N \) (see [6, Section 2]).

It is known that 
\[ \mu_q (x + p^N \mathbb{Z}_p) = \frac{q^x}{[p^N]_q} \]
is a distribution on \( X \) for \( q \in \mathbb{C}_p \) with \( |1 - q|_p \leq 1 \).

Let \( UD(\mathbb{Z}_p) \) be the set of uniformly differentiable function on \( \mathbb{Z}_p \). We say that \( f \) is a uniformly differentiable function at a point \( a \in \mathbb{Z}_p \), if the difference quotient
\[ F_f (x, y) = \frac{f(x) - f(y)}{x - y} \]
has a limit \( f'(a) \) as \((x, y) \to (a, a)\) and denote this by \( f \in UD(\mathbb{Z}_p) \).

The \( p \)-adic \( q \)-integral of the function \( f \in UD(\mathbb{Z}_p) \) is defined by
\[ (1.2) \quad I_q (f) = \int_{\mathbb{Z}_p} f(x) \, d\mu_q (x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x. \]

The bosonic integral is considered by Kim as the bosonic limit \( q \to 1 \),
\[ I_1 (f) = \lim_{q \to 1} I_q (f), \]
Similarly, the \( p \)-adic fermionic integration on \( \mathbb{Z}_p \) was defined by Kim as follows:
\[ I_{-q} (f) = \lim_{q \to -q} I_q (f) = \int_{\mathbb{Z}_p} f(x) \, d\mu_{-q} (x). \]

Let \( q \to 1 \). Then we have \( p \)-adic fermionic integral on \( \mathbb{Z}_p \) as follows:
\[ I_{-1} (f) = \lim_{q \to 1} I_{-q} (f) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x) (-1)^x. \]

Stirling asymptotic series are defined by
\[ (1.3) \quad \log \left( \frac{\Gamma (x + 1)}{\sqrt{2\pi}} \right) = \left( x - \frac{1}{2} \right) \log x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} \frac{B_n}{x^n} - x, \]
where \( B_n \) are familiar \( n \)-th Bernoulli numbers (cf. [5, 6, 21]).

Recently, Araci, Acikgoz and Seo defined \( q \)-Genocchi polynomials with weight \( \alpha \) in [1, 2] by the means of generating function:
\[ (1.4) \quad \sum_{n=0}^{\infty} \tilde{G}_{n,q}^{(\alpha)} (x) \frac{t^n}{n!} = t \int_{\mathbb{Z}_p} e^{[x + \xi]_q^{(\alpha)} t} \, d\mu_{-q} (\xi). \]

So from above, we easily get Witt’s formula of \( q \)-Genocchi polynomials with weight \( \alpha \) as follows:
\[ (1.5) \quad \frac{\tilde{G}_{n,q}^{(\alpha)} (x)}{n+1} = \int_{\mathbb{Z}_p} [x + \xi]_q^{(\alpha)} \, d\mu_{-q} (\xi), \]
where \( \widetilde{G}^{(\alpha)}_{n,q} (0) := \overline{G}^{(\alpha)}_{n,q} \) are called the \( q \)-extension of Genocchi numbers with weight \( \alpha \) (cf. [1, 2]).

For any non-negative integer \( n \), Ryoo [17] defined the \( q \)-Euler numbers with weight \( \alpha \) as follows:

\[
(1.6) \quad \widetilde{E}^{(\alpha)}_{n,q} = \int_{\mathbb{Z}_p} [\xi]_{q^n} \, d\mu - q(\xi).
\]

By (1.5) and (1.6), we get the following proposition:

**Proposition 1.** The following identity holds:

\[
(1.7) \quad \widetilde{E}^{(\alpha)}_{n,q} = \frac{\overline{G}^{(\alpha)}_{n+1,q}}{n+1}.
\]

In recent years, T. Kim studied the new formula of the \( p \)-adic \( q \)-analogue of \( \log \left( \Gamma \left( \frac{x+1}{2} \right) \right) \), in which he derived interesting properties of \( q \)-Euler and \( q \)-Bernoulli numbers. By the same motivation, we introduce the \( q \)-analogue of \( p \)-adic log gamma functions with weight alpha. Furthermore, we get interesting properties of \( q \)-extension of Genocchi numbers with weight alpha.

**On \( p \)-adic log \( \Gamma \) function with weight \( \alpha \)**

In this section, from (1.2), we start by the following expression:

\[
(1.8) \quad q^n I_{-q} (f_n) + (-1)^{n-1} I_{-q} (f) = [2]_q \sum_{l=0}^{n-1} q^l (-1)^{n-1-l} f (l),
\]

where \( f_n (x) = f (x+n) \) and \( n \in \mathbb{N} \) (see [3, 5, 7, 15]).

In particular for \( n = 1 \) into (1.8), we easily see that

\[
(1.9) \quad q I_{-q} (f_1) + I_{-q} (f) = [2]_q f (0).
\]

By the easy application, it is simple to indicate as follows:

\[
(1.10) \quad ((1 + x) \log (1 + x))' = 1 + \log (1 + x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} x^n,
\]

where \( ((1 + x) \log (1 + x))' = \frac{d}{dx} ((1 + x) \log (1 + x)) \).

By the expression of (1.10), we can derive

\[
(1.11) \quad (1 + x) \log (1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} x^{n+1} + x + c, \text{ where } c \text{ is a constant}.
\]

If we substitute \( x = 0 \), we have \( c = 0 \). By (1.10) and (1.11), we easily see that

\[
(1.12) \quad (1 + x) \log (1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} x^{n+1} + x.
\]
It is considered by T. Kim for $q$-analogue of $p$ adic locally analytic function on $\mathbb{C}_p \backslash \mathbb{Z}_p$ as follows:

$$G_{p,q}(x) = \int_{\mathbb{Z}_p} [x + \xi]_q \left( \log [x + \xi]_q - 1 \right) d\mu_{-q}(\xi) \quad \text{(for details, see [5, 6]).}$$

By the same motivation of (1.13), $q$-analogue of $p$-adic locally analytic function on $\mathbb{C}_p \backslash \mathbb{Z}_p$ with weight $\alpha$ as

$$G^{(\alpha)}_{p,q}(x) = \int_{\mathbb{Z}_p} [x + \xi]_q^\alpha \left( \log [x + \xi]_q^\alpha - 1 \right) d\mu_{-q}(\xi).$$

In particular $\alpha = 1$ into (1.14), we easily see that, $G^{(1)}_{p,q}(x) = G_{p,q}(x)$.

It is easy to show that,

$$[x + \xi]_q^\alpha = 1 + q^\alpha + q^{2\alpha} + \ldots + q^{\alpha(x + \xi - 1)}$$

$$= 1 + q^\alpha + q^{2\alpha} + \ldots + q^{\alpha(x-1)} + q^{\alpha x} \left( 1 + q^\alpha + q^{2\alpha} + \ldots + q^{\alpha(\xi-1)} \right)$$

$$= [x]_q^\alpha + q^{\alpha x} [\xi]_q^\alpha.$$

We set $x \to q^{\alpha x} [\xi]_q^\alpha$ into (1.12) and by using (1.15), we get an interesting formula:

$$[x + \xi]_q^\alpha \left( \log [x + \xi]_q^\alpha - 1 \right)$$

$$= \left( [x]_q^\alpha + q^{\alpha x} [\xi]_q^\alpha \right) \log [x]_q^\alpha + \sum_{n=1}^{\infty} \frac{(-q^{\alpha x})^{n+1} [\xi]_q^{n+1}}{n(n+1)(n+2)} [x]_q^{n+2} - [x]_q^\alpha.$$

If we substitute $\alpha = 1$ into (1.16), we get Kim’s $q$-analogue of $p$-adic log gamma function (for details, see [5]).

From expressions of (1.2) and (1.16), we obtain worthwhile and interesting theorems as follows:

**Theorem 1.** For $x \in \mathbb{C}_p \backslash \mathbb{Z}_p$ the following

$$G^{(\alpha)}_{p,q}(x) = \left( [x]_q^\alpha + \frac{q^{\alpha x} \widetilde{G}^{(\alpha)}_{2,q}}{2} \right) \log [x]_q^\alpha + \sum_{n=1}^{\infty} \frac{(-q^{\alpha x})^{n+1} [\xi]_q^{n+1}}{n(n+1)(n+2)} [x]_q^{n+2} - [x]_q^\alpha$$

is true.

**Theorem 2.** For $x \in \mathbb{C}_p \backslash \mathbb{Z}_p$ the following

$$G^{(\alpha)}_{p,q}(x) = \left( [x]_q^\alpha + q^{\alpha x} \widetilde{E}^{(\alpha)}_{1,q} \right) \log [x]_q^\alpha + \sum_{n=1}^{\infty} \frac{(-q^{\alpha x})^{n+1} \widetilde{E}^{(\alpha)}_{n+1,q}}{n(n+1)} [x]_q^n - [x]_q^\alpha$$

is true.
References

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