RIGIDITY THEOREMS IN THE HYPERBOLIC SPACE

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Abstract. As a suitable application of the well known generalized maximum principle of Omori-Yau, we obtain rigidity results concerning to a complete hypersurface immersed with bounded mean curvature in the $(n+1)$-dimensional hyperbolic space $H^{n+1}$. In our approach, we explore the existence of a natural duality between $H^{n+1}$ and the half $H^{n+1}$ of the de Sitter space $S_1^{n+2}$, which models the so-called steady state space.

1. Introduction

In this paper, we are interested in the study of complete non-compact hypersurfaces immersed with bounded mean curvature in the $(n+1)$-dimensional hyperbolic space $H^{n+1}$. Before giving details on our work, we present a brief outline of the main results related to our ones.

In [1], L. J. Alías and M. Dajczer studied complete surfaces properly immersed in $H^3$ which are contained between two horospheres, obtaining a Bernstein-type result for the case of constant mean curvature $-1 \leq H \leq 1$.

The author and A. Caminha have studied in [3] complete vertical graphs of constant mean curvature in $H^{n+1}$. Under appropriate restriction on the growth of the height function, they obtained necessary conditions for the existence of such a graph. Furthermore, for complete surfaces of nonnegative Gaussian curvature, they obtained a Bernstein-type theorem in $H^3$.

More recently, by applying a technique of S. T. Yau [13], the author jointly with F. E. C. Camargo and A. Caminha [2] have also obtained Bernstein-type results in $H^{n+1}$.

Here, under an appropriated restriction on the normal angle of the hypersurface (that is, the angle between the Gauss map of the hypersurface and the unitary vector field which determines on $H^{n+1}$ a codimension one foliation by horospheres; see Section 3), we obtain rigidity theorems concerning to a complete hypersurface immersed with bounded mean curvature in $H^{n+1}$. In our approach, we explore the existence of a natural duality between $H^{n+1}$ and the...
half $\mathcal{H}^{n+1}$ of the de Sitter space $S_1^{n+1}$, which models the so-called steady state space (cf. Sections 2 and 3).

We prove the following (cf. Theorem 3.3; see also Corollaries 3.5 and 3.6):

Let $\psi : \Sigma^n \to \mathbb{H}^{n+1}$ be a complete hypersurface, with bounded second fundamental form $A$. Suppose that the (not necessarily constant) mean curvature $H$ of $\Sigma^n$ is such that $0 \leq H \leq 1$. If $\Sigma^n$ is under a horosphere of $\mathbb{H}^{n+1}$ and its normal angle $\theta$ satisfies $\cos \theta \geq \sup_{\Sigma} H$, then $\Sigma^n$ is a horosphere and the image of its Lorentz Gauss map is exactly a hyperplane of $\mathbb{H}^{n+1}$.

We want to point out that our restriction on the normal angle of the hypersurface is motivated by a gradient estimate due to R. López and S. Montiel [6] (for more details, see Remark 3.4).

Furthermore, by applying a classical result due to A. Huber [4] concerned with parabolic surfaces, we also prove the following (cf. Theorem 3.7):

Let $\psi : \Sigma^2 \to \mathbb{H}^3$ be a complete surface of nonnegative Gaussian curvature and with (not necessarily constant) mean curvature $0 \leq H \leq 1$. If the normal angle $\theta$ of $\Sigma^2$ satisfies $\cos \theta \geq H$, then $\Sigma^2$ is a horosphere and the image of its Lorentz Gauss map is exactly a plane of $\mathbb{H}^3$.

2. The steady state space $\mathcal{H}^{n+1}$

In order to study the geometry of the Gauss map of a hypersurface immersed in the hyperbolic space, we need some preliminaries of Lorentz geometry.

Let $L^{n+2}$ denote the $(n+2)$-dimensional Lorentz-Minkowski space $(n \geq 2)$, that is, the real vector space $\mathbb{R}^{n+2}$ endowed with the Lorentz metric defined by

$$\langle v, w \rangle = \sum_{i=1}^{n+1} v_i w_i - v_{n+2} w_{n+2}$$

for all $v, w \in \mathbb{R}^{n+2}$. We define the $(n+1)$-dimensional de Sitter space $S_1^{n+1}$ as the following hyperquadric of $L^{n+2}$:

$$S_1^{n+1} = \{ p \in L^{n+2}; \langle p, p \rangle = 1 \}.$$  

The induced metric from $\langle \cdot, \cdot \rangle$ makes $S_1^{n+1}$ into a Lorentz manifold with constant sectional curvature one. Moreover, if $p \in S_1^{n+1}$, we can put

$$T_p (S_1^{n+1}) = \{ v \in L^{n+2}; \langle v, p \rangle = 0 \}.$$  

Let $a \in L^{n+2}$ be a non-zero null vector in the past half of the null cone (with vertex in the origin), that is, $\langle a, a \rangle = 0$ and $\langle a, e_{n+2} \rangle > 0$, where $e_{n+2} = (0, \ldots, 0, 1)$. Then the open region of the de Sitter space $S_1^{n+1}$, given by

$$\mathcal{H}^{n+1} = \{ x \in S_1^{n+1}; \langle x, a \rangle > 0 \}$$

is the so-called steady state space. Observe that $\mathcal{H}^{n+1}$ is extendible and, so, non-compact, being only half a de Sitter space. Its boundary, as a subset of $S_1^{n+1}$, is the null hypersurface

$$\{ x \in S_1^{n+1}; \langle x, a \rangle = 0 \},$$
whose topology is that of \( \mathbb{R} \times S^{n-1} \) (cf. [7]).

Now, we shall consider in \( H^{n+1} \) the timelike field
\[
\mathcal{K} = (x, a) \times x - a.
\]

We easily see that
\[
\nabla_V \mathcal{K} = (x, a) V \text{ for all } V \in \mathfrak{X}(H^{n+1}),
\]
that is, \( \mathcal{K} \) is closed and conformal field on \( H^{n+1} \) (cf. [5], Section 2). Then, from Proposition 1 of [9], we have that the \( n \)-dimensional distribution \( \mathcal{D} \) defined on \( H^{n+1} \) by
\[
p \in H^{n+1} \mapsto \mathcal{D}(p) = \{ v \in T_p H^{n+1}; \langle K(p), v \rangle = 0 \}
\]
determines a codimension one spacelike foliation \( \mathcal{F}(K) \) which is oriented by \( \mathcal{K} \).

Moreover (cf. [10], Example 1), the leaves of \( \mathcal{F}(K) \) are hyperplanes
\[
\mathcal{L}_\rho = \{ x \in S^{n+1}; \langle x, a \rangle = \rho \}, \quad \rho > 0,
\]
which are totally umbilical hypersurfaces of \( H^{n+1} \) isometric to the Euclidean space \( \mathbb{R}^n \), and having constant mean curvature 1 with respect to the unit past-directed normal fields
\[
\eta_\rho (x) = x - \frac{1}{\rho} a, \quad x \in \mathcal{L}_\rho.
\]

3. Rigidity results in \( H^{n+1} \)

In this section, instead of the more commonly used half-space model for the \((n+1)\)-dimensional hyperbolic space, we consider the warped product model
\[
H^{n+1} = \mathbb{R} \times_c \mathbb{R}^n.
\]
It can easily be seen that the fibers \( M_{t_0} = \{ t_0 \} \times \mathbb{R}^n \) of the warped product model are precisely the horospheres of \( H^{n+1} \). Moreover, these have constant mean curvature 1 if we take the orientation given by the unit normal vector field \( N = -\partial_t \) (cf. [8], Example 3 of Section 4).

Another useful model for \( H^{n+1} \) is the so-called Lorentz model, obtained by furnishing the hyperquadric
\[
\{ p \in \mathbb{L}^{n+2}; \langle p, p \rangle = -1, p_{n+2} > 0 \}
\]
with the (Riemannian) metric induced by the Lorentz metric of \( \mathbb{L}^{n+2} \). In this setting, if \( a \in \mathbb{L}^{n+2} \) denotes a fixed null vector as in the beginning of the previous section, a typical horosphere is
\[
L_\tau = \{ p \in H^{n+1}; \langle p, a \rangle = \tau \},
\]
where \( \tau \) is a positive real number. A straightforward computation shows that
\[
\xi_\rho = -p - \frac{1}{\tau} a \in H^{n+1}
\]
is a unit normal vector field along \( L_\tau \), with respect to which \( L_\tau \) has mean curvature 1 (cf. [6], Section 3).
In the context of the Lorentz model of $\mathbb{H}^{n+1}$, we say that a hypersurface $\psi: \Sigma \rightarrow \mathbb{H}^{n+1}$ is under a horosphere $L_\tau$ when $\langle \psi, a \rangle \leq \tau$. In this case, if we consider the warped model of $\mathbb{H}^{n+1}$, we easily see that the height function $h = \pi \circ \psi$ of $\Sigma^n$ is bounded from above.

Now, we present our analytical framework.

**Lemma 3.1** ([3], Proposition 3.2). Let $\psi: \Sigma \rightarrow \mathbb{R} \times fM^n$ be a hypersurface immersed into a Riemannian warped product $\mathbb{R} \times fM^n$, with Gauss map $N$. Then, by denoting $h = \pi \circ \psi$ the height function of $\Sigma^n$, we have

$$\Delta h = (\ln f)'(h)(n - |\nabla h|^2) + nH(N, \partial_t).$$

We also will need the well known generalized Maximum Principle due to H. Omori and S. T. Yau [11, 12].

**Lemma 3.2.** Let $\Sigma^n$ be an $n$-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below and $u: \Sigma^n \rightarrow \mathbb{R}$ be a smooth function which is bounded from above on $\Sigma^n$. Then there is a sequence of points $\{p_k\}$ in $\Sigma^n$ such that

$$\lim_{k \rightarrow \infty} u(p_k) = \sup_{\Sigma} u, \quad \lim_{k \rightarrow \infty} |\nabla u(p_k)| = 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} \Delta u(p_k) \leq 0.$$

In what follows, we will consider an isometry $\Phi$ between the warped product and Lorentz models of $\mathbb{H}^{n+1}$ which carries $(\partial_t)_q$ to $\Phi^*(\partial_t) = \xi_{\Phi(q)}$ (such isometry is given in [1]). In this setting, it is natural to consider the Lorentz Gauss map of $\Sigma$ with respect to $N$ as given by

$$\Sigma^n \rightarrow \mathcal{H}^{n+1}, \quad p \mapsto -\Phi_*(N_p).$$

Given a hypersurface $\Sigma^n$ in $\mathbb{H}^{n+1}$ whose Gauss map satisfies $\langle N, \partial_t \rangle < 0$, we define the normal angle $\theta$ of $\Sigma^n$ as being the smooth function $\theta: \Sigma^n \rightarrow [0, \frac{\pi}{2}]$ given by

$$0 \leq \cos \theta = -\langle N, \partial_t \rangle \leq 1.$$

Now, we can state and prove our main result.

**Theorem 3.3.** Let $\psi: \Sigma^n \rightarrow \mathbb{H}^{n+1}$ be a complete hypersurface, with bounded second fundamental form $A$. Suppose that the mean curvature $H$ of $\Sigma^n$ is such that $0 \leq H \leq 1$. If $\Sigma^n$ is under a horosphere of $\mathbb{H}^{n+1}$ and its normal angle $\theta$ satisfies $\cos \theta \geq \sup_{\Sigma^n} H$, then $\Sigma^n$ is a horosphere and the image of its Lorentz Gauss map is exactly a hyperplane of $\mathbb{H}^{n+1}$.

**Proof.** Initially, let us consider $X \in \mathcal{X}(\Sigma)$ with $|X| = 1$. It follows from Gauss equation that

$$\text{Ric}_\Sigma(X) = 1 - n + nH\langle AX, X \rangle - \langle AX, AX \rangle,$$

where $\text{Ric}_\Sigma$ stands for the Ricci curvature of $\Sigma^n$. Hence,

$$\text{Ric}_\Sigma \geq 1 - n - nH|A| - |A|^2.$$
Thus, since $H$ and $A$ are supposed to be bounded, we conclude that $\text{Ric}_{\Sigma}$ is bounded from below on $\Sigma^n$.

Now, from Lemma 3.1, we have that

$$\Delta h = n(1 + H\langle N, \partial_t \rangle) - |\nabla h|^2.$$ 

On the other hand, since $\Sigma^n$ is supposed to be under a horosphere of $\mathbb{H}^{n+1}$ and its Ricci curvature is bounded from below, we are in position to apply Lemma 3.2 to the function $h$, obtaining a sequence $\{p_k\}$ in $\Sigma^n$ such that

$$\lim_{k \to \infty} h(p_k) = \sup_{\Sigma} h, \quad \lim_{k \to \infty} |\nabla h(p_k)| = 0 \quad \text{and} \quad \lim_{k \to \infty} \Delta h(p_k) \leq 0.$$ 

Consequently, since the functions $H$ and $\langle N, \partial_t \rangle$ are bounded on $\Sigma^n$, we get a subsequence $\{p_{k_j}\}$ of $\{p_k\}$ such that

$$0 \geq \lim_{j \to \infty} \Delta h(p_{k_j}) \geq n \left(1 - \lim_{j \to \infty} H(p_{k_j})\right) \geq 0.$$ 

Then, $\lim_{j \to \infty} H(p_{k_j}) = 1$, and $\sup_{\Sigma} H = 1$. Thus, since we are supposing that the normal angle $\theta$ of $\Sigma^n$ satisfies $\cos \theta \geq \sup_{\Sigma} H$, we get that $\langle N, \partial_t \rangle = -\cos \theta = -1$ on $\Sigma^n$ and, hence, $\Sigma^n$ is a horosphere. Moreover, by considering an isometry $\Phi$ between the warped product and Lorentz models of $\mathbb{H}^{n+1}$, we get

$$\langle N, a \rangle = \langle -\partial_t, a \rangle = \langle -\xi_{\Phi}, a \rangle = \langle \psi, a \rangle$$

and, therefore, we conclude that $N(\Sigma)$ is exactly a hyperplane of $\mathbb{H}^{n+1}$. □

**Remark 3.4.** Let $\psi : \Sigma^n \to \mathbb{H}^{n+1}$ be a immersion from a compact manifold $\Sigma^n$ with mean convex boundary $\partial \Sigma$ contained into a horosphere $L_\tau$, for some $\tau > 0$. Suppose that $\psi$ has constant mean curvature $0 \leq H \leq 1$. From the gradient estimate (19) of [6], taking into account our choice of the orientation $N$ of $\Sigma^n$, we get

$$\langle N, a \rangle \geq H\tau.$$ 

Consequently, by supposing that $\Sigma^n$ is under the horosphere $L_\tau$, we conclude that its normal angle $\theta$ satisfies

$$\cos \theta = -\langle N, \partial_t \rangle = \frac{1}{\langle \psi, a \rangle} \langle N, a \rangle \geq \frac{1}{\tau} \langle N, a \rangle \geq H.$$ 

Since for a hypersurface $\Sigma^n$ immersed in $\mathbb{H}^{n+1}$ we have that

$$|A|^2 = n^2H^2 - n(n-1)(R + 1),$$

where $R$ denotes de scalar curvature of $\Sigma^n$, we get:

**Corollary 3.5.** Let $\psi : \Sigma^n \to \mathbb{H}^{n+1}$ be a complete hypersurface, with scalar curvature $R$ bounded from below. Suppose that the mean curvature $H$ of $\Sigma^n$ is such that $0 \leq H \leq 1$. If $\Sigma^n$ is under a horosphere of $\mathbb{H}^{n+1}$ and its normal angle $\theta$ satisfies $\cos \theta \geq \sup_{\Sigma} H$, then $\Sigma^n$ is a horosphere and the image of its Lorentz Gauss map is exactly a hyperplane of $\mathbb{H}^{n+1}$. 


By using once more the existence of a natural duality between $\mathbb{H}^{n+1}$ and $\mathcal{H}^{n+1}$, we obtain the following consequence of Theorem 3.3.

**Corollary 3.6.** Let $\psi: \Sigma^n \to \mathbb{H}^{n+1}$ be a complete hypersurface, with bounded second fundamental form $A$. Suppose that the mean curvature $H$ of $\Sigma^n$ is such that $0 \leq H \leq 1$. If $\Sigma^n$ is under a horosphere $L_\tau$ and the image of its Lorentz Gauss map $N(\Sigma)$ is contained in the closure of the interior domain enclosed by a hyperplane $L_\rho$ of $\mathcal{H}^{n+1}$, with $\frac{\rho}{\tau} \geq \sup_{\Sigma} H$, then $\Sigma^n$ is a horosphere and the image of its Lorentz Gauss map is exactly a hyperplane of $\mathcal{H}^{n+1}$.

**Proof.** By considering again an isometry $\Phi$ between the warped product and Lorentz models of $\mathbb{H}^{n+1}$, we get

$$\langle N, \partial_t \rangle = \langle N, -\psi - \frac{1}{\langle \psi, a \rangle}a \rangle = -\frac{1}{\langle \psi, a \rangle} (N, a).$$

Consequently, since we are supposing that $\Sigma^n$ is under the horosphere $L_\tau$ and that its Lorentz Gauss map $N(\Sigma)$ is contained in the closure of the interior domain enclosed by the hyperplane $L_\rho$,

$$\cos \theta = -\langle N, \partial_t \rangle \geq \frac{\rho}{\tau}.$$

Therefore, our hypothesis on the image of the Lorentz Gauss map of $\Sigma^n$ amounts to

$$\cos \theta \geq \sup_{\Sigma} H$$

and, hence, the result follows from Theorem 3.3. \qed

In the 3-dimensional case, we obtain the following rigidity result concerning to complete surfaces of nonnegative Gaussian curvature.

**Theorem 3.7.** Let $\psi: \Sigma^2 \to \mathbb{H}^3$ be a complete surface of nonnegative Gaussian curvature and with mean curvature $0 \leq H \leq 1$. If the normal angle $\theta$ of $\Sigma^2$ satisfies $\cos \theta \geq H$, then $\Sigma^2$ is a horosphere and the image of its Lorentz Gauss map is exactly a plane of $\mathcal{H}^3$.

**Proof.** By applying Lemma 3.1, we get

$$\Delta e^{-h} = e^{-h} (|\nabla h|^2 - \Delta h) = 2e^{-h} (|\nabla h|^2 - 1 - H \langle N, \partial_t \rangle).$$

On the other hand, since $h = \pi_{\mathbb{H}^3}$, one has

$$\nabla h = \nabla (\pi_{\mathbb{H}^3}) = (\nabla \pi_{\mathbb{H}^3})^\top = \partial_t^\top$$

$$= \partial_t - \langle N, \partial_t \rangle N,$$

where $\nabla$ denotes the gradient with respect to the metric of $\mathbb{H}^3$, and $(\cdot)^\top$ the tangential component of a vector field in $\mathfrak{X}(\mathbb{H}^3)$ along $\Sigma^2$. Consequently, $|\nabla h|^2 = 1 - \cos^2 \theta$. Thus,

$$\Delta e^{-h} = 2e^{-h} \cos \theta (H - \cos \theta)$$
and, hence, our hypothesis on the normal angle $\theta$ of $\Sigma^2$ guarantees that the function $e^{-h}$ is a superharmonic positive function on $\Sigma$. However, a classical result due to A. Huber [4] assures that complete surfaces of nonnegative Gaussian curvature must be parabolic. Therefore, $h$ is constant on $\Sigma^2$, that is, $\Sigma^2$ is a horosphere of $\mathbb{H}^3$ and the image of its Lorentz Gauss map $N(\Sigma)$ is exactly a plane of $\mathbb{H}^3$.

\[\square\]

References