MELTING OF THE EUCLIDEAN METRIC TO NEGATIVE SCALAR CURVATURE IN 3 DIMENSION

YUTAE KANG, JONGSU KIM, AND SEHO KWAK

Abstract. We find a $C^1$ one-parameter family of Riemannian metrics $g_t$ on $\mathbb{R}^3$ for $0 \leq t \leq \varepsilon$ for some number $\varepsilon$ with the following property: $g_0$ is the Euclidean metric on $\mathbb{R}^3$, the scalar curvatures of $g_t$ are strictly decreasing in $t$ in the open unit ball and $g_t$ is isometric to the Euclidean metric in the complement of the ball.

1. Introduction

In a remarkable paper of Lohkamp [3], where he proved the existence of negatively Ricci-curved metrics on a manifold, he constructed metrics on $\mathbb{R}^k$, $k \geq 3$ which have negative Ricci curvature on a ball and are Euclidean in the complement of the ball. These metrics played a central role in the proof of the existence. In fact, such metrics near the ball were quasi-isometrically embedded into any given manifold, and then he could prove by spreading argument that any manifold of dimension $\geq 3$ admit a Riemannian metric of negative Ricci curvature.

But he noted that the metrics on $\mathbb{R}^k$ can be only $C^0$-close to the Euclidean metric by the nature of the construction, e.g., see the metrics in [4, pp. 492–493]. Related to this, Lohkamp has made the following conjecture in [4].

Conjecture. Let $(M^n, g_0)$, $n \geq 3$, be a manifold and $B \subset M$ a ball. Then there are a metric $g_1$ and a $C^\infty$-continuous path $g_t$, on $M$ with

- (i) Ricci curvature of $g_t$ is strictly decreasing in $t$ on $B$.
- (ii) $g_t \equiv g_0$ on $M \setminus B$.

Through this conjecture, one wants to study how flexible a Riemannian metric can be with respect to Ricci curvature. Also one may want to study the topology of the space of negatively Ricci-curved metrics on a manifold. If the above $g_t$ exists, we call it a Ricci-curvature melting of $g_0$ on $B$. This

Received February 4, 2011; Revised May 13, 2011.

2010 Mathematics Subject Classification. 53B20, 53C20, 53C21.

Key words and phrases. scalar curvature.

This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(2010-0011704).
integral is zero. So we could not expect to get a melting in the open unit ball and property of them are constant function 1 away from the cylinder $C_R = g$.

Theorem 1.1. There exists a one-parameter family of Riemannian metrics $g_t$ on $\mathbb{R}^3$ which exists for $0 \leq t \leq \varepsilon$ for some number $\varepsilon$ with the following property: $g_0$ is the Euclidean metric on $\mathbb{R}^3$, $s(g_t) < s(g_0)$ for $0 \leq t < \varepsilon$, and $g_t$ is the Euclidean metric in the complement of the ball.

Note that at the Euclidean metric $g_0$, $dS_{g_0}(h) = \Delta(tr_h) + \delta\delta(h)$, whose integral is zero. So we could not expect to get a melting $g_t$ with $\frac{ds(g_t)}{dt}|_{t=0} = dS_{g_0}(g'(0)) < 0$ in a ball. In fact, we have $dS_{g_0}(g'(0)) \equiv 0$ whereas $\frac{d^2s(g_t)}{dt^2}|_{t=0} < 0$ in the open unit ball.

In Section 2, we construct Riemannian metrics on $\mathbb{R}^3$ that have negative scalar curvatures on a compact set near origin and are Euclidean away from it. In Section 3, we demonstrate a $C^\infty$ one-parameter family of metrics $g_t$ and prove that the scalar curvature $s(g_t)$ is monotonically decreasing in $t$ on $\mathbb{R}^3$. In Section 4, by a conformal deformation we spread the strict decreasing property of scalar curvature onto a ball.

2. Construction of the metric

We will deform the Euclidean metric $g_0 = dx^2 + dr^2 + r^2d\theta^2$ on $\mathbb{R}^3$ to $g = f^2dx^2 + h^2dr^2 + \frac{r^2}{f^2}d\theta^2$ where $f(x,r)$ and $h(x,r)$ are $C^\infty$ functions on $\mathbb{R}^3 = \{(x,r,\theta) \mid x \in \mathbb{R}, r \geq 0, 0 \leq \theta < 2\pi\}$, and we will require that both of them are constant function 1 away from the cylinder $C = \{(x,r,\theta) \mid |x| <$
Let \( w^1 = f dx, w^2 = h dr, w^3 = \frac{r}{h} d\theta \) be an orthonormal co-frame of \( g \). Then we can calculate the sectional curvatures \( -R_{ijij} \) as follows:

\[
R_{1212} = \frac{h_{xx}}{f^2 h} \left( \frac{f_x h_x}{f h} + \frac{f_r h_r}{f h^3} \right),
\]
\[
R_{2323} = \frac{3h^2}{h^3} - \frac{h_{rr}}{h^3} - \frac{3h_r}{h^3 r} - \frac{h^2_r}{f^2 h^2},
\]
\[
R_{1313} = \frac{2h^2}{f^2 h^2} + \frac{f_x h_x}{f^3 h} - \frac{h_{xx}}{f^2 h^2} + \frac{f_r h_r}{f h^2 r} - \frac{f^2 h_r}{f h^3}.
\]

The scalar curvature is as follows:

\[
s_g = \sum_{i,j} (-1)^i R_{ijij} = -2(R_{1212} + R_{1313} + R_{2323})
\]
\[
= -2 \left( \frac{f_{rr}}{f h^2} + \frac{f_r}{f^2 h^2} - \frac{h_{rr}}{h^3} - \frac{3h_r}{h^3 r} + \frac{h^2_r}{f^2 h^2} + \frac{3h^2_r}{h^4} - 2f_r h_r \right).
\]

We multiply \(-\frac{h^2}{h^2} \) to both sides of the above to get

\[
(1) \quad -s_g \frac{h^2}{2} = \left( \frac{f_{rr}}{f} - \frac{f^2}{f^2} + \frac{f_r}{f h} - \frac{h_{rr}}{h^3} - \frac{3h_r}{h^3 r} + \frac{h^2_r}{h^4} - 2f_r h_r \right) + \frac{f^2}{f^2} + \frac{h^2}{h^2}.
\]

Our strategy is to find \( f \) and \( h \) with support in \( C \) so that the sum of the terms in the parenthesis of (1) becomes zero. The remaining square terms \( \frac{f^2}{f^2} + \frac{h^2}{h^2} \) guarantee the non-positivity of \( s_g \) on \( C \). Now for convenience we denote the partial derivative in the \( f \) variable by prime (‘) as \( f' = f_r, f'' = f_{rr}, h' = h_r, h'' = h_{rr} \) and we let \( F = \frac{f'}{f} \) and \( H = \frac{h'}{h} \). Then \( F'' = F^2 + F' \), \( H'' = H^2 + H' \) and the sum of the terms in the parenthesis in (1) becomes \( F' + \frac{F}{F'} - (H^2 + H') - \frac{1}{r} H - 2FH + 3H^2 \), which we want to be zero, i.e.,

\[
F' + \left( \frac{1}{r} - 2H \right) F = \frac{3}{r} H + H' - 2H^2.
\]

Now we will solve the ordinary differential equation (2) with respect to \( r \). So we denote the functions \( f(x, r), h(x, r), \) etc. by \( f(r), h(r), \) etc. respectively for simplicity. We multiply an integrating factor \( e^{\int \frac{1}{r} - 2H ds} = e^{\int \frac{1}{r} - 3}\frac{H}{H'} ds = \frac{r}{H(r)} = \frac{r}{h(r)} \) to both sides of (2). Hereafter we require and assume that \( h(0) = h(1) = 1 \), which will be checked later. Then (2) becomes

\[
\frac{d}{dr} \left[ \frac{r}{h^2(r)} F(r) \right] = \left[ \frac{r}{h^2(r)} \left( \frac{3}{r} H(r) + H'(r) - 2H^2(r) \right) \right].
\]

Integrating both sides from 0 to \( r \),

\[
\int_0^r \frac{d}{ds} \left[ \frac{s}{h^2(s)} F(s) \right] ds = \int_0^r \left[ \frac{s}{h^2(s)} \left( \frac{3}{r} H(s) + H'(s) - 2H^2(s) \right) \right] ds.
\]
So we have
\[
F(r) = \frac{h^2(r)}{r} \int_0^r \left[ \frac{s}{h^2(s)} \left( \frac{3}{s} H(s) + H'(s) - 2H^2(s) \right) \right] ds, \ r \neq 0
\]
\[
= \frac{h^2(r)}{r} \int_0^r \left[ \frac{s}{h^2(s)} \left( \frac{h''(s)}{h(s)} - \frac{h'(s)}{h^2(s)} \right) - 2 \frac{h'(s)}{h^2(s)} \right] ds, \ r \neq 0
\]
\[
= \frac{h^2(r)}{r} \int_0^r \left( \frac{3h'(s)}{h^3(s)} + \frac{sh''(s)}{h^4(s)} - \frac{3sh'(s)}{h^4(s)} \right) ds, \ r \neq 0.
\]

Since \( \int_0^r \frac{sh''(s)}{h^4(s)} - \frac{3sh'(s)}{h^4(s)} ds = \int_0^r \frac{h''(s)}{h^3(\gamma)} \big|_0^r - \int_0^r \frac{h'(s)}{h^3(\gamma)} ds, \)
\[
F(r) = \frac{h^2(r)}{r} \left[ 2 \int_0^r \frac{h'(s)}{h^3(s)} ds + \frac{sh''(s)}{h^4(s)} \big|_0^r \right] = \frac{h^2(r)}{r} \left[ 1 - \frac{1}{h^2(r)} + r \frac{H(r)}{h^2(r)} \right]
\]
\[
= H(r) + \frac{h^2(r)}{r} - \frac{1}{r}, \quad r \in (0, 1).
\]

Hence we have found the following relation between \( f \) and \( h; \)

(3) \[
\frac{f'}{f} = \frac{h'}{h} + \frac{h^2}{r} - \frac{1}{r}.
\]

Now we are going to show the following lemma.

Lemma 2.1. Suppose that we are given any \( C^\infty \) function \( \psi(x, r) \) on \( \mathbb{R}^3 \) such that all its partial derivatives vanish at \( (x, 0) \), \( \int_0^1 \psi(x, r) dr = 0 \), \( \text{supp}(\psi) \subset C \) and \( |r\psi(x, r)| < 1 \). Then we can construct positive \( C^\infty \) functions \( f(x, r) \) and \( h(x, r) \) on \( \mathbb{R}^3 \) such that they have a constant value 1 on \( \mathbb{R}^3 - C \), \( f(x, 0) = h(x, 0) = 1 \) and satisfy \( \frac{f'}{f} = \frac{h'}{h} + \frac{h^2}{r} - \frac{1}{r}. \)

Proof. We set \( h(x, r) = \sqrt{r} \psi(x, r) + 1 \). Then \( h(x, r) \) is \( C^\infty \) on \( \mathbb{R}^3 \) which has a constant value 1 on \( \mathbb{R}^3 - C \) and \( h(x, 0) = 1 \). Define the function \( f(x, r) \) by \( f(x, r) = h(x, r)e^{\int_0^r \frac{h'(s)}{h^2(s)} ds} \). Since \( \int_0^r \frac{h^2(s)}{h(s)} ds = \int_0^r \psi(x, s) ds \) and \( e^{\int_0^r \frac{h'(s)}{h^2(s)} ds} \) are \( C^\infty \) on \( \mathbb{R}^3 \), the function \( f(x, r) \) is also \( C^\infty \) on \( \mathbb{R}^3 \). We can also check that \( f(x, r) \) has a constant value 1 on \( \mathbb{R}^3 - C \) and \( f(x, 0) = 1 \). Compute \( \frac{\partial}{\partial r} f(x, r) = \frac{\partial}{\partial r} \{ h(x, r)e^{\int_0^r \frac{h^2(s)}{h^4(s)} ds} \} \) and we get \( \frac{f'}{f} = \frac{h'}{h} + \frac{h^2}{r} - \frac{1}{r}. \)

Henceforth, we shall choose \( \psi(x, r) = \alpha(r) \beta(x) \) where \( \alpha \) should be a smooth function on \( \mathbb{R}^3 \) and may be described as in Figure 1 and \( \beta : \mathbb{R} \to \mathbb{R} \) is smooth with \( \text{supp} \beta = [-1, 1] \) and \( 0 < \beta' < 1 \) on the interval \((-1, 0)\) and \(-1 < \beta' < 0 \) on \((0, 1)\). Then \( \psi \) satisfies the hypothesis of Lemma 2.1. And with such a choice of \( \alpha, \psi > 0 \) and \( r \psi_r > 0 \) on \{ \( (x, r, \theta) \mid -1 < x < 1, \ 0 < r < 0.3 \} \).
Figure 1. An example of $\alpha(r)$

With this choice of $\psi$ and Lemma 2.1, we can conclude the following:

**Proposition 2.2.** There exist Riemannian metrics on $\mathbb{R}^3$ such that the scalar curvature is negative on $C$ except a thin subset and they are the Euclidean metric on the complement of $C$.

3. Decreasing property of the scalar curvature of metrics

We are going to show that there is a $C^1$ one-parameter family $g_t$ among the metrics in the previous section such that its scalar curvature $s(g_t)$ is decreasing for some interval $t \in (0, \varepsilon)$ on $C$ and $g_t$ is the Euclidean metric in the complement of $C$.

We set $g_t = f_t^2 dx^2 + h_t^2 dr^2 + \frac{r^2}{h_t} d\theta^2$, where $h_t, f_t$ are the functions defined by $h_t(x, r) = \sqrt{r \cdot T \cdot \psi(x, r) + 1}$ and $f_t(x, r) = h_t(x, r)e^{\int_0^r h_t^2(x, s) ds} = h_t(x, r)e^{\int_0^r \psi(x, s) ds}$ as in the proof of Lemma 2.1. By the argument of the previous section, $g_t$ is one of the metrics in Proposition 2.2. So its scalar curvature is

$$s(g_t) = -2 \left[ \frac{(h_t)_r^2}{h_t f_t^2} + \frac{(f_t)_r^2}{h_t^2 f_t^2} \right].$$

Differentiating $h_t^2 = r \cdot T \psi + 1$, we obtain $(h_t)_r^2 = \frac{r^2 \psi^2}{4 h_t^2}$ and $\frac{(h_t)_r}{h_t} = \frac{\psi + r \psi_r}{2 h_t}$. Since $\frac{(f_t)_r}{f_t} = \frac{(h_t)_r}{h_t} + T \psi$, the scalar curvature becomes

$$s(g_t) = -2t \left[ \frac{r^2 \psi^2}{4 f_t^2 h_t^4} + \left( \frac{\psi + r \psi_r}{2 h_t^3} \right)^2 \right].$$

Put

$$A = \left[ \frac{r^2 \psi^2}{4 f_t^2 h_t^4} + \left( \frac{\psi + r \psi_r}{2 h_t^3} \right)^2 \right].$$

Then

$$\frac{d}{dt}(s(g_t)) = -4tA - 2t^2 \frac{dA}{dt}$$

and

$$\frac{d^2}{dt^2}(s(g_t)) = -4A - 8t \frac{dA}{dt} - 2t^2 \frac{d^2 A}{dt^2}.$$
So we have
\[
d\frac{d}{dt}(s(g_t))|_{t=0} = 0
\]
and
\[
(5) \quad \frac{d^2}{dt^2}(s(g_t))|_{t=0} = -4A|_{t=0} = -r^2\psi_x^2 - (3\psi + r\psi_r)^2 \leq 0,
\]
with equality exactly where \(r\psi_x = 3\psi + r\psi_r = 0\). Note that inside \(C\) the set of points with \(d^2\frac{ds}{d\tau}(s(g_t))|_{t=0} = 0\) forms a thin subset.

Therefore \(s(g_t)\) is strictly decreasing on \(C\) except a thin subset, but it is not clear if there exists a constant \(\varepsilon\) such that \(s(g_t)\) is strictly decreasing for \(0 \leq t \leq \varepsilon\). Moreover it does not decrease on a ball. In order to have the right decreasing property, we need to diffuse the negativity (of scalar curvature) onto a ball.

4. Diffusion of negative scalar curvature onto a ball

We use the functions of [3]: \(f_{1,m}(\rho) \in C^\infty(\mathbb{R}, \mathbb{R}^{\geq 0})\) for \(m > 0, t \geq 0\) defined by \(f_{1,m}(\rho) = m \cdot t^2 \cdot \exp(-\frac{100}{\rho})\) on \(\mathbb{R}^{\geq 0}\) and \(f_{1,m} = 0\) on \(\mathbb{R}^{\leq 0}\). Also choose an \(H \in C^\infty([0,1])\) with \(H = 0\) on \(\mathbb{R}^{\leq 1}\), \(H = 1\) on \(\mathbb{R}^{\leq 0}\) and \(H_k(\rho) = H(\frac{1}{r}(\rho - b)), \text{ for } b > 0, \varepsilon > 0\).

Let \(B_r(p)\) be the open ball of radius \(r\) with respect to \(g_0\) centered at \(p\). In Section 2 we described the function \(\psi\) using Figure 1. So, \(\psi > 0\) and \(r\psi_r > 0\) on \(\{(x, r, \theta) \mid -1 < x < 1, 0 < r < 0.3\}\). We choose a point \(p\) so that \(B_{0.1}(p) \subset \{(x, r, \theta) \mid -1 < x < 1, 0.1 < r < 0.3\}\). Then in (4), \(s(g_t) < 0\) on \(B_{0.1}(p)\) for \(t > 0\).

Let \(f_{k,m} \in C^\infty(\mathbb{R}^3, \mathbb{R}^{\geq 0})\) be \(f_{k,m}(q) = f_{1,m}(\rho(q))\), where \(\rho\) is the distance from the above point \(p\) to \(q \in \mathbb{R}^{3}\) and let \(h_{k}^\phi \in C^\infty(\mathbb{R}^3, \mathbb{R}^{\geq 0})\) be \(h_{k}^\phi(q) = H_{k}(\rho(q))\). We choose \(b = 9\) and \(\varepsilon = 0.1\). We consider \(e^{2\phi_t}g_t\), where
\[
\phi_t(\rho) = f_{1,m}(9.1 - \rho) \cdot h_{0,1}(9.1 - \rho) = mt^2e^{-\frac{100}{\rho}}h_{0,1}(9.1 - \rho).
\]
Here \(m\) will be determined below. The scalar curvature is as follows:
\[
s(e^{2\phi_t}g_t) = e^{-2\phi_t}(s_{g_t} + 4\Delta_{g_t}\phi_t - 2|\nabla_{g_t}\phi_t|^2).
\]
In order to show that \(s(e^{2\phi_t}g_t)\) is strictly decreasing for some interval \((0, \varepsilon)\), we calculate \(\frac{ds(e^{2\phi_t}g_t)}{dt}|_{t=0}\) and \(\frac{d^2s(e^{2\phi_t}g_t)}{dt^2}|_{t=0}\). Put
\[
B = s_{g_t} + 4\Delta_{g_t}\phi_t - 2|\nabla_{g_t}\phi_t|^2.
\]
Then
\[
\frac{ds(e^{2\phi_t}g_t)}{dt} = -2\frac{d\phi_t}{dt}e^{-2\phi_t}B + e^{-2\phi_t}\left(\frac{ds_{g_t}}{dt} + 4\frac{\Delta_{g_t}\phi_t}{dt} - 2\frac{d|\nabla_{g_t}\phi_t|^2}{dt}\right)
\]
and
\[
\frac{d^2s(e^{2\phi_t}g_t)}{dt^2} = 4\left(\frac{d\phi_t}{dt}\right)^2e^{-2\phi_t}B - 2\frac{d^2\phi_t}{dt^2}e^{-2\phi_t}B - 4\frac{d\phi_t}{dt}e^{-2\phi_t}\left(\frac{ds_{g_t}}{dt} + 4\frac{\Delta_{g_t}\phi_t}{dt}\right)
\]
the following argument.

In sum, we have \((t,p) = (0,\emptyset)\) and \(|V| = 0\) so that 
\[
0 < 0, \quad \Delta \frac{\partial^2}{\partial t^2} (9 - \rho) = 0, \Delta \frac{\partial^2}{\partial t^2} (g_{0,1}(9 - \rho)) = 0,
\]
and 
\[
0 < 0, \quad \exists \epsilon > 0 \text{ such that } s(e^{2\phi_t} g_t) \text{ strictly decreasing for } 0 \leq t \leq \epsilon.
\]

As we have \(\phi_0 = B|_{t=0} = 0, \Delta \phi_t = m^2 \Delta \phi_t e^{-\frac{100}{9-\rho}} h_{0,1}^3(9 - \rho) \) and \(|V| = m^2 \Delta \phi_t e^{-\frac{100}{9-\rho}} h_{0,1}^3(9 - \rho)\) we get 
\[
\left. \frac{ds(e^{2\phi_t} g_t)}{dt}\right|_{t=0} = 4 \left. \frac{d\Delta \phi_t}{dt}\right|_{t=0} - 2 \left. \frac{d|\nabla \phi_t|^2}{dt}\right|_{t=0} = 0
\]
and 
\[
\left. \frac{d^2 s(e^{2\phi_t} g_t)}{dt^2}\right|_{t=0} = 4 \left. \frac{d^2 \Delta \phi_t}{dt^2}\right|_{t=0} + 4 \left. \frac{d^2 |\nabla \phi_t|^2}{dt^2}\right|_{t=0} = 0.
\]

Now we will show that \(\frac{d^2 s(e^{2\phi_t} g_t)}{dt^2}|_{t=0} < 0\) on \(B_{0,1}(p)\). Let \(U = \{q \in B_{0,1}(p) \mid \left. \frac{\partial^2 s(e^{2\phi_t} g_t)}{\partial t^2}\right|_{t=0} = 0\}\) and let \(V = B_{0,1}(p) - \{U \cup B_{0,1}(p)\}\). Note that 
\(U \cap B_{0,1}(p) = \emptyset\) by our choice of \(\psi\) and (5). On \(U\), since \(h_{0,1}^3(9 - \rho) = 1\) we have 
\[
\left. \frac{d^2 s(e^{2\phi_t} g_t)}{dt^2}\right|_{t=0} = 4 \left. \frac{d^2 \Delta \phi_t}{dt^2}\right|_{t=0} = 8m \Delta \phi_t e^{-\frac{100}{9-\rho}} h_{0,1}^3(9 - \rho) > 0.
\]
On \(V\), \(\frac{\partial^2 s(e^{2\phi_t} g_t)}{\partial t^2}|_{t=0} < 0\) and \(\frac{\partial^2 s(e^{2\phi_t} g_t)}{\partial t^2}|_{t=0} < 0\), so we have 
\[
\left. \frac{d^2 s(e^{2\phi_t} g_t)}{dt^2}\right|_{t=0} = 4 \left. \frac{d^2 \Delta \phi_t}{dt^2}\right|_{t=0} = 8m \Delta \phi_t e^{-\frac{100}{9-\rho}} h_{0,1}^3(9 - \rho),
\]
so choose \(m > 0\) small so that 
\[
\left. \frac{d^2 s(e^{2\phi_t} g_t)}{dt^2}\right|_{t=0} < 0.
\]

In sum, we have 
\[
\left. \frac{ds(e^{2\phi_t} g_t)}{dt}\right|_{t=0} = 0 \text{ and } \left. \frac{d^2 s(e^{2\phi_t} g_t)}{dt^2}\right|_{t=0} < 0 \text{ on } B_{0,1}(p) \text{ and } e^{2\phi_t} g_t = g_0 \text{ on } \mathbb{R}^3 - B_{0,1}(p).
\]
This does not seem to guarantee the existence of a constant \(\epsilon\) such that 
\(s(e^{2\phi_t} g_t)\) strictly decreasing for \(0 \leq t \leq \epsilon\). So we add the following argument.

On \(B_{0,1}(p)\), there exists \(\hat{\epsilon} > 0\) such that \(s(e^{2\phi_t} g_t)\) strictly decreasing for 
\(0 \leq t \leq \hat{\epsilon}\). On \(B_{0,1}(p) - B_{0,1}(p)\), \(g_t \) is Euclidean and \(s_{g_t} = 0\), so 
\[
s(e^{2\phi_t} g_t) = e^{-2\phi_t} (4\Delta g_t - 2|\nabla g_t|^2)
\]
\[
= e^{-2\phi_t} 400 m^2 e^{-\frac{100}{9-\rho}} \left\{ \left( 100 \frac{9-\rho}{9-\rho} - 2 \right) - \frac{50 m^2 e^{-\frac{100}{9-\rho}}}{9-\rho} \right\}.
\]

The term \(\left\{ \left( 100 \frac{9-\rho}{9-\rho} - 2 \right) - \frac{50 m^2}{9-\rho} e^{-\frac{100}{9-\rho}} \right\} \) is strictly decreasing with respect to \(t\). Since 
\(2m e^{-\frac{100}{9-\rho}} < \frac{2m}{\sqrt{m}}\), we have 
\[
\frac{d}{dt} e^{-2\phi_t} 400 m^2 e^{-\frac{100}{9-\rho}} = 400 m e^{-\frac{100}{9-\rho}} e^{-2\phi_t} (1 - 2m e^{-\frac{100}{9-\rho}} t^2) > 0
\]
for \(0 < t \leq \frac{\sqrt{m}}{\sqrt{2m}}\). Hence \(s(e^{2\phi_t} g_t)\) is strictly decreasing for 
\(0 \leq t \leq \frac{\sqrt{m}}{\sqrt{2m}}\). Setting \(\epsilon = \min\{\hat{\epsilon}, \frac{\sqrt{m}}{\sqrt{2m}}\}\), we conclude that on \(\mathbb{R}^3\) there exists \(\epsilon > 0\) such that 
\(s(e^{2\phi_t} g_t)\) is strictly increasing for \(0 \leq t \leq \epsilon\).
Finally, with the affine transformation $\nu : \mathbb{R}^3 \to \mathbb{R}^3$, $\nu(x) = 9.1x + p$, we get the pulled-back metric $\nu^*(e^{2\alpha}g_t)$, which yields a melting on the unit ball. This proves Theorem 1.1.

Remark 4.1. From Theorem 1.1, one may suspect that melting of the Euclidean metric on a ball should hold in any dimension bigger than two. We already have the metrics of [2] on $\mathbb{R}^{2n}, n \geq 2$ which have negative scalar curvature on a compact set and are Euclidean on its complement. It is very interesting to find a scalar curvature melting of a general metric on a ball.

References