LIST EDGE AND LIST TOTAL COLORINGS OF PLANAR GRAPHS WITHOUT 6-CYCLES WITH CHORD

Aijun Dong, Guizhen Liu, and Guojun Li

Abstract. Giving a planar graph $G$, let $\chi'(G)$ and $\chi''(G)$ denote the list edge chromatic number and list total chromatic number of $G$ respectively. It is proved that if a planar graph $G$ without 6-cycles with chord, then $\chi'(G) \leq \Delta(G) + 1$ and $\chi''(G) \leq \Delta(G) + 2$ where $\Delta(G) \geq 6$.

1. Introduction

The terminology and notation used but undefined in this paper can be found in [1]. Let $G = (V, E)$ be a graph. We use $V(G)$, $E(G)$, $F(G)$, $\Delta(G)$ and $\delta(G)$ to denote the vertex set, edge set, face set, maximum degree, and minimum degree of $G$, respectively. Let $d_G(x)$ or simply $d(x)$, denote the degree of a vertex (face) $x$ in $G$. $A$ vertex (face) $x$ is called a $k$-vertex ($k$-face), $k^+$-vertex ($k^+$-face), $k^-$-vertex, if $d(x) = k$, $d(x) \geq k$, $d(x) \leq k$. We use $(d_1, d_2, \ldots, d_n)$ to denote a face $f$ if $(d_1, d_2, \ldots, d_n)$ are the degree of vertices incident to the face $f$. If $u_1, u_2, \ldots, u_n$ are the vertices on the boundary walk of a face $f$, then we write $f = u_1u_2\cdots u_n$. Let $\delta(f)$ denote the minimal degree of vertices incident to $f$. We use $f_i(v)$ denote the number of $i$-faces incident to $v$ for each $v \in V(G)$. Let $n_i(f)$ denote the number of $i$-vertices incident to $f$ for each $f \in F(G)$. A cycle $C$ of length $k$ is called $k$-cycle, if $xy \in E(G) \setminus E(C)$ and $x, y \in V(C)$, the cycle $C$ is called $k$-cycle with chord.

The mapping $L$ is said to be a total assignment for a graph $G$ if it assigns a list $L(x)$ of possible colors to each element $x \in V \cup E$. If $G$ has a proper total coloring $\phi(x) \in L(x)$ for all $x \in V \cup E$, then we say that $G$ is total-L-colorable. Let $f : V \cup E \to N$ be a function into the positive integers. We say that $G$ is total-$f$-choosable if it is total-L-colorable for every total assignment $L$ satisfying $|L(x)| = f(x)$ for all $x \in V \cup E$. The list total coloring number $\chi''(G)$ of $G$ is the smallest integer $k$ such that $G$ is total-$f$-choosable when $f(x) = k$ for each $x \in V \cup E$. The list edge coloring number $\chi'(G)$ of $G$ is

\[\chi'(G) \leq \Delta(G) + 1\]
\[\chi''(G) \leq \Delta(G) + 2\]

Received November 5, 2010.
2010 Mathematics Subject Classification. 05C15.
Key words and phrases. list coloring, planar graph, choosability.
This work was supported by the National Science Foundation of China (264062, 202799, 269960).
Similarly, the color $j$ and, edge-

Now there are at least three colors available for $uv\vDash x$.

Lemma 2.1. Let $L_u$ for $vw\vDash x$.

with chord, then $\chi''(G) = \chi''(G)$.

Part (a) of the above conjecture was formulated independently by Vizing, by Gupta, by Alberston and Collins, and by Bollobás and Harris [5, 11]. It is well known as the List Coloring Conjecture. Part (b) was formulated by Borodin, Kostochka and Woodall [2]. Part (a) and Part (b) has been proved for outerplanar graphs [13], and graphs with $\Delta \geq 12$ which can be embedded in a surface of nonnegative characteristic [2]. List Coloring Conjecture has been proved for a few other special graphs, such as bipartite multigraphs [4], complete graphs of odd order [6]. There are several related results for planar graphs without some short cycles or by adding girth restrictions [7, 8, 9, 3, 16, 14, 15, 10].

In this paper, we shall show that if $G$ is a planar graph without 6-cycles with chord, then $\chi'_l(G) \leq \Delta(G) + 1$ and $\chi''(G) \leq \Delta(G) + 2$ where $\Delta(G) \geq 6$.

2. Planar graphs without 6-cycles with chord

First let us introduce an important lemma.

Lemma 2.1. Let $G$ be a critical planar graph without 6-cycles with chord. If $\Delta(G) \geq 6$, then there is an edge $uv \in E(G)$ such that $\min\{d(u), d(v)\} \leq \left\lfloor \frac{\Delta(G)+1}{2} \right\rfloor$ and $d(u) + d(v) \leq \max(8, \Delta(G) + 2)$.

Proof. For $G$ is a critical planar graph with $\Delta(G) \geq 6$, then $G$ contains no (4,4,5)-face $f = uvw$. By contradiction, let $L'$ and $L''$ be any list assignments such that $|L'(e)| = \Delta(G) + 1$ for each $e \in E(G)$ and $|L''(x)| = \Delta(G) + 2$ for each $x \in V(G) \cup E(G)$.

Let $G' = G - \{uv, vw, wu\}$. By $G$ is a critical graph, $G'$ is edge-$L'$-colorable. Now there are at least three colors available for $uv$, and at least two colors for $vw$ and $wu$. We can easily color $vw$, $uw$, and $uv$ successively. So $G$ is edge-$L'$-colorable, a contradiction.

For the same reason, $G''$ is total-$L''$-colorable. Erase the colors of the vertices $u, v, w$. For each element $x$ incident with $f$, we define a reduced total list $L''(x)$ such that $L''(x) = L''(x) \setminus \{\phi(x') \mid x'$ is incident with or adjacent to $x$, and $x'$ is not incident with $f\}$ where $\phi(x')$ denotes the color of the element $x'$. Then $|L''(u)| \geq 4$, $|L''(v)| \geq 4$, $|L''(w)| \geq 2$, $|L''(uw)| \geq 4$, $|L''(vw)| \geq 3$, $|L''(wu)| \geq 3$. If there is a color $\alpha \in L''(uw) \setminus L''(u)$, then we can color $uw$ with the color $\alpha$, and color $w, v, u, w, v, u$ successively. So $L''(uw) \subseteq L''(u)$. Similarly, $L''(vw) \subseteq L''(v)$. If there is a color $\beta \in L''(u) \setminus L''(v)$, then we can
color \( u \) with \( \beta \), and color \( w, uw, vw, uv \) and \( v \) successively. So \( L'(u) = L'(v) \). Thus there is a color \( \gamma \in L''(uw) \setminus L''(v) \). We color \( uw \) and \( v \) with \( \gamma \), and color \( w, uw, vw, uv \) and \( v \) successively. From the above discussion, in any case, \( f \) is total-\( L'' \)-colorable. So \( G \) is total-\( L'' \)-colorable, a contradiction.

In the following, we show that for the critical planar graph without 6-cycles with chord, if \( \Delta(G) \geq 6 \), then there is an edge \( uv \in E(G) \) such that \( \min\{d(u), d(v)\} \leq \left\lfloor \frac{\Delta(G)+1}{2} \right\rfloor \) and \( d(u) + d(v) \leq \max\{8, \Delta(G) + 2\} \). By contradiction, we have \( d(u) + d(v) \geq \max\{9, \Delta(G) + 3\} \) for each edge \( uv \in E(G) \) such that \( \min\{d(u), d(v)\} \leq \left\lfloor \frac{\Delta(G)+1}{2} \right\rfloor \). It is clear that \( \delta(G) \geq 3 \).

By Euler’s formula \( |V| - |E| + |F| = 2 \) and \( \sum_{v \in V(G)} d(v) = \sum_{f \in F(G)} d(f) = 2|E| \), we have

\[
\sum_{v \in V(G)} (2d(v) - 6) + \sum_{f \in F(G)} (d(f) - 6) = -6(|V| - |E| + |F|) = -12.
\]

Define an initial charge function \( w \) on \( V(G) \cup F(G) \) by setting \( w(v) = 2d(v) - 6 \) if \( v \in V(G) \) and \( w(f) = d(f) - 6 \) if \( f \in F(G) \), so that \( \sum_{v \in V(G) \cup F(G)} W(x) = -12 \). Now redistribute the charge according to the following discharging rules.

For convenience, let \( w(v) \) denote the total charge transferred from a vertex \( v \) to all its incident 4- and 5-faces where \( d(v) = 4 \).

**D1.** If \( f \) is a 3-face incident with a vertex \( v \), then \( v \) gives \( f \) charge \( \frac{2 - w(v)}{f_3(v)} \) if \( d(v) = 4 \), \( \frac{4}{3} \) if \( d(v) = 5 \), \( \frac{5}{2} \) if \( d(v) \geq 6 \).

**D2.** If \( f \) is a 4-face incident with a vertex \( v \), then \( v \) gives \( f \) charge \( \frac{1}{2} \) if \( d(v) = 4 \) or \( 5 \), \( 1 \) if \( d(v) \geq 6 \).

**D3.** If \( f \) is a 5-face incident with a vertex \( v \), then \( v \) gives \( f \) charge \( \frac{1}{5} \) if \( d(v) = 4 \) or \( 5 \), \( \frac{1}{4} \) if \( d(v) \geq 6 \).

Let the new charge of each element \( x \) be \( w'(x) \) for each \( x \in V(G) \cup F(G) \).

In the following, let us check the new charge of each element \( x \in V(G) \cup F(G) \).

Suppose \( d(v) = 3 \). Then \( w'(v) = w(v) = 0 \).

Suppose \( d(v) = 4 \). Then \( w(v) = 2 \), \( f_3(v) \leq 4 \). If \( f_3(v) \geq 1 \), then \( w'(v) \geq 2 - \frac{2 - w(v)}{f_3(v)} f_3(v) - w(v) = 0 \) by D1. Otherwise, i.e., \( f_3(v) = 0 \), then \( f_4(v) \leq 2 \), \( f_5(v) \leq 4 \) for \( G \) contains no 6-cycles with chord. We have \( w'(v) > 2 - \frac{1}{2} x 2 - \frac{1}{5} x 4 = \frac{1}{2} > 0 \) by D2 and D3.

Suppose \( d(v) = 5 \). Then \( w(v) = 4 \), \( f_3(v) \leq 3 \) for \( G \) contains no 6-cycles with chord. If \( f_3(v) = 3 \), then \( f_4(v) = 0 \) and \( f_5(v) = 0 \) for \( G \) contains no 6-cycles with chord. We can get \( w'(v) \geq 4 - \frac{4}{3} x 3 = 0 \) by D1. If \( f_3(v) = 2 \), then \( f_4(v) \leq 1 \) and \( f_5(v) \leq 1 \) for \( G \) contains no 6-cycles with chord. We can get \( w'(v) \geq 4 - \frac{4}{5} x 2 - \frac{2}{5} x 2 - \frac{1}{5} x 2 = \frac{13}{5} > 0 \) by D1, D2 and D3. If \( f_3(v) = 1 \), then \( f_4(v) \leq 2 \) and \( f_5(v) \leq 2 \) for \( G \) contains no 6-cycles with chord. We can get \( w'(v) \geq 4 - \frac{4}{5} - \frac{1}{5} x 2 - \frac{2}{5} x 2 = \frac{13}{5} > 0 \) by D1, D2 and D3. If \( f_3(v) = 0 \),
then $f_3(v) \leq 2$ or $f_3(v) \leq 5$ for $G$ contains no 6-cycles with chord, we have $w'(v) > 4 - \frac{1}{3} \times 2 - \frac{1}{5} \times 5 = 2 > 0$ by $D2$ and $D3$.  

Suppose $d(v) = 6$. Then $w(v) = 6$, $f_3(v) \leq 4$ for $G$ contains no 6-cycles with chord. If $f_3(v) = 4$, then $f_4(v) = 0$ and $f_3(v) = 0$ for $G$ contains no 6-cycles with chord. We can get $w'(v) \geq 6 - \frac{3}{2} \times 4 = 0$ by $D1$. If $f_3(v) = 3$, then $f_4(v) \leq 1$ and $f_5(v) \leq 1$ for $G$ contains no 6-cycles with chord. We can get $w'(v) \geq 6 - \frac{3}{2} \times 3 - 1 = \frac{3}{5} > 0$ by $D1$, $D2$ and $D3$. If $f_3(v) = 2$, then $f_4(v) \leq 2$ and $f_5(v) \leq 2$ for $G$ contains no 6-cycles with chord. We can get $w'(v) \geq 6 - \frac{3}{2} \times 2 - 1 \times 2 - \frac{1}{3} \times 2 = \frac{1}{4} > 0$ by $D1$, $D2$ and $D3$. If $f_3(v) = 1$, then $f_4(v) \leq 3$ and $f_5(v) \leq 3$ for $G$ contains no 6-cycles with chord. We can get $w'(v) \geq 6 - \frac{3}{2} - 1 \times 3 - \frac{1}{3} \times 3 = \frac{1}{5} > 0$ by $D1$, $D2$ and $D3$. If $f_3(v) = 0$, then $f_4(v) \leq 3$ or $f_5(v) \leq 6$ for $G$ contains no 6-cycles with chord, we have $w'(v) > 6 - 1 \times 3 - \frac{1}{3} \times 6 = 1 > 0$ by $D2$ and $D3$.  

Suppose $d(v) = 7$. Then $w(v) = 8$, $f_3(v) \leq 5$ for $G$ contains no 6-cycles with chord. If $f_3(v) = 5$, then $f_4(v) = 0$ and $f_5(v) = 0$ for $G$ contains no 6-cycles with chord. We can get $w'(v) \geq 8 - \frac{3}{2} \times 5 = \frac{1}{4} > 0$ by $D1$. If $f_3(v) = 4$, then $f_4(v) \leq 1$ and $f_5(v) \leq 0$ for $G$ contains no 6-cycles with chord. We can get $w'(v) \geq 8 - \frac{3}{2} \times 4 - 1 = 1 > 0$ by $D1$ and $D2$. If $f_3(v) = 3$, then $f_4(v) \leq 2$ and $f_5(v) \leq 2$ for $G$ contains no 6-cycles with chord. We can get $w'(v) \geq 8 - \frac{3}{2} \times 3 - 1 \times 2 - \frac{1}{3} \times 2 = \frac{2}{3} > 0$ by $D1$, $D2$ and $D3$. If $f_3(v) = 2$, then $f_4(v) \leq 3$ and $f_5(v) \leq 2$ for $G$ contains no 6-cycles with chord. We can get $w'(v) \geq 8 - \frac{3}{2} \times 2 - 1 \times 3 - \frac{1}{3} \times 3 = \frac{1}{2} > 0$ by $D1$, $D2$ and $D3$. If $f_3(v) \leq 1$, then $f_4(v) \leq 3$ or $f_5(v) \leq 7$ for $G$ contains no 6-cycles with chord, we have $w'(v) > 8 - \frac{3}{2} - 1 \times 3 - \frac{1}{4} \times 7 = \frac{1}{5} > 0$ by $D1$, $D2$ and $D3$.  

Suppose $d(v) = 8$. Then $w(v) = 10$, $f_3(v) \leq 6$ for $G$ contains no 6-cycles with chord. If $f_3(v) = 6$ or 5, then $f_4(v) = 0$ and $f_5(v) = 0$ for $G$ contains no 6-cycles with chord. We can get $w'(v) \geq 10 - \frac{3}{2} \times 6 = 1 > 0$ by $D1$. If $f_3(v) = 4$, then $f_4(v) \leq 1$ and $f_5(v) \leq 1$ for $G$ contains no 6-cycles with chord. We can get $w'(v) \geq 10 - \frac{3}{2} \times 4 - 1 = \frac{5}{2} > 0$ by $D1$, $D2$ and $D3$. If $f_3(v) = 3$, then $f_4(v) \leq 2$ and $f_5(v) \leq 2$ for $G$ contains no 6-cycles with chord. We can get $w'(v) \geq 10 - \frac{3}{2} \times 3 - 1 \times 2 - \frac{1}{3} \times 2 = \frac{3}{5} > 0$ by $D1$, $D2$ and $D3$. If $f_3(v) = 2$, then $f_4(v) \leq 4$ and $f_5(v) \leq 4$ for $G$ contains no 6-cycles with chord. We can get $w'(v) \geq 10 - \frac{3}{2} \times 2 - 1 \times 4 - \frac{1}{3} \times 4 = \frac{1}{2} > 0$ by $D1$, $D2$ and $D3$. If $f_3(v) \leq 1$, then $f_4(v) \leq 4$ or $f_5(v) \leq 8$ for $G$ contains no 6-cycles with chord, we have $w'(v) > 10 - \frac{3}{2} - 1 \times 4 - \frac{1}{3} \times 8 = \frac{11}{3} > 0$ by $D1$, $D2$ and $D3$.  

Suppose $d(v) \geq 9$. Then $w(v) = 2d(v) - 6$. $f_3(v) \leq d(v) - \frac{1}{4} \times f_3(v)$, $f_5(v) \leq d(v) - \frac{1}{2} f_3(v)$ for $G$ contains no 6-cycles with chord. So $w'(v) \geq 2d(v) - 6 - \frac{3}{4} f_3(v) - f_4(v) - \frac{1}{2} f_3(v) \geq 2d(v) - 6 - \frac{3}{4} f_3(v) - d(v) + \frac{1}{4} f_3(v) - \frac{1}{4} d(v) + \frac{5}{4} f_3(v) \geq \frac{5}{4} d(v) - 6 + \frac{5}{18} f_3(v)$ by $D1$, $D2$ and $D3$. For $0 \leq f_3(v)$, we have $w'(v) \geq \frac{5}{4} d(v) - 6 \geq 0$.  

Suppose $d(f) = 3$. Then $w(f) = -3$.  

Case 1. $d(f) = 3$, then $f$ is a (3,6⁺,6⁺)-face by assumption. We have $w'(f) = -3 + \frac{1}{2} \times 2 = 0$ by $D1$. 

Case 2. \( \delta(f) = 4 \), then \( f \) is a \((4, 4, 6)\)- or \((4, 5, 5)\)-face for \( G \) contains no \((4, 4, 5)\)-face.

Case 2.1. \( f \) is a \((4, 4, 6)\)-face. For convenience, let \( f = uvw \) where \( d(u) = d(v) = 4 \). If one of the 4-vertex is incident to at least three 3-faces, without loss of generality, let \( f_3(u) = 3 \), then \( f_4(u) = f_5(u) = 0 \) and \( f_3(v) + f_4(v) + f_5(v) \leq 2 \) for \( G \) contains no 6-cycles with chord. We have \( w'(f) \geq -3 + \frac{1}{2} + \frac{3}{2} + \frac{2 - w(v)}{f_3(v)} \geq -3 + \frac{1}{2} + \frac{3}{2} + 1 = 0 \) by \( D_1, D_2 \) and \( D_3 \). Otherwise, i.e., \( f_3(u) \leq 2, f_3(v) \leq 2 \), then we have \( \frac{w'(v) - w(v)}{f_3(v)} \geq \frac{4}{5} \) for \( G \) contains no 6-cycles with chord and by \( D_1, D_2 \) and \( D_3 \). So \( w'(f) \geq -3 + \frac{1}{2} + \frac{3}{2} + 2 = \frac{2}{3} > 0 \) by \( D_1 \).

Case 2.2. \( f \) is a \((4, 5, 5)\)-face. By \( D_1, D_2 \) and \( D_3 \), we have \( \frac{w'(v) - w(v)}{f_3(v)} \geq \frac{1}{2} \). So \( w'(f) \geq -3 + \frac{1}{2} + \frac{3}{2} = \frac{1}{2} > 0 \) by \( D_1 \).

Case 3. \( \delta(f) \geq 5 \), then we have \( w'(f) = -3 + \frac{1}{2} \times 3 = 1 > 0 \) by \( D_1 \).

Suppose \( d(f) = 4 \). Then \( w(f) = -2 \). If \( \delta(f) = 3 \), then \( f \) is a \((3, 3^+, 6^+, 6^+)\)-face by assumption. We have \( w'(f) \geq -2 + 1 \times 2 = 0 \) by \( D_2 \). If \( \delta(f) \leq 4 \), then \( f \) is a \((4^+, 4^+, 4^+, 4^+)\)-face. We have \( w'(f) \geq -2 + \frac{1}{2} \times 4 = 0 \) by \( D_2 \).

Suppose \( d(f) = 5 \). Then \( w(f) = -1 \).

Case 4. \( \delta(f) = 3 \), then \( n_3(f) \leq 2 \) by assumption. If \( n_3(f) = 2 \), then \( f \) is a \((3, 3^+, 6^+, 6^+)\)-face by assumption. We have \( w'(f) \geq -1 + \frac{1}{2} \times 3 = 0 \) by \( D_3 \). If \( n_3(f) = 1 \), then \( f \) is a \((3, 4^+, 4^+, 6^+, 6^+)\)-face by assumption. We have \( w'(f) \geq -1 + \frac{1}{2} \times 2 + \frac{1}{2} \times 2 = \frac{1}{10} > 0 \) by \( D_3 \).

Case 5. \( \delta(f) = 4 \), then we have \( w'(f) \geq -1 + \frac{1}{2} \times 5 = 0 \) by \( D_3 \).

Suppose \( d(f) \geq 6 \). Then \( w'(f) = w(f) > 0 \).

From the above discussion, we can obtain \( \sum_{x \in V(G) \cup F(G)} w'(x) \geq 0 > -12 \), a contradiction. \( \square \)

In the following, let us give the proof of the main theorem.

**Theorem 2.2.** If \( G \) is a planar graph without 6-cycles with chord, then \( \chi'_1(G) \leq \Delta(G) + 1 \) and \( \chi''_1(G) \leq \Delta(G) + 2 \) where \( \Delta(G) \geq 6 \).

**Proof.** By contradiction, let \( G' \) and \( G'' \) be minimal counterexamples (i.e., critical graphs) to the conclusions for \( \chi'_1 \) and \( \chi''_1 \) respectively, and let \( L' \) and \( L'' \) be list assignments such that \( |L'(e)| = \Delta + 1 \) for each \( e \in E(G) \) and \( G' \) is not edge-\( L' \)-colorable, and \( |L''(x)| = \Delta + 2 \) for each \( x \in V(G) \cup E(G) \) and \( G'' \) is not total-\( L'' \)-colorable. By Lemma 2.1, \( G' \) and \( G'' \) contains an edge \( uv \in E(G) \) such that \( \min\{d(u), d(v)\} \leq \frac{\Delta(G) + 1}{2} \) and \( d(u) + d(v) \leq \max\{8, \Delta(G) + 2\} \).

Let \( G' = G' - uv \). Then \( G' \) is edge-\( L' \)-colorable by assumption. For \( d(u) + d(v) \leq \max\{8, \Delta(G) + 2\} \), there is at most \( \Delta(G) \) edges which are adjacent with \( uv \) in \( G' \). So there is at least one color in \( L'(uv) \) which we can use to color \( uv \). Then \( G' \) is edge-\( L' \)-colorable, a contradiction.

Let \( G'' = G'' - uv \). Then \( G'' \) is total-\( L'' \)-colorable by assumption. Without loss of generality, let \( d(u) = \min\{d(u), d(v)\} \). Erase the color of \( u \), then there
is at least one color in $L''(uv)$ which we can use to color $uv$ for $d(u) + d(v) \leq \max\{8, \Delta(G)+2\}$. For $d(u) \leq \lfloor \frac{\Delta(G)+1}{2} \rfloor$, then $u$ is adjacent to at most $\lfloor \frac{\Delta(G)+1}{2} \rfloor$ vertices or is incident to $\lfloor \frac{\Delta(G)+1}{2} \rfloor$ edges. So there is at least one color in $L''(u)$ which we can use to color $u$. Then $G''$ is total-$L''$-colorable, a contradiction. So $\chi'(G) \leq \Delta(G) + 1$ and $\chi''(G) \leq \Delta(G) + 2$.

References

Guizhen Liu
School of Mathematics
Shandong University
Jinan, 250100, P. R. China
E-mail address: gzliu@sdu.edu.cn

Guojun Li
School of Mathematics
Shandong University
Jinan, 250100, P. R. China
E-mail address: gjl@sdu.edu.cn