VALUE SHARING RESULTS OF A MEROMORPHIC FUNCTION $f(z)$ AND $f(qz)$

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Abstract. In this paper, we investigate sharing value problems related to a meromorphic function $f(z)$ and $f(qz)$, where $q$ is a non-zero constant. It is shown, for instance, that if $f(z)$ is zero-order and shares two values CM and one value IM with $f(qz)$, then $f(z) = f(qz)$.

1. Introduction

In what follows, a meromorphic function will mean meromorphic in the whole complex plane. We say that two meromorphic functions $f$ and $g$ share a value $a \in \mathbb{C} \cup \{\infty\}$ IM (ignoring multiplicities) when $f - a$ and $g - a$ have the same zeros. If $f - a$ and $g - a$ have the same zeros with the same multiplicities, then we say that $f$ and $g$ share the value $a$ CM (counting multiplicities). We assume that the reader is familiar with the standard symbols and fundamental results of Nevanlinna theory, as found in [5, 10].

As usual, by $S(r, f)$ we denote any quantity satisfying $S(r, f) = o(T(r, f))$ for all $r$ outside of a possible exceptional set of finite linear measure. In addition, denote by $S(f)$ the family of all meromorphic functions $a(z)$ that satisfy $T(r, a) = o(T(r, f))$, for $r \to \infty$ outside a possible exceptional set of finite logarithmic measure. In particular, we denote by $S_1(r, f)$ any quality satisfying $S_1(r, f) = o(T(r, f))$ for all $r$ on a set of logarithmic density 1.

The classical results due to Nevanlinna [9] in the uniqueness theory of meromorphic functions are the five-point, resp. four-point, theorems:

Theorem A. If two meromorphic functions $f$ and $g$ share five distinct values $a_1, a_2, a_3, a_4, a_5 \in \mathbb{C} \cup \{\infty\}$ IM, then $f \equiv g$.

Theorem B. If two meromorphic functions $f$ and $g$ share four distinct values $a_1, a_2, a_3, a_4 \in \mathbb{C} \cup \{\infty\}$ CM, then $f \equiv g$ or $f \equiv T \circ g$, where $T$ is a Möbius transformation.

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It is well-known that 4 CM cannot be improved to 4 IM, see [3]. Further, Gundersen [4, Theorem 1] has improved the assumption 4 CM to 2 CM + 2 IM, while 1 CM + 3 IM is still an open problem.

In recent papers [6], Heittokangas et al. started to consider the uniqueness of a finite order meromorphic function sharing values with its shift. They concluded that:

**Theorem C.** Let $f$ be a meromorphic function of finite order, let $c \in \mathbb{C}$, and let $a_1, a_2, a_3 \in S(f) \cup \{\infty\}$ be three distinct periodic functions with period $c$. If $f(z)$ and $f(z + c)$ share $a_1, a_2$ CM and $a_3$ IM, then $f(z) = f(z + c)$ for all $z \in \mathbb{C}$.

Closely related to difference expressions are $q$-difference expressions, where the usual shift $f(z + c)$ of a meromorphic function will be replaced by the $q$-shift $f(qz)$, $q \in \mathbb{C} \setminus \{0\}$. The Nevanlinna theory of $q$-difference expressions and its applications to $q$-difference equations have recently been considered, see [1, 7]. In addition, some results about solutions of zero-order for complex $q$-difference equations, can be found in the introduction in [1].

A natural question is: what is the uniqueness result in the case when $f(z)$ shares values with $f(qz)$ for a zero-order meromorphic function $f(z)$. Corresponding to this question, we get the following result:

**Theorem 1.1.** Let $f$ be a zero-order meromorphic function, and $q \in \mathbb{C} \setminus \{0\}$, and let $a_1, a_2, a_3 \in \mathbb{C} \cup \{\infty\}$ be two distinct values. If $f(z)$ and $f(qz)$ share $a_1, a_2$ CM and $a_3$ IM, then $f(z) = f(qz)$.

**Remark 1.** Indeed, from the proof of Theorem 1.1, we know the assumption that share $a_3$ IM can be replaced by one of the following assumptions:

1. if there exists a point $z_0$ such that $f(z_0) = f(qz_0) = a_3$; or
2. if $a_3$ is a Picard exceptional value of $f$.

However, we give Theorem 1.1 just as a $q$-difference analogue of Theorem C.

If $f$ is an entire function in Theorem 1.1, then the conclusion will be improved.

**Theorem 1.2.** Let $f$ be a zero-order entire function, and $q \in \mathbb{C} \setminus \{0\}$, and let $a_1, a_2 \in \mathbb{C}$ be two distinct values. If $f(z)$ and $f(qz)$ share $a_1$ and $a_2$ IM, then $f(z) = f(qz)$.

**Remark 2.** As a corollary of Theorem 1.1, we just know that $f(z) = f(qz)$ provided that $f(z)$ and $f(qz)$ share values under the condition that “1 CM + 1 IM”.

In the following, we consider the value sharing problems relative to $F(z) = f^n$ and $F(qz)$, and we obtain the following results:
Theorem 1.3. Let $f$ be a zero-order meromorphic function, and $q \in \mathbb{C} \setminus \{0\}$, $n \geq 4$ be an integer, and let $F = f^n$. If $F(z)$ and $F(qz)$ share $a \in \mathbb{C} \setminus \{0\}$ and $\infty$ CM, then $f(z) = tf(qz)$ for a constant $t$ that satisfies $t^n = 1$.

Remark 3. Theorem 1.3 is not true, if $a = 0$. This can be seen by considering $f(z) = z$ and $f(\frac{1}{2}z) = \frac{1}{2}z$. Then $f(z)^n$ and $f(\frac{1}{2}z)^n$ share 0 and $\infty$ CM, however, $f(z) = 2f(\frac{1}{2}z)$, $2^n \neq 1$, where $n$ is a positive integer.

Corollary 1.4. Let $f$ be a zero-order entire function, and $q \in \mathbb{C} \setminus \{0\}$, $n \geq 3$ be an integer, and let $F = f^n$. If $F(z)$ and $F(qz)$ share 1 CM, then $f(z) = tf(qz)$ for a constant $t$ that satisfies $t^n = 1$.

Corollary 1.5. Let $f$ be a zero-order meromorphic function, and $q \in \mathbb{C} \setminus \{0\}$, $n \geq 4$ be an integer, and let $F = f^n$. If $F(z)$ and $F(qz)$ share 0 and 1 CM, then $f(z) = tf(qz)$ for a constant $t$ that satisfies $t^n = 1$.

Remark 4. By simply calculations, we get $|q| = 1$ in above results. And some ideas of this paper are from [8].

2. Some lemmas

Lemma 2.1 ([1, Theorem 1.1]). Let $f$ be a zero-order meromorphic function, and $q \in \mathbb{C} \setminus \{0\}$. Then

$$m\left(r, \frac{f(qz)}{f(z)}\right) = S_1(r, f).$$

Lemma 2.2 ([1, Theorem 2.1]). Let $f$ be a zero-order meromorphic function, let $q \in \mathbb{C} \setminus \{0, 1\}$, and let $a_1, \ldots, a_p \in \mathbb{C}$, $p \geq 2$, be distinct points. Then

$$m(r, f) + \sum_{k=1}^{p} m\left(r, \frac{1}{f - a_k}\right) \leq 2T(r, f) - N_{\text{pair}}(r, f) + S_1(r, f),$$

where

$$N_{\text{pair}}(r, f) = 2N(r, f) - N(r, \Delta q f) + N\left(r, \frac{1}{\Delta q f}\right)$$

and $\Delta_q f = f(qz) - f(z)$.

Lemma 2.3 ([11, Theorem 1.1 and Theorem 1.3]). Let $f$ be a zero-order meromorphic function, and $q \in \mathbb{C} \setminus \{0\}$. Then

(2.1) \hspace{1cm} T(r, f(qz)) = (1 + o(1))T(r, f(z))

and

(2.2) \hspace{1cm} N(r, f(qz)) = (1 + o(1))N(r, f(z))

on a set of lower logarithmic density 1.

Remark. From Remark 1 after Theorem 1.1 in [11], we know that $f(z)$ and $f(qz)$ are simultaneously of order zero.
Lemma 2.4 ([10, Theorem 2.17]). Let $f$ and $g$ be meromorphic functions, and the order of $f$ and $g$ is less than 1. If $f$ and $g$ share $0$ and $\infty$ CM, then $f \equiv kg$, where $k$ is a non-zero constant.

3. Proof of Theorem 1.1

If $q = 1$, then the conclusion holds. Now we consider the case that $q \neq 1$. Suppose first that $a_1, a_2, a_3 \in \mathbb{C}$. Denote

$$g(z) = \frac{f(z) - a_1 a_2 - a_2}{f(z) - a_2 a_3 - a_1},$$

then

$$g(qz) = \frac{f(qz) - a_1 a_3 - a_2}{f(qz) - a_2 a_3 - a_1}.$$

From the assumption of Theorem 1.1, we know $g(z)$ and $g(qz)$ share $0, \infty$ CM.

Suppose first that 1 is not a Picard exceptional value of $g(z)$ and $g(qz)$. Assume that $g(z) \neq g(qz)$, and from Lemma 2.2, we obtain

$$m(r, g) + m\left(r, \frac{1}{g}\right) + m\left(r, \frac{1}{g - 1}\right)$$

$$\leq 2T(r, g) - 2N(r, g) + N(r, \Delta qg) - N\left(r, \frac{1}{\Delta qg}\right) + S_1(r, g),$$

and so

$$T(r, g) \leq N(r, g) + N\left(r, \frac{1}{g}\right) + N\left(r, \frac{1}{g - 1}\right) + N(r, g(qz))$$

$$+ N(r, g) - 2N(r, g) - N\left(r, \frac{1}{\Delta qg}\right) + S_1(r, g).$$

(3.1)

Since 1 is a Picard exceptional value of $g(z)$, by combining (2.2) and (3.1), it follows that

$$T(r, g) \leq N(r, g) + N\left(r, \frac{1}{g}\right) - N\left(r, \frac{1}{\Delta qg}\right) + S_1(r, g).$$

(3.2)

Since $g(z)$ and $g(qz)$ share $0, \infty$ CM, we get

$$N(r, g) + N\left(r, \frac{1}{g}\right) \leq N\left(r, \frac{1}{\Delta qg}\right).$$

(3.3)

From (3.2) and (3.3), we conclude that

$$T(r, g) = S_1(r, g),$$

which is impossible. Hence, we conclude that $f(z) = f(qz)$.
It remains to consider the case that one of \( a_j (j = 1, 2, 3) \) is infinite. Without loss of generality, we suppose that \( a_1 = \infty \), while \( a_2, a_3 \in \mathbb{C} \). Take \( d \in \mathbb{C} \setminus \{ a_2, a_3 \} \) and denote \( h(z) = \frac{1}{f(z)-d} \), \( b_2 = \frac{1}{a_2-d} \) and \( b_3 = \frac{1}{a_3-d} \). Then \( b_2, b_3 \in \mathbb{C} \setminus \{0\} \) are two distinct values. Hence \( h(z) \) and \( h(qz) \) share 0, \( b_2 \) CM and \( b_3 \) IM. By the above argument, we get \( h(z) = h(qz) \), and therefore \( f(z) = f(qz) \).

4. Proof of Theorem 1.2

From the fact that a non-constant meromorphic function of zero-order can have at most one Picard exceptional value (see, e.g., [2, p. 114]), we obtain that \( N(r, \frac{1}{f(z)-a_1}) \neq 0 \) and \( N(r, \frac{1}{f(z)-a_2}) \neq 0 \). Let

\[
F(z) = \frac{f(z) - a_1}{a_2 - a_1} \quad \text{and} \quad F(qz) = \frac{f(qz) - a_1}{a_2 - a_1}.
\]

Then \( F(z) \) and \( F(qz) \) share 0 and 1 IM. Clearly, neither 0 nor 1 is a Picard exceptional value of \( F(z) \). From Lemma 2.3, we obtain that

\[
T(r, F(qz)) = T(r, F(z)) + S_1(r, F).
\]

Denote

\[
V(z) = \frac{F'(z)(F(qz) - F(z))}{F(z)(F(z) - 1)}.
\]

Lemma 2.1 and the lemma on logarithmic derivative yield that \( m(r, V) = S_1(r, F) \). From (4.3), we know the poles of \( V(z) \) are at the zeros and 1-points of \( F(z) \). Since \( F(z) \) and \( F(z + c) \) share 0 and 1, we get \( N(r, V) = S(r, F) \). Therefore, \( T(r, V) = S_1(r, F) \).

Case 1. If \( V \neq 0 \), then \( F(z) \neq F(qz) \). From (4.3) and Lemma 2.1, we have

\[
N \left( r, \frac{1}{F(z)} \right) + \overline{N} \left( r, \frac{1}{F(z) - 1} \right) = N \left( r, \frac{F'(z)}{F(z)(F(z) - 1)} \right) + S(r, F)
\]

\[
= N \left( r, \frac{V}{F(qz) - F(z)} \right) + S(r, F)
\]

\[
\leq T(r, F(qz) - F(z)) + S_1(r, F) = m(r, F(qz) - F(z)) + S_1(r, F)
\]

\[
\leq m \left( r, \frac{F(qz) - F(z)}{F(z)} \right) + m(r, F(z)) + S_1(r, F)
\]

\[
\leq T(r, F) + S_1(r, F).
\]

According to second main theorem and above inequality, we get

\[
T(r, F) = N \left( r, \frac{1}{F} \right) + \overline{N} \left( r, \frac{1}{F - 1} \right) + S_1(r, F).
\]
Now we define
\[(4.5)\]
\[U(z) = \frac{F'(qz)(F(qz) - F(z))}{F(qz)(F(qz) - 1)}\]

Using the same argument as above, we know that \(T(r, U) = S_1(r, F(qz)) = S_1(r, F(z))\).

In what follows, we denote \(S_f g(m, n)\) for the set of those points \(z \in \mathbb{C}\) such that \(z\) is an \(a\)-point of \(f\) with multiplicity \(m\) and an \(a\)-point of \(g\) with multiplicity \(n\). Let \(N(m, n)(r, \frac{1}{F(z)})\) and \(N(m, n)(r, \frac{1}{F(qz)})\) denote the counting function and reduced counting function of \(f\) with respect to the set \(S_f g(m, n)\), respectively.

For any point \(z_0 \in S_{F(z) \sim F(qz)}(m, n)(0)\), we have \(mn \neq 0\), since 0 is not a Picard exceptional value of \(F(z)\) as we discuss above. From (4.3), (4.5) and the Taylor expansion of \(F(z)\) and \(F(qz)\) at \(z_0\), by calculating carefully, we get that
\[(4.6)\]
\[-V(z_0) = n\left(\frac{F'(qz_0) - F'(z_0)}{m}\right)\]
and
\[(4.7)\]
\[-U(z_0) = m\left(\frac{F'(qz_0) - F'(z_0)}{n}\right)\]

From (4.6) and (4.7), we know \(nV(z_0) = mU(z_0)\).

If \(nV = mU\), then we deduce that
\[n\left(\frac{F'(z)}{F(z) - 1} - \frac{F'(z)}{F(z)}\right) = m\left(\frac{F'(qz)}{F(qz) - 1} - \frac{F'(qz)}{F(qz)}\right),\]
which implies that
\[\left(\frac{F - 1}{F}\right)^n = b\left(\frac{F(qz) - 1}{F(qz)}\right)^m,\]
where \(b\) is a non-zero constant. If \(m \neq n\), then we get from above equality and (4.2) that
\[nT(r, F(z)) = nT(r, F(qz)) + S_1(r, F) = mT(r, F(z)) + S_1(r, F),\]
which is a contradiction. If \(m = n\), then we get
\[\left(\frac{F'(z)}{F(z) - 1} - \frac{F'(z)}{F(z)}\right) = \left(\frac{F'(qz)}{F(qz) - 1} - \frac{F'(qz)}{F(qz)}\right).\]

Hence
\[(4.8)\]
\[\frac{F(z) - 1}{F(z)} = d\frac{F(qz) - 1}{F(qz)},\]
where \(d\) is a non-zero constant. If \(d = 1\), then we obtain \(F(z) = F(qz)\), which contradicts the assumption of Case 1. It remains to consider the case that
\[ d \neq 1. \] It follows from (4.8) that
\[ \frac{d - 1}{d} \frac{F(z)}{F(z)} = \frac{1}{F(qz)}. \]
Since \( N(r, F(z)) = N(r, F(qz)) = 0 \), we get \( N(r, \frac{1}{F(z) - 1}) = 0 \). Clearly, \( \frac{1}{1 - \frac{1}{d}} \neq 0 \) and \( \frac{1}{1 - \frac{1}{d}} \neq 1 \), then apply the second main theorem, resulting in
\[ 2T(r, F) \leq N\left(r, \frac{1}{F}\right) + N\left(r, \frac{1}{F - 1}\right) + S(r, F), \]
which contradicts (4.4).

Hence \( nV \neq mU \). By the above argument, we know any point \( z_0 \in S_{F(z) - F(qz)}(0) \) satisfies that \( nV(z_0) = mU(z_0) \). Therefore,
\[ \mathcal{N}_{(m, n)} \left(r, \frac{1}{F}\right) \leq N \left(r, \frac{1}{nU - mV}\right) = S_1(r, F). \]
Using the same reason, we get
\[ \mathcal{N}_{(m, n)} \left(r, \frac{1}{F - 1}\right) \leq N \left(r, \frac{1}{mU - nV}\right) = S_1(r, F). \]
It follows that
\[ (4.9) \quad \mathcal{N}_{(m, n)} \left(r, \frac{1}{F}\right) + \mathcal{N}_{(m, n)} \left(r, \frac{1}{F - 1}\right) = S_1(r, F). \]
From Lemma 2.3, (4.4) and (4.9), we obtain that
\[ T(r, F) = \mathcal{N}(r, \frac{1}{F}) + \mathcal{N}(r, \frac{1}{F - 1}) + S_1(r, F) \]
\[ = \sum_{m, n} (\mathcal{N}_{(m, n)}(r, \frac{1}{F}) + \mathcal{N}_{(m, n)}(r, \frac{1}{F - 1})) + S_1(r, F) \]
\[ = \sum_{m + n \geq 5} (\mathcal{N}_{(m, n)}(r, \frac{1}{F}) + \mathcal{N}_{(m, n)}(r, \frac{1}{F - 1})) + S_1(r, F) \]
\[ \leq \frac{1}{5} \sum_{m + n \geq 5} (N_{(m, n)}(r, \frac{1}{F}) + N_{(m, n)}(r, \frac{1}{F - 1}) \]
\[ + N_{(m, n)}(r, \frac{1}{F(qz)}) + N_{(m, n)}(r, \frac{1}{F(qz) - 1})) + S_1(r, F) \]
\[ \leq \frac{2}{5} T(r, F) + \frac{2}{5} T(r, F(qz)) + S_1(r, F) \]
\[ = \frac{4}{5} T(r, F) + S_1(r, F), \]
which is a contradiction.

Case 2. If \( V = 0 \), then \( F(z) = F(qz) \). Clearly, \( f(z) = f(qz) \). This completes the proof of Theorem 1.2.
5. Proof of Theorem 1.3

Let \( G(z) = \frac{F(z)}{z} \), then we know \( G(z) \) and \( G(qz) \) share 1 and \( \infty \) CM, and since the order of \( f \) is zero, it follows that

\[
\frac{G(qz) - 1}{G(z) - 1} = \tau,
\]

where \( \tau \) is a non-zero constant. Rewriting the above equation, gives

\[
G(z) + \frac{1}{\tau} - 1 = \frac{G(qz)}{\tau}.
\]

Assume that \( \tau \neq 1 \). Noting (2.2) and (5.1), the second main theorem yields

\[
nT(r, f(z)) = T(r, G(z)) \leq N(r, G(z)) + N\left(r, \frac{1}{G(z)}\right) + N\left(r, \frac{1}{G(z) - 1 + \frac{1}{\tau}}\right) + S(r, f)
\]

\[
(5.2)
\]

\[
\leq N(r, f(z)) + N\left(r, \frac{1}{f(z)}\right) + N\left(r, \frac{1}{f(qz)}\right) + S(r, f)
\]

\[
\leq N(r, f(z)) + 2N\left(r, \frac{1}{f(z)}\right) + S_1(r, f)
\]

\[
\leq 3T(r, f(z)) + S_1(r, f),
\]

which contradicts the assumption that \( n \geq 4 \). Hence, we get \( \tau = 1 \), which implies that \( G(z) = G(qz) \), that is, \( f^n(z) = f^n(qz) \). So we have \( f(z) = tf(qz) \) for a constant \( t \) with \( t^n = 1 \).

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