SEMI-DIVISORIALITY OF HOM-MODULES AND INJECTIVE COGENERATOR OF A QUOTIENT CATEGORY

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Abstract. In this paper, we study $w$-nullity and (co-)semi-divisoriality of Hom-modules and the semi-divisorial envelope of $\text{Hom}_R(M,N)$ under suitable conditions on $R, M,$ and $N$. We also investigate an injective cogenerator of a quotient category.

1. Introduction

Let $R$ be an integral domain. In [17] Wang and McCasland defined semi-divisorial closure, or $w$-closure for torsion-free $R$-modules. In [7], H. Kim extended this notion to any $R$-module and introduced and studied the related notions of co-semi-divisoriality and $w$-nullity. In [7, 8, 9] these concepts were then used to give new module-theoretic characterizations of $t$-linkative domains, generalized GCD domains, and strong Mori domains, classes of domains widely considered in multiplicative ideal theory.

Earlier, in [1, 12, 13], Beck, Nishi and Shinagawa investigated injective modules over a Krull domain in terms of co-divisorial modules, pseudo-null modules, and divisorial modules and investigated pseudo-nullity and (co-)divisoriality of Home-modules. In particular, it was shown that in the case of a Krull domain $R$ with quotient field $K$, the injective envelope $E(K/R)$ of $K/R$ is a cogenerator of the quotient category $\text{Mod}(R)//\mathcal{M}_0$, where $\text{Mod}(R)$ is the category of all unitary $R$-modules and $\mathcal{M}_0$ is the thick subcategory of the modules with trivial maps into the codivisorial modules. Recently, in [11] Moucouf characterized the rings of Krull type $R$ with quotient field $K$ such that the (canonical) functorial image of $E(K/R)$ is an injective cogenerator of the quotient category $\text{Mod}(R)//\mathcal{M}_0$. Also in [16], Wang investigated the case when Hom-modules are semi-divisorial in torsion-free.

Received July 31, 2009.
2010 Mathematics Subject Classification. Primary 13A15; Secondary 13D30.
Key words and phrases. (co-)semi-divisorial, $w$-null, cogenerator, Hom-module, H-domain, Krull domain, torsion theory.

This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(2010-0011996).

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In this paper, we study an injective cogenerator of a quotient category and \( w \)-nullity and (co-)semi-divisoriality of Hom-modules using methods developed in [1, 11, 12, 13]. As a corollary, for the class of completely integrally closed domains, we characterize Krull domains in terms of an injective cogenerator of a quotient category. We also investigate the semi-divisorial envelope of \( \text{Hom}_R(M, N) \) under suitable conditions on \( R, M, \) and \( N \).

Throughout this paper, \( R \) denotes an integral domain with quotient field \( K \).

Let \( \mathcal{F}(R) \) denote the set of nonzero fractional ideals of \( R \). Recall that the function on \( \mathcal{F}(R) \) defined by \( A \mapsto (A^{-1})^{-1} = A_w \) is a star operation called the \( w \)-operation, where \( A^{-1} = R :_K A = \{ x \in K \mid xA \subseteq R \} \). An ideal \( J \) of \( R \) is called a Glaz-Vasconcelos ideal if \( J \) is a finitely generated ideal of \( R \) with \( J^{-1} = R \). We abbreviate this as \( GV \)-ideal, denoted by \( J \in \text{GV}(R) \). Following [17], a torsion-free \( R \) module \( M \) is called a \( w \)-module if \( Jx \subseteq M \) for \( J \in \text{GV}(R) \) and \( x \in M \otimes K \) implies that \( x \in M \), which is said to be semi-divisorial in [4]. For a torsion-free \( R \)-module \( M \), Wang and McCasland defined the \( w \)-envelope of \( M \) in [17] as \( M_w = \{ x \in M \otimes K \mid Jx \subseteq M \) for some \( J \in \text{GV}(R) \} \). In particular, if \( I \) is a nonzero fractional ideal, then \( I_w = \{ x \in K \mid Jx \subseteq I \) for some \( J \in \text{GV}(R) \}\). The canonical map \( I \mapsto I_w \) on \( \mathcal{F}(R) \) is a star-operation, denoted \( w \). It was shown in [17] that a prime ideal \( P \) of \( R \) is a \( w \)-ideal if and only if \( P_w \neq R \). Therefore, all prime ideals contained in a proper \( w \)-ideal of \( R \) are also \( w \)-ideals.

We denote by \( w\text{-Max}(R) \) the set of \( w \)-maximal ideals of \( R \). It is also worth noting that \( w \) distributes over (finite) intersections [17, Proposition 2.5]. For unexplained terminology and notation, we refer to [2, 3, 14].

2. \( w \)-null and (co-)semi-divisorial Hom-modules

In [7], H. Kim introduced the notions of “co-semi-divisoriality” and “\( w \)-nullity” of a module as follows. Let \( M \) be a module over an integral domain \( R \) and let \( \tau(M) := \{ x \in M \mid (O(x))_w = R \} \), where \( O(x) := (0 :_R x) = \text{ann}_R(x) \) is the order ideal of \( x \). Then \( \tau(M) \) is a submodule of \( M \). \( M \) is said to be co-semi-divisorial (resp., \( w \)-null) if \( \tau(M) = 0 \) (resp., \( \tau(M) = M \)). Note that the notions of co-semi-divisoriality and \( w \)-nullity can be interpreted in terms of a suitable torsion theory [2, Proposition IX.6.2 and Proposition IX.6.4] (with \( \mathcal{P} = w\text{-Max}(R) \)).

Let \( R \) be an integral domain, let \( \mathcal{F}_r(R) \) denote the full subcategory of \( \text{Mod}(R) \) consisting of all modules \( M \) such that \( M_P = 0 \) for all \( P \in w\text{-Max}(R) \), and let \( \mathcal{F}_c(R) \) denote the full subcategory of all \( R \)-modules \( M \) have no subobject other than zero belonging to \( \mathcal{F}_r(R) \). Finally let \( \mathcal{E}_r(R) \) be the full subcategory of \( \text{Mod}(R) \) consisting of all co-semi-divisorial and semi-divisorial \( R \)-modules.

In an abelian category \( \mathcal{A} \), we have the following definitions:

(a) An injective object \( E \) is called an injective cogenerator if \( \text{Hom}_{\mathcal{A}}(M, E) \neq 0 \) for every \( M \in \mathcal{A} \) that is not a zero object.
(b) A nonempty full subcategory $C$ of $A$ is said to be thick if, for each short exact sequence $0 \to L \to M \to N \to 0$ in $A$, $M$ is an object of $C$ if and only if $L$ and $N$ are objects of $C$. It is also called a Serre subcategory of $A$.

It is clear that $\mathcal{T}(R)$ is a thick subcategory of $\text{Mod}(R)$. Then we can now consider the quotient category $\text{Mod}(R) = \mathcal{T}(R)$ and the canonical functor $T: \text{Mod}(R) \to \text{Mod}(R)/\mathcal{T}(R)$.

As usual, we denote by $E(M)$ the injective envelope of an $R$-module $M$. The following result will be useful later on.

**Proposition 2.1.** The following statements are equivalent for an $R$-module $M$.

1. $M$ is co-semi-divisorial, i.e., $M \in \mathcal{T}(R)$.
2. $\mathcal{O}(x)$ is a $w$-ideal for every element $x \in M$.
3. $(O(x))_w \neq R$ for every nonzero element $x \in M$.
4. $\text{Hom}_R(N, M) = 0$ for every $w$-null $R$-module $N$.
5. $\text{Hom}_R(N, E(M)) = 0$ for every $w$-null $R$-module $N$.

**Proof.** The equivalences of (1), (2), (3), and (4) are given in [7, Proposition 2.6], while the equivalence of (1) and (5) follows from [6, Proposition 1.2]. □

Note from [17, Proposition 1.4] that the annihilator ideal of any submodule of a co-semi-divisorial module is a $w$-ideal. Recall from [1] that a module $M$ is said to be codivisorial if the annihilator of every nonzero element of $M$ is a divisorial ideal. Thus in a Krull domain, the notions of co-semi-divisoriality and codivisoriality are the same.

Recall from [16, Definition 4.5] that an $R$-module $M$ is said to be $w$-vanishing if $M_P = 0$ for any maximal $w$-ideal $P$ of $R$.

**Proposition 2.2.** Let $N$ be an $R$-module. Then the following statements are equivalent.

1. $N$ is $w$-null, i.e., $M \in \mathcal{T}(R)$.
2. For each $x \in N$, $\mathcal{O}(x)$ is not contained in any maximal $w$-ideal.
3. $N$ is $w$-vanishing.
4. There is a torsion-free $R$-module $F$ with $N \cong F_w/F$.
5. $\text{Hom}_R(N, E(M)) = 0$ for every co-semi-divisorial $R$-module $M$.

**Proof.** The equivalences of (1), (2), (3), and (4) are given in [7, Proposition 9.3], while the equivalence of (1) and (5) follows from [6, Proposition 1.2]. □

Now we study $w$-nullity and (co-)semi-divisoriality of Hom-modules. It was shown in [7, Proposition 3.1] that an $R$-module $M$ is co-semi-divisorial if and only if $\text{Hom}_R(\mathcal{Z}(R), M) = 0$, where $\mathcal{Z}(R) := \bigoplus_{I \leq R \mid I_w = R} R/I$.

**Proposition 2.3.** Let $R$ be an integral domain and let $M$ and $N$ be $R$-modules. If $M$ is co-semi-divisorial, then so is $\text{Hom}_R(N, M)$. 


Proof. By [7, Proposition 2.6], it suffices to show that $\text{Hom}(L, \text{Hom}_R(N, M)) \cong \text{Hom}_R(N, \text{Hom}_R(L, M)) = 0$ since $M$ is co-semi-divisorial.

Proposition 2.4. Let $R$ be an integral domain and let $M$ and $N$ be any $R$-module. If $M$ is $w$-null, then so is $\text{Tor}_n^R(N, M)$ for all $n \geq 0$.

Proof. First we consider the case $n = 0$. For every co-semi-divisorial $R$-module $L$ we have $\text{Hom}(N \otimes_R M, E(L)) \cong \text{Hom}_R(N, \text{Hom}_R(M, E(L))) = 0$ since $M$ is $w$-null; therefore $N \otimes_R M$ is $w$-null by Proposition 2.2. For the case when $n \geq 1$, we consider a projective resolution of $N$:

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_2 \to P_1 \to P_0 \to N \to 0.$$

Then, since each $P_i \otimes M$ is $w$-null, we can see that $\text{Tor}_n^R(N, M)$ is $w$-null for every $n \geq 0$ by noting that the submodules and homomorphic images of $w$-null modules are also $w$-null.

Now we recall some definitions from [7]: Let $M$ be an $R$-module. Then $W(M) := \pi^{-1}(\tau(E(M)/M))$ is called the semi-divisorial envelope of $M$, where $\pi : E(M) \to E(M)/M$ is the canonical projection, $M$ is said to be semi-divisorial if $W(M) = M$, and $M$ is said to be weakly $w$-flat if $\text{Tor}_1^R(Z(R, M)) = 0$. It is clear from the definition that every injective $R$-module is semi-divisorial.

Let $N$ be an $R$-module. Then we denote $U_w(N) := \{L \mid L$ is a submodule of $N$ such that $(L :_R x) = R$ for every $x \in N\}$.

Proposition 2.5. The following statements are equivalent for an $R$-module $M$.

1. $M$ is weakly $w$-flat.
2. $M^p := \text{Hom}_Z(M, \mathbb{Q}/\mathbb{Z})$ is semi-divisorial.
3. $I \otimes_R M \to M$ is a monomorphism for all $I \in U_w(R)$.
4. $L \otimes_R M \to N \otimes_R M$ is a monomorphism for all $L \in U_w(N)$.

Proof. The equivalence of (1) and (2) is given in [7, Proposition 4.3], while the equivalences of (2), (3), and (4) are given in [14, IX, Exercise 25].

Let $M$ be a semi-divisiorial $R$-module and $N$ be an $R$-module. Then it was shown in [7, Corollary 3.4] that if $\text{Hom}_R(\text{Tor}_1^R(Z(R), N), M) = 0$, then $\text{Hom}_R(N, M)$ is semi-divisorial.

Theorem 2.6. Let $R$ be an integral domain, $M$ be a semi-divisorial $R$-module, and $N$ be an $R$-module. Then $\text{Hom}_R(N, M)$ is semi-divisorial if one of the following conditions is satisfied:

1. $M$ is co-semi-divisorial;
2. $N$ is weakly $w$-flat.

Proof. It suffices to show that $\text{Hom}_R(\text{Tor}_1^R(Z(R), N), M) = 0$ by [7, Corollary 3.4].
(i) Note that \( R/I \) is \( w \)-null for every \( I \in \mathcal{U}_w(R) \) ([7, Proposition 2.5]). Thus we have that \( \text{Tor}_1^R(R/I, N) \) is \( w \)-null for every \( I \in \mathcal{U}_w(R) \). Now since \( \text{Tor} \) commutes with direct sums and \( w \)-nullity is closed under direct sums, we have \( \text{Tor}_1^R(\mathcal{Z}(R), N) \) is \( w \)-null. Therefore \( \text{Hom}_R(\text{Tor}_1^R(\mathcal{Z}(R), N), M) = 0 \) by the co-semi-divisoriality of \( M \) ([7, Proposition 2.6]).

(ii) This follows from the definition of “weakly \( w \)-flat”.

It was shown in [5, Proposition 2.2] that for a rank one \( \mathcal{N} \) flat ideal \( I \subset K \), the endomorphism \( \text{End}_R(I)(= I : I) \) of \( I \) is semi-divisorial. We extend this result to any \( \mathcal{N} \) flat module in the following corollary. Note that flat \( R \)-modules are torsion-free (and so co-semi-divisorial) for every integral domain \( R \).

**Corollary 2.7.** Let \( R \) be an integral domain.

1. If \( M \) is a flat \( R \)-module, then \( \text{End}_R(M) \) is a semi-divisorial \( R \)-module.
2. If \( M \) is a co-semi-divisorial and semi-divisorial \( R \)-module, then so is \( \text{End}_R(M) \).
3. If \( M \) is co-semi-divisorial, then \( M^* = \text{Hom}_R(M, R) \) is semi-divisorial.

### 3. Semi-divisorial equivalence

In this section, we investigate the semi-divisorial envelope of \( \text{Hom}_R(M, N) \) under suitable conditions on \( R, M, \) and \( N \). To do so, we need some definitions and results.

**Lemma 3.1** ([15, Proposition 1.1]). Let \( R \) be an integral domain and let \( L \to M \to N \) be an exact sequence of \( R \)-modules. If \( L \) and \( N \) are \( w \)-null, then so is \( M \).

Let \( M \) and \( N \) be \( R \)-modules and let \( f : M \to N \) be an \( R \)-homomorphism. Then \( f \) is said to be \( w \)-injective (resp., \( w \)-surjective) if \( \ker(f) \) (resp., \( \coker(f) \)) is \( w \)-null. And \( f \) is said to be \( w \)-isomorphic if \( f \) is both \( w \)-injective and \( w \)-surjective.

**Lemma 3.2** ([15, Lemma 1.2]). Let \( R \) be an integral domain and let \( f : L \to M \) and \( g : M \to N \) be homomorphisms of \( R \)-modules. If \( f \) and \( g \) are \( w \)-injective (resp., \( w \)-surjective or \( w \)-isomorphic), then so is \( gf \).

**Theorem 3.3** ([7, Theorem 8.1]). The following statements are equivalent for an integral domain \( R \).

1. If an \( R \)-module \( M \) is injective, then so is \( \tau(M) \).
2. \( E(\tau(M)) = \tau(E(M)) \) for any \( R \)-module \( M \).
3. Let \( N \) be an essential extension of \( M \). If \( M \) is \( w \)-null, then so is \( N \).
4. Let \( I \leq R \) such that \( I_w \neq R \). Then \( I :_R a \) is a \( w \)-ideal for some \( a \in R \setminus I_w \).
5. If \( M \) is not \( w \)-null, then \( M \) has a nonzero co-semi-divisorial submodule.
6. If \( I \leq R \), then there exists an ideal \( J \) of \( R \) such that \( J_w = R \) and \( I = I_w \cap J \).
Recall that an integral domain $R$ is said to be pseudo-t-linkative if $R$ satisfies one of the equivalent conditions of Theorem 3.3.

**Proposition 3.4.** Let $R$ be a pseudo-t-linkative domain with quotient field $K(\neq R)$. Let $f : M \to N$ be a homomorphism of $R$-modules and $p : M \to M/\tau(M)$, $q : N \to N/\tau(N)$ be the canonical projections.

1. There is a unique homomorphism $f_* : M/\tau(M) \to N/\tau(N)$ such that $f_*p = qf$.
2. If $f$ is $w$-injective, then $f_*$ is injective, and if $f$ is $w$-isomorphic, then so is $f_*$. 
3. If $f$ is $w$-isomorphic and $M$ is semi-divisorial, then $f_*$ is an isomorphism.

**Proof.** (1) The existence of $f_*$ follows from [7, Proposition 2.8] and its uniqueness is clear.

(2) Suppose first that $f$ is $w$-injective. Since $(M)$ is $w$-null, we have the following exact sequence

$$0 \to \ker(f) \to f^{-1}(\tau(N)) \to \tau(N).$$

This implies, by Lemma 3.1, that $f^{-1}(\tau(N))$ is $w$-null; therefore $\tau(M) = f^{-1}(\tau(N))$. Thus $f_*$ must be injective. If, moreover, $f$ is $w$-surjective, then $\operatorname{coker}(f)$ is $w$-null. Since the induced homomorphism of $\operatorname{coker}(f)$ to $\operatorname{coker}(f_*)$ is surjective, $\operatorname{coker}(f_*)$ must be $w$-null.

(3) Suppose that $M$ is semi-divisorial. Then $M \cong \tau(M) \oplus M/\tau(M)$ by [7, Corollary 8.9], and hence $M/\tau(M)$ is also semi-divisorial. Now the assertion follows from [7, Corollary 5.3].

It was shown in [16, Proposition 2.1] that $\operatorname{Hom}_R(M, N) = \operatorname{Hom}_R(M_w, N)$ for a torsion-free $R$-module $M$ and a $w$-module $N$. It follows from this result that $w$, as a functor from the category of all torsion-free $R$-modules to the category of all $w$-modules, is a reflector. The following result shows that the functor $W$ is a reflector from the category $\mathcal{F}_e(R)$ to the category $\mathcal{E}_e(R)$. By the $R$-dual of an $R$-module $M$ is meant the $R$-module $M^* = \operatorname{Hom}_R(M, R)$.

**Proposition 3.5.** Let $R$ be an integral domain and let $M, N$ be $R$-modules. Let $i$ be the canonical injection of $M$ to $W(M)$. If $N$ is co-semi-divisorial, then

$$\operatorname{Hom}_R(i, W(N)) : \operatorname{Hom}_R(W(M), W(N)) \to \operatorname{Hom}_R(M, W(N))$$

is an isomorphism. In particular, we have $M^* = (W(M))^*$.

**Proof.** Since $N$ is co-semi-divisorial, so is $W(N)$ by [7, Proposition 2.9]. On the other hand, $W(M)/M$ is $w$-null by the definition of a semi-divisorial envelope $W$. Therefore $\operatorname{Hom}(W(M)/M, W(N)) = 0$, which implies that $\operatorname{Hom}_R(i, W(N))$ is an injection. By [7, Proposition 3.2], we can see that $\operatorname{Hom}_R(i, W(N))$ is a surjection. □
Corollary 3.6. Let $R$ be a pseudo-$t$-linkative domain with quotient field $K(\neq R)$. Let $f : M \to N$ be a homomorphism of $R$-modules. Then there exists a unique homomorphism $T(f) : T(M) \to T(N)$ such that $T(f)i = jf$, where $i$ (resp., $f$) is the canonical homomorphism of $M$ (resp., $N$) to $T(M)$ (resp., $T(N)$). Moreover, if $f$ is a $w$-isomorphism, then $T(f)$ is an isomorphism.

Proof. The homomorphism $f$ induces the homomorphism $f_*$ of $M/\tau(M)$ to $N/\tau(N)$ by Proposition 3.4. Applying Proposition 3.5 to $f_*$, we can obtain a homomorphism $T(f) : T(M) \to T(N)$ such that $T(f)i = jf$.

It is easy to show that, similarly to the proof of Proposition 3.5, $\text{Hom}(i,T(N))$ is an injection. This shows the uniqueness of $T(f)$.

Suppose now that $f$ is a $w$-isomorphism. Then by Proposition 3.4, $f_*$ is a $w$-isomorphism ($f_*$ is necessarily injective). Since the canonical injection of $M/\tau(M)$ to $T(M)$ is an essential extension, $T(f)$ must be an injection. Since both $f_*$ and the canonical injection of $N/\tau(N)$ to $T(N)$ are $w$-surjective, so is the composition of them by Lemma 3.2. We can conclude from this fact that $T(f)$ is a $w$-surjection. Since a $w$-isomorphism of co-semi-divisorial and semi-divisorial modules is an isomorphism by [7, Corollary 5.3], $T(f)$ must be an isomorphism.

□

It was shown in [16, Proposition 2.3] that $(\text{Hom}_R(M,N))_w = \text{Hom}_R(M,N_w)$ for a torsion-free finitely generated $R$-module $M$ and a torsion-free $R$-module $N$. As a corollary, Wang obtained that $(\text{End}_R(M))_w = \text{End}_R(M_w)$ for a torsion-free finitely generated $R$-module $M$ ([16, Corollary 2.4]).

Theorem 3.7. Let $R$ be a pseudo-$t$-linkative domain. Let $M$ and $N$ be co-semi-divisorial $R$-modules. If $M$ is a submodule of a finitely generated $R$-module $L$, then we have

\[ W(\text{Hom}_R(M,N)) \cong \text{Hom}_R(W(M),W(N)). \]

Proof. By Proposition 3.5, we have only to prove

\[ W(\text{Hom}_R(M,N)) \cong \text{Hom}_R(M,W(N)). \]

Consider the following exact sequence

\[ 0 \to \text{Hom}_R(M,N) \to \text{Hom}_R(M,W(N)) \to \text{Hom}_R(M,W(N)/N). \]

Since $N$ is co-semi-divisorial, so is $W(N)$; thus, by Proposition 2.3, $\text{Hom}_R(M,N)$ and $\text{Hom}_R(M,W(N))$ are co-semi-divisorial. Also we have that $\text{Hom}_R(M,W(N))$ is semi-divisorial by Theorem 2.6. Since a $w$-isomorphism of co-semi-divisorial modules is an essential extension, it suffices to show that $\text{Hom}_R(M,W(N)/N)$ is $w$-null.

In general, for a submodule $M_2$ of the finitely generated $R$-module $M_2$ and a $w$-null $R$-module $N_1$, we will show that $\text{Hom}_R(M_1,N_1)$ is $w$-null. Set $N_2 := E(N_1)$. Then $N_2$ is $w$-null by [7, Theorem 8.1], since $R$ is pseudo-$t$-linkative. Let $\{x_1, \ldots, x_n\}$ be a system of generators of $M_2$ and let $f \in \text{Hom}_R(M_2,N_2)$. Then $\mathcal{O}(f) = \mathcal{O}(f(x_1)) \cap \cdots \cap \mathcal{O}(f(x_n))$. Since each $(\mathcal{O}(f(x_i)))_w = R$, we
have \((\mathcal{O}(f))_w = R\) by the distributivity of the star-operation \(w\) over finite intersection. Hence \(\text{Hom}_R(M_2, N_2)\) is \(w\)-null. Therefore, \(\text{Hom}_R(M_1, N_2)\) is \(w\)-null, since it is a homomorphic image of \(\text{Hom}_R(M_2, N_2)\). Thus \(\text{Hom}_R(M_1, N_1)\) is \(w\)-null since it is isomorphic to a submodule of \(\text{Hom}_R(M_1, N_2)\).

**Corollary 3.8.** Let \(R\) be a pseudo-\(t\)-linkative domain with quotient field \(K(\neq R)\) and let \(M\) and \(N\) be co-semi-divisorial and semi-divisorial \(R\)-modules. If \(M\) is a submodule of a finitely generated \(R\)-module, then \(\text{Hom}_R(M, N)\) is semi-divisorial.

Let \(M\) and \(N\) be an \(R\)-modules. We say that \(M\) is semi-divisorially equivalent to \(N\) if there exists a \(w\)-isomorphism of \(W(M)\) to \(W(N)\).

**Proposition 3.9.** Let \(R\) be a pseudo-\(t\)-linkative domain with quotient field \(K(\neq R)\). Let \(M\) and \(N\) be \(R\)-modules.

1. \(M\) is semi-divisorially equivalent to \(N\) if and only if \(W(M/\tau(M))\) is isomorphic to \(W(N/\tau(N))\). In particular, the “semi-divisorial equivalence” is an equivalence relation.

2. If \(M\) is \(w\)-isomorphic to \(N\), then \(M\) is semi-divisorially equivalent to \(N\).

**Proof.** (1) The necessity follows from the facts that \(W(M) \cong W(\tau(M)) \oplus W(M/\tau(M))\) and \(W(N) \cong W(\tau(N)) \oplus W(N/\tau(N))\) by [7, Corollary 8.9] and \(W(\tau(M))\) and \(W(\tau(N))\) are \(w\)-null by [7, Theorem 8.1] since \(R\) is pseudo-\(t\)-linkative. The sufficiency follows from Proposition 3.4.

2. The assertion follows immediately from Corollary 3.6. \(\square\)

**4. Injective cogenerator of a quotient category**

In this section, we generalize some results of \([1, 11]\) related to an injective cogenerator in a quotient category. We recall from \([4]\) that a domain \(R\) is said to be an \(H\)-domain if every ideal \(I\) of \(R\) with \(I^{-1} = R\) is quasi-finite (i.e. \(I^{-1} = J^{-1}\) for some finitely generated subideal \(J\) of \(I\)).

**Theorem 4.1.** Let \(R\) be an \(H\)-domain with quotient field \(K(\neq R)\), and let \(M\) be any \(R\)-module. Then \(M\) is \(w\)-null if and only if \(\text{Hom}_R(M, E(K/R)) = 0\).

**Proof.** (\(\Rightarrow\)): This follows from Proposition 2.1 since \(E(K/R)\) is co-semi-divisorial by [7, Corollary 2.11].

(\(\Leftarrow\)): Suppose that \(M\) is not \(w\)-null and let \(N = M/\tau(M)\). By Proposition 2.1 and [7, Proposition 2.8], there is a non-zero element of \(x \in N\) such that \(\mathcal{O}(x)\) is a proper \(w\)-ideal and hence \(R : \mathcal{O}(x) \nsubseteq R\) (since \(R\) is an \(H\)-domain). Let \(a \in R : \mathcal{O}(x) \setminus R\). Then \(R :_R a \supset \mathcal{O}(x)\). Let \(f : R \rightarrow K/R\) be the homomorphism defined by \(f(b) = ab\), where \(ab\) is the class of \(ab\) in \(K/R\). Since \(\ker(f) = R :_R a \supset \mathcal{O}(x)\), there is a non-zero homomorphism \(g : R/\mathcal{O}(x) \rightarrow K/R\) such that \(f = gp\), where \(p\) is the canonical projection of \(R\) to \(R/\mathcal{O}(x)\). Let \(i\) be the canonical injection of \(R/\mathcal{O}(x)(\cong Rx)\) to \(N\). Then there is a non-zero homomorphism \(h\) of \(N\) to \(E(K/R)\) such that \(ih = hj\), and hence \(hq\) is a
non-zero homomorphism of $M$ to $E(K/R)$, where $q$ is the canonical projection of $M$ to $N$. \hfill \Box

Since $K/R \in \mathcal{T}_e(R)$, i.e., $K/R$ has no subobject other than zero belonging to $\mathcal{T}_e(R)$, then $T(E(K/R))$ is the injective envelope of the object $T(K/R)$ of $\Mod(R)/\mathcal{T}_e(R)$.

**Corollary 4.2.** If $R$ is an H-domain, then $T(E(K/R))$ is an injective cogenerator in the quotient category $\Mod(R)/\mathcal{T}_e(R)$. Hence every co-semi-divisorial and semi-divisorial module over an H-domain can be embedded in an injective module.

**Proof.** Let $T(N) \in \Mod(R)/\mathcal{T}_e(R)$ with $\Hom_{\Mod(R)/\mathcal{T}_e(R)}(T(N), T(E(K/R))) = 0$. Then by [11, Lemma 2.6] we have $\Hom_{\Mod(R)}(N, E(K/R)) = 0$, and by Theorem 4.1 we have $N \in \mathcal{T}_e(R)$, and then $T(N) = 0$. It is clearly seen that $T(E(K/R))$ is a cogenerator object of $\Mod(R)/\mathcal{T}_e(R)$. The last assertion follows from [14, Proposition I.6.6]. \hfill \Box

**Lemma 4.3.** Let $R$ be an integral domain, let $P \in w\text{-Max}(R)$, let $M$ be a co-semi-divisorial $R$-module and let $f : R/P \to M$ a homomorphism. Then either $f \equiv 0$ or $f$ is injective.

**Proof.** Suppose that $f \neq 0$ and let $f(1) = x$. Then we have $x \in M$. Since $M$ is co-semi-divisorial, then $\mathcal{O}(x)$ is a $w$-ideal, and since $x \neq 0$, there exists $Q \in w\text{-Max}(R)$ such that $\mathcal{O}(x) \subset Q$, but since $P \subset \mathcal{O}(x)$, we have $P \subset Q$ and hence $P = Q$, so $\mathcal{O}(x) = P$ and $f$ is injective. \hfill \Box

We recall from [10, III.1.4] two facts related to $\mathcal{C}_e(R)$, $\Mod(R)/\mathcal{T}_e(R)$, and $T$.

(a) The subcategory $\mathcal{C}_e(R)$ of $\Mod(R)$ may be identified with $\Mod(R)/\mathcal{T}_e(R)$.

(b) Let $M$ be an $R$-module. Then $T(M) = W(M/\tau(M))$.

Therefore, we have that $T(E(K/R)) = W(E(K/R)/\tau(E(K/R))) \cong E(K/R)$.

**Theorem 4.4.** Let $R$ be an integral domain with quotient field $K$ satisfying $(R:v)_v = (R:v)_x$ for every $x \in K$. If $T(E(K/R))$ is an injective cogenerator in the quotient category $\Mod(R)/\mathcal{T}_e(R)$, then $R$ is an H-domain.

**Proof.** Note that if $R$ satisfies that $(R:v)_v = (R:v)_x$ for every $x \in K$, then $K/R$ is co-divisorial. Suppose that $R$ is not an H-domain. Then by [17, Proposition 5.7] there exists a prime ideal $P$ which is $w$-maximal but not a $v$-ideal. First we show that the module $R/P$ can not be injected in $E(K/R)$. If this were not so, then the kernel of the composition $R \xrightarrow{\Pi} R/P \to E(R/K)$ is $P$, where $\Pi$ is the canonical projection. Then by [1, Corollary 1.7] $P$ is a $v$-ideal, which is a contradiction. Thus by Lemma 4.3, $\Hom_{\Mod(R)}(R/P, E(K/R)) \neq 0$. So $\Hom_{\Mod(R)/\mathcal{T}_e(R)}(T(R/P), T(E(K/R))) \cong \Hom_{\Mod(R)}(W(R/P), E(K/R)) \cong \Hom_{\Mod(R)}(R/P, E(K/R)) = 0$ (note that the last isomorphism follows from Proposition 3.5). Since $T(E(K/R))$ is a cogenerator object in $\Mod(R)/\mathcal{T}_e(R)$,
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\[ T(R/P) = 0, \text{ and thus } R/P \in \mathcal{T}_r(R), \text{ i.e., } R/P \text{ is } w\text{-null. Hence } P_w = R, \text{ which is a contradiction. Therefore } R \text{ is an } H\text{-domain.} \]

It is well known that if \( R \) is a completely integrally closed domain, then \( R \) satisfies the hypothesis of Theorem 4.4. Now the following result follows from Corollary 4.2, Theorem 4.4, and the fact that an integral domain \( R \) is a Krull domain if and only if \( R \) is a completely integrally closed H-domain ([4, 3.2(d)]).

**Corollary 4.5.** Let \( R \) be a completely integrally closed domain. Then \( R \) is a Krull domain if and only if \( E(K/R) \) is an injective cogenerator in the quotient category \( \text{Mod}(R)/\mathcal{T}_r(R) \).

Let \( M \) be any \( R \)-module. We have a canonical mapping:

\[ \lambda_M : M \to \text{Hom}_R(\text{Hom}_R(M, E(K/R)), E(K/R)). \]

Let \( f \in \text{Hom}_R(M, E(K/R)) \). Then define \( \lambda_M(f) \) by the equation \( \lambda_M(m)(f) = f(m) \) for all \( m \in M \).

**Theorem 4.6.** Let \( R \) be an \( H \)-domain with quotient field \( K(\neq R) \), and let \( M \) be any \( R \)-module. Then \( M \) is co-semi-divisorial if and only if \( \lambda_M \) is injective.

**Proof.** (\( \Leftarrow \)): This follows from the facts that \( E(K/R) \) is co-semi-divisorial and \( \text{Hom}_R(L, N) \) is co-semi-divisorial whenever \( N \) is co-divisorial.

(\( \Rightarrow \)): Let \( x \in M \setminus \{0\} \). Since \( Rx \) is not \( w \)-null, we can find a homomorphism \( f : Rx \to E(K/R) \) such that \( f(x) \neq 0 \) by Theorem 4.1. Since \( E(K/R) \) is injective, we can lift \( f \) to a mapping \( \tilde{f} : M \to E(K/R) \). This shows that \( \lambda_M \) is injective, since \( \lambda_M(x)(\tilde{f}) = f(x) = f(x) \neq 0 \) and hence \( \lambda_M(x) \neq 0 \). \( \square \)

**References**


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