PRECONDITIONED GAUSS-SEIDEL ITERATIVE METHOD
FOR Z-MATRICES LINEAR SYSTEMS

HAILONG SHEN, XINHUI SHAO, ZHENXING HUANG, AND CHUNJI LI

Abstract. For $Ax = b$, it has recently been reported that the convergence of the preconditioned Gauss-Seidel iterative method which uses a matrix of the type $P = I + S(\alpha)$ to perform certain elementary row operations on is faster than the basic Gauss-Seidel method. In this paper, we discuss the adaptive Gauss-Seidel iterative method which uses $P = I + S(\alpha) + K(\beta)$ as a preconditioner. We present some comparison theorems, which show the rate of convergence of the new method is faster than the basic method and the method in [7] theoretically. Numerical examples show the effectiveness of our algorithm.

1. Introduction

We consider iterative methods for solving a linear system

\[(1)\quad Ax = b,\]

where $A$ is an $n \times n$ matrix with unit diagonal elements, $x$ and $b$ are $n$-dimensional vectors.

If we write $A = M - N$ with a nonsingular matrix $M$, then the basic iterative scheme for (1) is defined by

\[(2)\quad Mx_{k+1} = Nx_k + b, \quad k = 0, 1, 2, \ldots.\]

Let $T = M^{-1}N$ and $c = M^{-1}b$. Then (2) can be also be written as

\[(3)\quad x_{k+1} = Tx_k + c, \quad k = 0, 1, 2, \ldots.\]

Writing $A = I - L - U$ and taking $M = I - L$ in (2) or (3) yield the classic Gauss-Seidel method, where $I$ is the identity matrix, and $L$ and $D$ are strictly lower and strictly upper triangular matrices, respectively. The matrix $T = (I - L)^{-1}U$ is then called the Gauss-Seidel iteration matrix.
We now transform the original system (1) into the preconditioned form

\[ PAx = Pb. \]

Then the corresponding basic iterative scheme is

\[ M_p x_{k+1} = N_p x_k + Pb, \quad k = 0, 1, 2, \ldots, \]

where \( PA = M_p - N_p \) is a regular splitting of \( PA \), and \( M_p \) is a nonsingular matrix, the nonsingular matrix \( P \) is called a preconditioner, the iterative method used in linear equations \( PAx = Pb \) is called the preconditioned iterative method.

In 1987, Milaszewiez [8] used the following preconditioner

\[ P = I + C = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -a_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & 0 & \cdots & 1 \end{pmatrix}. \]

In 2003, Hadjidimos et al. [3] presented the following preconditioner

\[ P(\alpha) = I + C(\alpha) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -\alpha_2a_{21} & 1 & 0 & \cdots & 0 \\ -\alpha_3a_{31} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -\alpha_na_{n1} & 0 & 0 & \cdots & 1 \end{pmatrix}. \]

The preconditioner \( P(\alpha) \) in (7) is the general preconditioner of the matrix \( P \) in (6), when \( \alpha_i = 1, i = 2, 3, \ldots \), the preconditioner became the matrix \( P \).

In 1991, Gunawardena et al. [2] employed the following preconditioner

\[ P_S = I + S = \begin{pmatrix} 1 & -a_{12} & 0 & \cdots & 0 \\ 0 & 1 & -a_{23} & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & -a_{n-1,n} \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}. \]

In 1997, Kohno et al. [5] employed the general preconditioned form of the matrix

\[ P_S(\alpha) = I + S(\alpha) = \begin{pmatrix} 1 & -\alpha_1a_{12} & 0 & \cdots & 0 \\ 0 & 1 & -\alpha_2a_{23} & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & -\alpha_{n-1}a_{n-1,n} \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}. \]

When \( \alpha_i = 1, i = 1, 2, \ldots, n - 1 \), the preconditioner \( P_S(\alpha) \) is equal to \( P_S \).
In [6], \( P = I + S(\alpha) + K(\beta) \) is presented, where
\[
K(\beta) = \begin{cases} 
-\beta a_{ij}, & i = j + 1, \\
0, & \text{otherwise.}
\end{cases}
\]

The comparison theorem and numerical examples show that the method is the improvement for the iterative method in [5].

In this paper, we first consider the new preconditioner \( P = I + S(\alpha) + \tilde{K}(\beta) \), where
\[
\tilde{K}(\beta) = \begin{cases} 
-\beta a_{i1}, & i = 2, 3, \ldots, n, \\
0, & \text{otherwise.}
\end{cases}
\]

Next we discuss its convergence. Finally, we show with numerical examples that this method yields a considerable improvement in the rate of convergence for the iterative method.

2. Basic results

First, we need the following definitions and results.

**Definition 1 ([4]).** An \( n \times n \) real matrix \( A = (a_{ij}) \) is called Z-matrix, if \( a_{ij} \leq 0, \forall i \neq j \), and \( a_{ii} > 0 \); and an \( A \) is called diagonal dominant (or strictly diagonal dominant) Z-matrix, if \( \sum_{j=1}^{n} a_{ij} \geq 0 \) (or \( \sum_{j=1}^{n} a_{ij} > 0 \)).

**Definition 2 ([4]).** A matrix \( A \) is irreducible if there exists a permutation matrix \( P \) such that \( PAP^T = \begin{pmatrix} A_1 & 0 \\
A_2 & A_3 \end{pmatrix} \), where \( A_1 \) and \( A_3 \) are square matrices.

**Definition 3 ([1]).** Let \( A = (a_{ij})_{n \times n}, B = (b_{ij})_{n \times n} \). We say \( A > B \) if \( a_{ij} > b_{ij}, \forall 1 \leq i, j \leq n \).

**Definition 4 ([1]).** Let \( A = (a_{ij})_{n \times n}, B = (b_{ij})_{n \times n} \). We say \( A \geq B \) if \( a_{ij} \geq b_{ij}, \forall 1 \leq i, j \leq n \).

**Lemma 1 ([4]).** Let \( A \) be a nonnegative matrix. Then
(1) \( A \) has a positive real eigenvalue equal to its spectral radius.
(2) To the spectral radius \( \rho(A) \) there corresponds an eigenvector \( x > 0 \).
(3) If \( A \) is an irreducible matrix, then \( \rho(A) > 0 \) and there exists \( x > 0 \) such that \( Ax = \rho(A)x \).

**Lemma 2 ([4]).** Let \( A \) be a nonnegative matrix. Then
(1) If \( Ax \leq A \) for some nonnegative vector \( x \neq 0 \), then \( \alpha \leq \rho(A) \).
(2) \( A \) is a irreducible matrix if and only if \( (I + A)^{n-1} > 0 \).
(3) If \( Ax \leq \beta x \) for some positive vector \( x \), then \( \rho(A) \leq \beta \). Moreover, if \( A \) is irreducible and if \( 0 \neq ax \leq Ax \leq \beta x \), then \( \alpha < \rho(A) < \beta \), and \( x \) is a positive vector.
(4) If \( B \) is an \( n \times n \) matrix, \( 0 \leq A \leq B, A \neq B, A + B \) are all irreducible matrices, then \( \rho(A) < \rho(B) \).
Lemma 3 ([6]). Let $A = D - E - F$ be a $Z$-matrix, where $D, -E$ and $-F$ are diagonal, strictly lower triangular and strictly upper triangular parts of $A$, respectively. Then an upper bound of the spectral radius for the Gauss-Seidel iterative matrix $T$ is given by
\[
\rho(T) \leq \max \frac{\tilde{f}_i}{d_i - \tilde{e}_i}, \forall i,
\]
where $\tilde{d}_i, \tilde{e}_i$ and $\tilde{f}_i$ are sums of elements in the $i$-th row of $D, E$ and $F$, respectively.

3. New preconditioned iterative method and convergence analysis

We propose a preconditioned iterative method with
\[
P = I + S_\alpha^2 = \begin{pmatrix}
1 & -\alpha_1 a_{12} & -\alpha_2 a_{23} & \cdots & -\alpha_n a_{n,1} \\
-\beta_2 a_{21} & 1 & \cdots & \cdots & \cdots \\
-\beta_2 a_{21} & 1 & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-\beta_n a_{n1} & 1 & \cdots & \cdots & 1
\end{pmatrix},
\]
where $S_\alpha^2 = S(\alpha) + \tilde{K}(\beta)$,
\[
S(\alpha) = \begin{pmatrix}
0 & -\alpha_1 a_{12} & 0 & \cdots & 0 \\
0 & 0 & -\alpha_2 a_{23} & \cdots & \cdots \\
0 & 0 & 0 & \ddots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \cdots \\
0 & 0 & 0 & \cdots & -\alpha_n a_{n,1}
\end{pmatrix},
\]
\[
\tilde{K}(\beta) = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
-\beta_2 a_{21} & 0 & 0 & \cdots & \cdots \\
-\beta_3 a_{31} & 0 & 0 & \ddots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \cdots \\
-\beta_n a_{n1} & 0 & 0 & \cdots & 0
\end{pmatrix}
\]

Then
\[
A_\alpha^2 = (I + S_\alpha^2) A = I - L - U + S_\alpha^2 - S_\alpha^2 (L + U),
\]
\[
b_\alpha^2 = (I + S_\alpha^2) b.
\]
Let $S_\alpha^2 (L + U) = D_0 + L_0 + U_0$, where $D_0, L_0$ and $U_0$ are diagonal, strictly lower triangular and strictly upper triangular parts of $S_\alpha^2 (L + U)$, respectively, and
\[
D_0 = \text{diag} \{ \beta_1 a_{12} a_{21}, \alpha_2 a_{21} a_{12} + \beta_2 a_{23} a_{32}, \ldots, \alpha_{n-1} a_{n-1,1} a_{1,n-1} \}$
and matrix and $A$ satisfies the following conditions, then a matrix with unit diagonal elements. If for 

$$S = \alpha = \text{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n), \quad \beta = \text{diag}(\beta_1, \beta_2, \ldots, \beta_n)$$

and

$$A = I - D_0 - L + S\alpha - L_0 - (U - S^2 + U_0).$$

If

$$\begin{cases} 
\alpha_1 a_{12} a_{21} \neq 1, & i = 1, \\
\alpha_i a_{i,i+1} a_{i+1,i} + \beta_i a_{i,1} a_{1,i} \neq 1, & i = 2, \ldots, n - 1, \\
\beta_n a_{n,1} a_{1,n} \neq 1, & i = n,
\end{cases}$$

then matrix $(I - D_0 - L + S\alpha - L_0)^{-1}$ exists and the Gauss-Seidel iteration matrix $T_\alpha$ for $A_\alpha$ is defined by

$$T_\alpha = (I - D_0 - L + S\alpha - L_0)^{-1} (U - S^2 + U_0).$$

Let

$$S = \alpha = \text{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n), \quad \beta = \text{diag}(\beta_1, \beta_2, \ldots, \beta_n)$$

and

$$A = (I + S + K) A = \bar{D} - L - U, \quad \bar{T} = (\bar{D} - \bar{L})^{-1} U.$$

Obviously, when $\alpha_i = 0, i = 1, 2, \ldots, n$, the new method in this paper becomes the method in [3]; when $\beta_i = 0, i = 1, 2, \ldots, n$, the new method in this paper becomes the method in [5].

**Theorem 1.** Let $A \in \mathbb{R}^{n \times n}$ ($n \geq 3$) be a nonsingular diagonal dominant $Z$-matrix with unit diagonal elements. If $0 < \alpha_i \leq 1, 0 < \beta_i \leq 1$ for $i = 1, 2, \ldots, n$, $A$ satisfies the following conditions, then $A_\alpha$ is a strictly diagonal dominant $Z$-matrix and $\rho(T_\alpha) < 1$.

1. $a_{ii} - |a_{i1}| - |a_{i,i+1}| > 0, a_{11} - |a_{12}| > 0, a_{nn} - |a_{n1}| > 0$ for any $i = 2, 3, \ldots, n - 1$.

2. If $\sum_{j=1}^{n} a_{ij} = 0$ for $i = 1, 2, \ldots, n - 1$, then $a_{i,1} a_{i,i+1} \neq 0$.

3. There exists at least a row for $i$-th row and $(i + 1)$-th row such that it is strictly diagonal dominant row for any $i = 1, 2, \ldots, n - 1$.

4. There exists at least a strictly diagonal dominant row for the first row and $n$-th row, and if $\sum_{j=1}^{n} a_{nj} = 0$, then $a_{n1} \neq 0$.

**Proof.** In fact,

$$(A_\alpha)^{\beta}_{ii} = a_{ii} - \alpha_i a_{i,i+1} a_{i+1,i} - \beta_i a_{i,1} a_{1,i} \geq a_{ii} - |a_{i,i+1}| - |a_{i,1}| > 0.$$
For $i > j, i = 2, \ldots, n$, when $j = 1$,

\[(A_{ij})_1 = a_{i1} - \alpha_i a_{i,i+1} a_{i+1,i} - \beta_i a_{i1} a_{i1} \leq 0,\]

when $j = 2, \ldots, n - 1$,

\[(A_{ij})_j = a_{ij} - \alpha_i a_{i,i+1} a_{i+1,j} - \beta_i a_{i1} a_{1j} \leq 0.\]

For $i < j, i = 1, 2, \ldots, n - 1, j = 2, \ldots, n$, when $i + 1 = j$,

\[(A_{ij})_{i+1} = (1 - \alpha_i) a_{i,i+1} - \beta_i a_{i1} a_{1i+1} \leq 0.\]

Otherwise

\[(A_{ij})_{ij} = a_{ij} - \alpha_i a_{i,i+1} a_{i+1,j} - \beta_i a_{i1} a_{1j} \leq 0.\]

Hence, $A_{ij}$ is a $Z$-matrix.

From (2), (3) and (4), there exists at least an inequality such that it holds for the following inequalities:

$$
\sum_{j=1}^{n} a_{ij} > 0, \quad -\alpha_i a_{i,i+1} \sum_{j=1}^{n} a_{i+1,j} > 0, \quad -\beta_i a_{i1} \sum_{j=1}^{n} a_{1j} > 0.
$$

Then

\[(A_{ij})_j = \sum_{j=1}^{n} (A_{ij})_j = a_{ij} - \alpha_i a_{i,i+1} \sum_{j=1}^{n} a_{i+1,j} - \beta_i a_{i1} \sum_{j=1}^{n} a_{1j} > 0.

Hence, we know that $A_{ij}$ is a strictly diagonal dominant $Z$-matrix.

Now we decompose $A_{ij}$, let $A_{ij} = D_{ij} - L_{ij} - U_{ij}$, where $D_{ij}, L_{ij}$ and $U_{ij}$ are diagonal, strictly lower triangular and strictly upper triangular parts of $A_{ij}$, respectively, and let $d_{\alpha,i}^{ij}$, $l_{\alpha,i}^{ij}$ and $u_{\alpha,i}^{ij}$ are the sum of $i$-th row elements of $D_{ij}$, $L_{ij}$ and $U_{ij}$, respectively. From (18), we have $d_{\alpha,i}^{ij} - l_{\alpha,i}^{ij} - u_{\alpha,i}^{ij} > 0$; and from (17) and (18), we have $u_{\alpha,i}^{ij} \geq 0$. So

$$
\frac{u_{\alpha,i}^{ij}}{d_{\alpha,i}^{ij} - l_{\alpha,i}^{ij}} < 1.
$$

From Lemma 3, $\rho(T_{\alpha}^{ij}) < 1$. The proof is complete.

**Theorem 2.** Let $A = I - L - U$ be a $Z$-matrix, $0 < a_{i1} a_{i1} + a_{i,i+1} a_{i+1,i} < 1$ for $i = 2, \ldots, n - 1$, and $0 < a_{i1} a_{1n} < 1$. If $\rho(T) = 1$, then $\rho(T) = 1$.

**Proof.** It is obviously that $\tilde{A}$ in (14) is a $Z$-matrix. Since

$$
\tilde{A} = \left( I + \tilde{K} + \tilde{S} \right) A = \left( I + \tilde{K} + \tilde{S} \right) (I - L - U)
$$

$$
= I + \tilde{K} - L - SL - KL - \tilde{K}U - U - SU + S,
$$

$$
\rho(T) = 1.
$$

□
we have \(-KL = 0, I + \tilde{K} - L - SL\) is a lower triangular matrix; \(-U - SU + S\) is an upper triangular matrix. But
\[
\tilde{K}U = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & -a_{21}a_{12} & -a_{21}a_{13} & \cdots & -a_{21}a_{1n} \\
0 & -a_{31}a_{12} & -a_{31}a_{13} & \cdots & -a_{31}a_{1n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & -a_{n1}a_{12} & -a_{n1}a_{13} & \cdots & -a_{n1}a_{1n}
\end{pmatrix}.
\]

If \(-\tilde{K}U = M_1 + M_2\), where \(M_2\) is a strictly upper triangular part of \(-\tilde{K}U\), then
\[
\tilde{T} = \begin{pmatrix} \tilde{D} - \tilde{L} \end{pmatrix}^{-1} \tilde{U} = (I + \tilde{K} - L - SL + M_1)^{-1} (U + SU - S - M_2).
\]

Hence \(T\) and \(\tilde{T}\) are nonnegative matrices, and
\[
T = \begin{pmatrix} 0 & T_{12} \\ 0 & T_{22} \end{pmatrix}, \quad \tilde{T} = \begin{pmatrix} 0 & \tilde{T}_{12} \\ 0 & \tilde{T}_{22} \end{pmatrix},
\]
where \(T_{22}\) and \(\tilde{T}_{22}\) are all \((n - 1) \times (n - 1)\) irreducible matrices. From \(\rho(T) = 1\), we have \(\rho(T_{22}) = 1\).

From Lemma 4, \(T_{22}x' = x'\) for some vector \(x' > 0\). Now we construct the positive vector \(x = \begin{pmatrix} T_{12}x' \end{pmatrix}\), it satisfies \(Tx = x\).

Since \((I - L)^{-1} U x = x, U x = (I - L) x\) and
\[
\tilde{T}x = \begin{pmatrix} I + \tilde{K} - L - SL + M_1 \end{pmatrix}^{-1} (U + SU - S - M_2) x,
\]
so we have
\[
(U + SU - S - M_2) x = \begin{pmatrix} U + SU - S + \tilde{K}U + M_1 \end{pmatrix} x
\]
\[
= U x + SU x - S x + \tilde{K}U x + M_1 x
\]
\[
= (I - L) x + S (I - L) x - S x + \tilde{K} (I - L) x + M_1 x
\]
\[
= \begin{pmatrix} I + \tilde{K} - L - SL + M_1 \end{pmatrix} x.
\]

Hence \(\tilde{T}x = x\). From Lemma 2, we have \(\rho(\tilde{T}) = 1\). The proof is complete. \(\square\)

**Theorem 3.** Let \(A = I - L - U\) be an \(n \times n\) \((n \geq 3)\) \(Z\)-matrix. \(0 < a_{i1}a_{1i} + a_{i,i+1}a_{i+1,i} < 1\) for \(i = 2, \ldots, n-1\), and \(0 < a_{12}a_{21} < 1, 0 < a_{n1}a_{1n} < 1\). Then

(1) \(\rho(\tilde{T}) < \rho(T)\) for \(\rho(T) < 1\),

(2) \(\rho(\tilde{T}) > \rho(T)\) for \(\rho(T) > 1\).

**Proof.** From Theorem 2, there exists a positive vector \(x = \begin{pmatrix} T_{12}x' \end{pmatrix}_{\rho(T)x'}\) such that \(Tx = \rho(T)x\). We have
\[
\tilde{T}x - Tx
\]
\[
\begin{align*}
&= (\tilde{D} - \tilde{L})^{-1} (U + SU - S - M_2)x - (I - L)^{-1} Ux \\
&= (\tilde{D} - \tilde{L})^{-1} [(U + SU - S - M_2)x - (\tilde{D} - \tilde{L}) \rho(T) x] \\
&= (\tilde{D} - \tilde{L})^{-1} [(U + SU - S - M_2)x - (I + \tilde{K} - L - SL + M_1) \rho(T) x] \\
&= (\tilde{D} - \tilde{L})^{-1} [\rho(T) Sx - Sx + M_1x - \rho(T) M_1x] \\
&= (\rho(T) - 1) (\tilde{D} - \tilde{L})^{-1} (S - M_1) x.
\end{align*}
\]

We know easily that \((\tilde{D} - \tilde{L})^{-1} (S - M_1)\) is a nonnegative matrix, so \(\tilde{T}x \leq Tx = \rho(T)x\) when \(\rho(T) < 1\); \(\tilde{T}x \geq Tx = \rho(T)x\) when \(\rho(T) > 1\). From (20), Lemma 4 and Lemma 5, \(\rho(T) < \rho(T)\) for \(\rho(T) < 1\), and \(\rho(T) > \rho(T)\) for \(\rho(T) > 1\). The proof is complete. \(\square\)

**Theorem 4.** Let \(A = I - L - U\) be a Z-matrix. \(0 < a_{i1}a_{11} + a_{i,i+1}a_{i+1,i} < 1\) for \(i = 2, \ldots, n - 1\), and \(0 < a_{12}a_{21} < 1, 0 < a_{n1}a_{1n} < 1\). Then for any \(0 \leq \alpha_i \leq 1, 0 \leq \beta_i \leq 1\), we have

1. \(\rho(T_\alpha^3) < \rho(T)\) for \(\rho(T) < 1\),
2. \(\rho(T_\alpha^3) > \rho(T)\) for \(\rho(T) > 1\).

**Proof.** Since
\[
T_\alpha^3x -Tx = (D_\alpha^3 - L_\alpha^3)^{-1} (U + \alpha SU - \alpha S - \beta M_2)x - (I - L)^{-1} Ux
\]

\[
= (\rho(T) - 1) (D_\alpha^3 - L_\alpha^3)^{-1} (S - \beta M_1)x,
\]

and \(0 \leq \alpha_i \leq 1, 0 \leq \beta_i \leq 1\), \((D_\alpha^3 - L_\alpha^3)^{-1} (S - \beta M_1)\) is also a nonnegative matrix. Then we have \(T_\alpha^3x \leq Tx = \rho(T)x\) for \(\rho(T) < 1\); \(T_\alpha^3x \geq Tx = \rho(T)x\) for \(\rho(T) > 1\). Hence we have our conclusion. \(\square\)

**4. Numerical examples**

**Example 1.** Let
\[
A = \begin{pmatrix}
1 & -0.4 & -0.3 \\
-0.5 & 1 & -0.2 \\
-0.6 & -0.5 & 1
\end{pmatrix}, \quad b = \begin{pmatrix}
0.3 \\
0.3 \\
-0.1
\end{pmatrix}.
\]

It is obviously that \(A\) is a nonsingular Z-matrix, but it is not diagonal dominant matrix, so it does not satisfy the conditions in [5]. At the same time,
\[
(I + S)A = \begin{pmatrix}
0.8 & 0 & -0.38 \\
-0.62 & 0.9 & 0 \\
-0.6 & -0.5 & 1
\end{pmatrix}
\]
is not strictly diagonal dominant $Z$-matrix. But

$$\tilde{A} = (I + S + \tilde{K}) A = \begin{pmatrix} 0.8 & 0 & -0.38 \\ -0.12 & 0.7 & -0.15 \\ 0 & -0.74 & 0.82 \end{pmatrix}$$

is a strictly diagonal dominant $Z$-matrix.

Now the classical Gauss-Seidel iterative method and the preconditioned Gauss-Seidel iterative method are considered for solving the linear system of equation (1). The stopping criterion $\|r_k\|_\infty < 10^{-3}$ was used in the computations, where $r_k = b - Ax^{(k)}$ and $x^{(k)}$ is the $k$-th iteration for each of the methods. The number of iterations (IT) for convergence needed for the classical Gauss-Seidel iterative method and the preconditioned Gauss-Seidel iterative method are listed in Table 1 and Table 2, respectively. Thus it can be seen that the preconditioned Gauss-Seidel method proposed in this paper has a faster convergence rate than that of the classical Gauss-Seidel iterative method. Let $x^{(0)} = (0, 0, 0)^T$, the classical Gauss-Seidel iterative method took 16 iterations for the convergence, while it took 6 iterations for the convergence of the preconditioned Gauss-Seidel method. But the two methods took the same CPU times.

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<td>0.9911</td>
<td>0.9891</td>
<td>0.0109</td>
</tr>
<tr>
<td>11</td>
<td>0.9932</td>
<td>0.9944</td>
<td>0.9931</td>
<td>0.0069</td>
</tr>
<tr>
<td>12</td>
<td>0.9957</td>
<td>0.9965</td>
<td>0.9957</td>
<td>0.0044</td>
</tr>
<tr>
<td>13</td>
<td>0.9973</td>
<td>0.9978</td>
<td>0.9973</td>
<td>0.0027</td>
</tr>
<tr>
<td>14</td>
<td>0.9983</td>
<td>0.9986</td>
<td>0.9983</td>
<td>0.0017</td>
</tr>
<tr>
<td>15</td>
<td>0.9989</td>
<td>0.9991</td>
<td>0.9989</td>
<td>0.0011</td>
</tr>
<tr>
<td>16</td>
<td>0.9993</td>
<td>0.9994</td>
<td>0.9993</td>
<td>0.0007</td>
</tr>
</tbody>
</table>
Table 2. Numerical results of preconditioned Gauss-Seidel method with $P = I + S + \bar{K}$.

<table>
<thead>
<tr>
<th>It(e).</th>
<th>$x_1^{(k)}$</th>
<th>$x_2^{(k)}$</th>
<th>$x_3^{(k)}$</th>
<th>$|x^{(k)} - x^*|_\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0.5250</td>
<td>0.7043</td>
<td>0.7331</td>
<td>0.4750</td>
</tr>
<tr>
<td>2</td>
<td>0.8732</td>
<td>0.9211</td>
<td>0.9288</td>
<td>0.1268</td>
</tr>
<tr>
<td>3</td>
<td>0.9662</td>
<td>0.9789</td>
<td>0.9810</td>
<td>0.0338</td>
</tr>
<tr>
<td>4</td>
<td>0.9910</td>
<td>0.9944</td>
<td>0.9949</td>
<td>0.0090</td>
</tr>
<tr>
<td>5</td>
<td>0.9976</td>
<td>0.9985</td>
<td>0.9986</td>
<td>0.0024</td>
</tr>
<tr>
<td>6</td>
<td>0.9994</td>
<td>0.9996</td>
<td>0.9996</td>
<td>0.0006</td>
</tr>
<tr>
<td>7</td>
<td>0.9998</td>
<td>0.9999</td>
<td>0.9999</td>
<td>0.0002</td>
</tr>
<tr>
<td>8</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

Example 2. Let

$$A = \begin{pmatrix} 1 & -0.1 & -0.2 & -0.01 & 0 \\ -0.4 & 1 & -0.4 & 0 & -0.1 \\ -0.2 & -0.5 & 1 & -0.1 & -0.1 \\ -0.45 & -0.1 & -0.2 & 1 & -0.1 \\ -0.1 & -0.12 & -0.2 & -0.6 & 1 \end{pmatrix}.$$ 

Then we report the spectral radius of three iterative methods in Table 3.

Table 3. Comparisons of the three iterative methods in [5], [7] and this paper

<table>
<thead>
<tr>
<th>$(\alpha_i, \beta_i)$</th>
<th>$0.7, 0.7$</th>
<th>$0.9, 0.9$</th>
<th>$0.8, 1$</th>
<th>$1.1$</th>
<th>$0.8, 0.9$</th>
<th>$0.9, 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>GS1: $\rho(I + \alpha S)$</td>
<td>0.4059</td>
<td>0.3629</td>
<td>0.3848</td>
<td>0.3403</td>
<td>0.3848</td>
<td>0.3629</td>
</tr>
<tr>
<td>GS2: $\rho(I + \alpha S + \beta K)$</td>
<td>0.3785</td>
<td>0.3442</td>
<td>0.3540</td>
<td>0.3309</td>
<td>0.3572</td>
<td>0.3422</td>
</tr>
<tr>
<td>GS3: $\rho(I + \alpha S + \beta \bar{K})$</td>
<td>0.3590</td>
<td>0.2947</td>
<td>0.2385</td>
<td>0.2651</td>
<td>0.3196</td>
<td>0.2851</td>
</tr>
</tbody>
</table>

The above matrix $S$ is the form in [5], matrices $K$ and $\bar{K}$ are as follows:

$$K = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0.4 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 \end{pmatrix}, \quad \bar{K} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0.4 & 0 & 0 & 0 & 0 \\ 0.2 & 0 & 0 & 0 & 0 \\ 0.45 & 0 & 0 & 0 & 0 \\ 0.1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$ 

$\rho(\cdot)$ is the spectral radius of the corresponding iterative method. Numerical results for Example 1 are shown in Fig. 1.
From the above numerical examples, we see that the preconditioning effectiveness of \((I + \alpha S + \beta K)\)-type preconditioners constructed in this paper is obvious.

5. Conclusion

In this paper, we have presented the new preconditioned Gauss-Seidel iterative method for the Z-matrices linear systems. It remains to construct the comparison theorems for the iterative methods. We conclude that the rate of convergence of the new method in this paper is faster than the rate of convergence of the methods in [5] and [7] by theoretical analysis and numerical examples.

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