COMPLEX MOMENT MATRICES VIA HALMOS-BRAM AND EMBRY CONDITIONS

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Abstract. By considering a bridge between Bram-Halmos and Embry characterizations for the subnormality of cyclic operators, we extend the Curto-Fialkow and Embry truncated complex moment problem, and solve the problem finding the finitely atomic representing measure \( \mu \) such that

\[
\gamma_{ij} = \int \bar{z}^i z^j d\mu, \quad (0 \leq i+j \leq 2n, |i-j| \leq n+s, 0 \leq s \leq n);
\]

the cases of \( s = n \) and \( s = 0 \) are induced by Bram-Halmos and Embry characterizations, respectively. The former is the Curto-Fialkow truncated complex moment problem and the latter is the Embry truncated complex moment problem.

1. Introduction and preliminaries

In [5, Proposition 2.8], it was shown that Bram-Halmos characterization for the subnormality of a cyclic operator on a complex Hilbert space induces a moment matrix \( M(n) \) which was considered in [1] and [2]. As a parallel study, in [5] they discussed a matrix \( E(n) \) corresponding to the Embry characterization of such an operator. The moment matrices \( M(n) \) and \( E(n) \) are contained in our new classes of moment matrices \( M(n, s) \) (which will be defined below). Let

\[
\Gamma_{n,s} = \{ \gamma_{ij} \in \mathbb{C} : 0 \leq i+j \leq 2n, |i-j| \leq n+s, 0 \leq s \leq n \},
\]

where \( \gamma_{00} > 0, \; \gamma_{jj} = \overline{\gamma_{ij}} \). Notice that the data in \( \Gamma_{n,s} \) lie in the gray pentagon in Figure 1.

The truncated complex moment problem for \( \Gamma_{n,s} \) entails finding a positive Borel measure \( \mu \) supported in the complex plane \( \mathbb{C} \) such that

\[
\gamma_{ij} = \int \bar{z}^i z^j d\mu, \quad (0 \leq i+j \leq 2n, |i-j| \leq n+s).
\]

And \( \mu \) is said to be a representing measure for \( \Gamma_{n,s} \). In particular, \( \Gamma_{n,n} \) induces the Curto-Fialkow moment matrix \( M(n) \) ([1], [2]); \( \Gamma_{n,0} \) induces the Embry

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moment matrix $E(n)$ ([5]). If $n = 2k$, let
\[
\eta(n, s) = \begin{cases} 
(k + 1)^2 + 2mk - m(m - 1) & \text{if } s = 2m, \\
(k + 1)^2 + (2m - 1)k - (m - 1)^2 & \text{if } s = 2m - 1,
\end{cases}
\]
and if $n = 2k + 1$, let
\[
\eta(n, s) = \begin{cases} 
(k + 1)(k + 3) + 2mk - m(m - 1) & \text{if } s = 2m + 1, \\
(k + 1)(k + 3) + (2m - 1)k - (m - 1)^2 & \text{if } s = 2m.
\end{cases}
\]
Let $\mathcal{M}_k(\mathbb{C})$ be the set of all $k \times k$ matrices. For $A \in \mathcal{M}_n(n, s)(\mathbb{C})$, we introduce the order on the rows and columns of $A$. For example, if $n = 4$ and $s = 3$, i.e., $\eta(4, 3) = 14$, then the order is as follows:
\[
1, Z, Z^2, Z\bar{Z}, Z^2\bar{Z}, Z^3, Z\bar{Z}^2, Z^2\bar{Z}, Z^4, Z\bar{Z}^3, Z^2\bar{Z}^2, Z^3\bar{Z}.
\]
Let $\Lambda_{(n, s)} = \{(i, j) : 0 \leq i + j \leq n, \max\{i - s, 0\} \geq 0, 0 \leq s \leq n\}$ and let $\mathcal{P}_{n, s}$ be the set of polynomials $p(z, \bar{z}) = \sum_{(i, j) \in \Lambda_{(n, s)}} a_{ij}z^i\bar{z}^j$, where $a_{ij} \in \mathbb{C}$. Then it is clear that $\mathcal{P}_{n, s}$ is a subspace of $\mathbb{P}_n[z, \bar{z}]$, the vector space of all complex polynomials in $z, \bar{z}$ of total degree $\leq n$. Let $\{e_{ij}\}_{(i, j) \in \Lambda_{(n, s)}}$ be a basis for $\mathbb{C}^n(n, s)$ as follows: $e_{ij} \equiv e_{ij}^{(\eta(n, s))}$ is the vector with 1 in the $Z^i\bar{Z}^j$ entry and 0 in all other positions. For $p(z, \bar{z}) = \sum_{(i, j) \in \Lambda_{(n, s)}} a_{ij}z^i\bar{z}^j$, let $\hat{p} := \sum_{(i, j) \in \Lambda_{(n, s)}} a_{ij}e_{ij}$. We define a sesquilinear form $\langle \cdot, \cdot \rangle_A$ on $\mathcal{P}_{n, s}$ by $\langle p, q \rangle_A := \langle \hat{p}, \hat{q} \rangle$ ($p, q \in \mathcal{P}_{n, s}$). In particular, $\langle z^i\bar{z}^j, z^k\bar{z}^l \rangle_A = a_{ij}\delta_{ik}\delta_{jl}$, where $a_{ij} \in \Lambda_{(n, s)}$ and $(i, j, k, l) \in \Lambda_{(n, s)}$. We define the moment matrix $M(n, s)$ be a $\eta(n, s) \times \eta(n, s)$ matrix that the entry in row $Z^k\bar{Z}^l$ and column $Z^i\bar{Z}^j$ is $M(n, r)(k, l)_{(i, j)} = \gamma_{i + j + k}$, where $(k, l, (i, j)) \in \Lambda_{(n, s)}$. (Observe that $M(n, n) = E(n)$ whose definitions are in [1] and [5], resp.) For example, if $n = 2, s = 1$, i.e., for
\[
\gamma : 700, 701, 710, 702, 711, 720, 703, 712, 721, 730, 713, 722, 731,
\]
the associated moment matrix is
\[
M(2, 1) = \begin{pmatrix}
700 & 701 & 710 & 702 & 711 \\
710 & 711 & 720 & 712 & 721 \\
701 & 702 & 711 & 703 & 712 \\
720 & 721 & 730 & 722 & 731 \\
711 & 712 & 721 & 713 & 722
\end{pmatrix}.
\]
In particular, $M(2, s)$ is referred to the quartic moment matrix here.

The paper consists of five sections. In Section 2 we consider a bridge between Bram-Halmos’ and Embry characterizations for cyclic subnormal operators, which is related to complex moment matrices $M(n, s)$. In Section 3, we prove that if $\Gamma_{n, s}$ is double flat (i.e., rank $M(n, s)$ = rank $M(n - 2, s)$) and $M(n, s) \geq 0$, then $M(n, s)$ admits a unique flat extension of the form $M(n + k, s)$ for all $k \in \mathbb{N}$. And also we consider several useful examples. Let $M(1, 0)$ be any positive quadratic moment matrix. Then it always has a flat extension $M(1, 1)$. 

But, in general the case of $M(2,0)$ is independent to the case $M(1,s)$. For example, in Section 4 we show that there exists a moment matrix $M(2,0)$ with a representing measure $\mu$ such that the number of atoms is different from rank $M(2,0)$. In addition, we discuss singular quartic moment matrix $M(2,s)$ and related examples. Finally, in Section 5 we obtain an algorithm finding a moment measure from the nonsingular quartic moment matrix $M(2,0)$ which will be applied to $M(2,s)$ by a similar method.

Some of the calculations in this article were obtained through computer experiments using the software tool Mathematica [8].

2. A bridge between Bram-Halmos and Embry characterizations

Lemma 2.1. Let $A := \{\gamma_{ij}\}_{i,j=0}^{\infty}$ be an infinite matrix of complex numbers. Suppose $n \in \mathbb{N}$ and $0 \leq s \leq n$. Then the following assertions are equivalent:

(i) there exists a linear functional $\Lambda : \mathbb{P}[z,\bar{z}] \to \mathbb{C}$ defined by $\Lambda(\bar{z}^i z^j) = \gamma_{ij}$ such that

$$\Lambda\left(\sum_{(i,j) \in \Lambda(n,s)} a_{ij} \bar{z}^i z^j\right)^2 \geq 0$$

for any $p(z,\bar{z}) = \sum_{(i,j) \in \Lambda(n,s)} a_{ij} \bar{z}^i z^j \in \mathbb{P}_n$;

(ii) $M(n,s) \geq 0$.

Proof. (i) $\Rightarrow$ (ii): Let $p(z,\bar{z}) = \sum_{(i,j) \in \Lambda(n,s)} a_{ij} \bar{z}^i z^j$. Then

$$\Lambda\left(|p(z,\bar{z})|^2\right) = \Lambda\left(\sum_{(k,l) \in \Lambda(n,s)} a_{kl} \bar{z}^k \bar{z}^l, \sum_{(i,j) \in \Lambda(n,s)} a_{ij} \bar{z}^i z^j\right)$$

$$= \sum_{(k,l), (i,j) \in \Lambda(n,s)} \pi_{kl} a_{ij} \Lambda(\bar{z}^{i+l} z^{j+k})$$
Let \( \Lambda(\zeta^i \zeta^j) = \gamma_{i,j} \) for any \( i, j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \). Suppose \( n \in \mathbb{N} \) and \( 0 \leq s \leq n \). Then the following assertions are equivalent:

(i) for any \( p_i(z) \in \mathbb{P}[z] \) with \( \deg p_i(z) \leq n - i - \max\{i - s, 0\} \) \( (i = 1, \ldots, n) \),

\[
\sum_{0 \leq i,j \leq n} (T^i T^j p_i(x_0, p_j(T)x_0) \geq 0;
\]

(ii) \( M(n,s) \geq 0 \).

Proof. Let \( \Lambda : \mathbb{P}[z, \bar{z}] \to \mathbb{C} \) be a linear functional satisfying \( \Lambda(\zeta^i \zeta^j) = (T^i T^j x_0, x_0) \) for any \( i, j \in \mathbb{N}_0 \). By Lemma 2.1, \( M(n,s) \geq 0 \) is equivalent to

\[
\Lambda(|p(z, \bar{z})|^2) \geq 0 \quad \text{for any polynomial } p(z, \bar{z}) \in \mathbb{P}_{n,s}.
\]

Since \( 0 \leq i + j \leq n \) and \( \max\{i - s, 0\} \leq j \) if and only if \( 0 \leq i \leq n \) and \( \max\{i - s, 0\} \leq j \leq n - i \), (2.1) is equivalent to

\[
\Lambda(|p_0(z) + \bar{z} p_1(z) + \cdots + \bar{z}^n p_n(z)|^2) \geq 0
\]

for any polynomial \( p_i(z) \) with \( \max\{i - s, 0\} \leq \deg p_i(z) \leq n - i \) \( (i = 0, \ldots, n) \), which is equivalent to

\[
\sum_{0 \leq i,j \leq n} (T^i T^j p_i(T)x_0, p_j(T)x_0) \geq 0
\]

for any polynomial \( p_i(z) \) with \( \max\{i - s, 0\} \leq \deg p_i(z) \leq n - i \) \( (i = 0, \ldots, n) \). That is

\[
\sum_{0 \leq i,j \leq n} (T^i + \max\{j - s, 0\} T^j + \max\{i - s, 0\}) p_i(T)x_0, p_j(T)x_0) \geq 0
\]

for any polynomial \( p_i(z) \) with \( \deg p_i(z) \leq n - i - \max\{i - s, 0\} \) \( (i = 0, \ldots, n) \). \( \square \)
Given an infinite matrix $A := \{\gamma_{ij}\}_{i,j=0}^{\infty}$ of complex numbers, the full complex moment problem (write: CMP) entails finding a positive Borel measure $\mu$ on the closed unit disk $D$ in $\mathbb{C}$ such that $\gamma_{ij} = \int_{D} z^i \overline{z}^j \, d\mu(z)$. Let $\{\gamma_{ij}\}_{i,j=0}^{\infty}$ and $\{\delta_{ij}\}_{i,j=0}^{\infty}$ be two infinite matrices of complex numbers. For brevity, we write $\{\gamma_{ij}\}_{i,j=0}^{\infty} \geq \{\delta_{ij}\}_{i,j=0}^{\infty}$ if $\sum_{0 \leq i,j \leq n} (\gamma_{ij} - \delta_{ij})a_i a_j \geq 0$ for any $a_i \in \mathbb{C}$ and all $n \in \mathbb{N}$. In [5], ones obtained that if $\{\gamma_{ij}\}_{i,j=0}^{\infty}$ is an infinite matrix of complex numbers, then $\{\gamma_{ij}\}_{i,j=0}^{\infty}$ solves CMP if and only if $\{\gamma_{ij}\}_{i,j=0}^{\infty} \geq \{\gamma_{i+1,j+1}\}_{i,j=0}^{\infty}$ and $M(n,n) \geq 0$ for all $n \in \mathbb{N}$ if and only if $\{\gamma_{ij}\}_{i,j=0}^{\infty} \geq \{\gamma_{i+1,j+1}\}_{i,j=0}^{\infty}$ and $M(n,0) \geq 0$, for all $n \in \mathbb{N}$.

**Corollary 2.3.** Let $T$ be an operator with a cyclic vector $x_0$ in $\mathcal{H}$ and let $\gamma_{ij} := (T^* T^j x_0, x_0)$, for any $i, j \in \mathbb{N}_0$. Then the following assertions are equivalent:

(i) for any $p_i(z) \in \mathcal{P}[z]$ ($i = 1, \ldots, n$) and any $n \in \mathbb{N}$,

$$\sum_{0 \leq i,j \leq n} (T^{i+\max\{j-s,0\}} T^{j+\max\{i-s,0\}}) p_i(T)x_0, p_j(T)x_0 \geq 0;$$

(ii) $M(n,s) \geq 0$ for all $n \in \mathbb{N}$ and any $s = 0, \ldots, n$,

(iii) $M(n,s) \geq 0$ for all $n \in \mathbb{N}$ and some $s = 0, \ldots, n$.

Moreover, the following two assertions are equivalent:

(iv) $\{\gamma_{ij}\}_{i,j=0}^{\infty}$ solves CMP,

(v) $\{\gamma_{ij}\}_{i,j=0}^{\infty} \geq \{\gamma_{i+1,j+1}\}_{i,j=0}^{\infty}$ and $M(n,s) \geq 0$ for all $n \in \mathbb{N}$ and some $s = 0, \ldots, n$.

3. Double flat extension theorem

We review some useful properties which can be obtained by the similar proofs in [3] and [5]. We omit the detail proof here.

(P1) If $\mu$ is a representing measure, then

$$\langle M(n,s) \hat{p}, \hat{p} \rangle = \int |p(z,\bar{z})|^2 \, d\mu, p(z,\bar{z}) \in \mathcal{P}_{n,s}.$$

(P2) If $\mu$ is a representing measure, then $M(n,s) \geq 0$.

(P3) If $\mu$ is a representing measure, then sup $\mu \subseteq \mathcal{Z}(p) \iff p(Z,\bar{Z}) = 0$ for $p \in \mathcal{P}_{n,s}$, where $\mathcal{Z}(p) = \{z \in \mathbb{C} : p(z,\bar{z}) = 0\}$.

(P4) Let $M(n,s)$ be a moment matrix. If $p(z,\bar{z}) \in \mathcal{P}_{n,s}$ with $p(z,\bar{z}) \in \mathcal{P}_{n,s}$, then $p(Z,\bar{Z}) = 0$ if and only if $\hat{p}(Z,\bar{Z}) = 0$.

(P5) Let $M(n,s) \geq 0$. If $f, g, fg \in \mathcal{P}_{n-1,s}$ and $f(Z,\bar{Z}) = 0$, then $\langle fg \rangle (Z,\bar{Z}) = 0$ in the column space $\mathcal{C}_{M(n,s)}$.

(P6) If $\mu$ is a representing measure for $\gamma$, then card sup $\mu \geq$ rank $M(n,s)$.

(P7) Let $M(\infty,s)$ be an infinite moment matrix with representing measure $\mu$. Then card sup $\mu = \text{rank } M(\infty,s)$.

(P8) Let $M(\infty,s)$ be a finite-rank positive infinite moment matrix. Then $M(\infty,s)$ has a unique representing measure, which is rank $M(\infty,s)$-atomic. In this case, let $r := \text{rank } M(\infty,s)$; there exist unique scalars $a_0, \ldots, a_{r-1}$ such
that $Z' = \alpha_0 I + \cdots + \alpha_{r-1} Z^{r-1}$. The unique representing measure for $M(\infty, s)$ has support equal to the $r$ distinct roots $z_0, \ldots, z_{r-1}$ of the polynomial $z^r - (\alpha_0 + \cdots + \alpha_{r-1} z^{r-1})$, and densities $\rho_0, \ldots, \rho_{r-1}$ determined by the Vandermonde equation

$$
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
z_0 & z_1 & \cdots & z_{r-1} \\
\vdots & \vdots & & \vdots \\
z_0^{-1} & z_1^{-1} & \cdots & z_{r-1}^{-1}
\end{pmatrix}
\begin{pmatrix}
\rho_0 \\
\rho_1 \\
\vdots \\
\rho_{r-1}
\end{pmatrix}
= 
\begin{pmatrix}
\gamma_{00} \\
\gamma_{01} \\
\vdots \\
\gamma_{0,r-1}
\end{pmatrix}.
$$

The followings is a parallel theorem corresponding to [1].

**Theorem 3.1.** If $A \in M_{\Gamma(n,s)}(\mathbb{C})$, then there exists a truncated moment sequence $\Gamma_{n,s}$ with $\gamma_{00} > 0$ and $\gamma_{ij} = \frac{\rho_{ij}}{1}$ such that $A = M(n,s)(\Gamma_{n,s})$ if and only if

1. $\langle 1, 1 \rangle_A > 0$;
2. $A = A^*$;
3. $\langle p, \bar{q} \rangle_A = \langle \bar{p}, q \rangle_A$ ($p, \bar{p}, q, \bar{q} \in \mathcal{P}_{n,s}$);
4. $\langle zp, \bar{q} \rangle_A = \langle \bar{z}p, q \rangle_A$ ($p, q \in \mathcal{P}_{n-1,s}, \bar{z}q \in \mathcal{P}_{n,s}$);
5. $\langle zp, zq \rangle_A = \langle \bar{z}p, \bar{q} \rangle_A$ ($p, q \in \mathcal{P}_{n-1,s}, \bar{z}p, \bar{q} \in \mathcal{P}_{n,s}$).

**Proof.** The proof is very similar to [1] and [5], and so we will omit the proof. □

**Proposition 3.2.** Suppose $n$ is even number. If $\Gamma_{n,s}$ is flat and $M(n,s) \geq 0$, $0 \leq s \leq n$, then $M(n,s)$ admits a unique flat extension of the form $M(n+k,s)$ for all $k \in \mathbb{N}$.

**Proof.** We want construct a moment matrix $M(n+1,s)$ of the form

$$
\tilde{A} = \begin{pmatrix}
A & B \\
B^* & C
\end{pmatrix},
$$

where $A = M(n,s), B = AW,$ and $C = W^*AW.$

If $n = 2w$, we denote the columns of $B$ by the flatness of

$\begin{pmatrix}
Z^{2w+1} & Z^{2w} & \cdots & Z^w \\
Z^w & Z^{w+1} & \cdots & Z^{2w}
\end{pmatrix},
\begin{pmatrix}
Z^{2w+1} & Z^{2w} & \cdots & Z^w \\
Z^w & Z^{w+1} & \cdots & Z^{2w}
\end{pmatrix},
\begin{pmatrix}
Z^{2w+1} & Z^{2w} & \cdots & Z^w \\
Z^w & Z^{w+1} & \cdots & Z^{2w}
\end{pmatrix},
\begin{pmatrix}
Z^{2w+1} & Z^{2w} & \cdots & Z^w \\
Z^w & Z^{w+1} & \cdots & Z^{2w}
\end{pmatrix},
\begin{pmatrix}
Z^{2w+1} & Z^{2w} & \cdots & Z^w \\
Z^w & Z^{w+1} & \cdots & Z^{2w}
\end{pmatrix}.

And so $M(2w,s)$ is flat, i.e.,

$Z^i Z^{2w-i} = p_i(Z, \bar{Z}), \ i = 0, 1, \ldots, w, w+1, \ldots, w + \left[\frac{s}{2}\right], \ p_i(z, \bar{z}) \in P_{2w-1,s}.$

Let

$Z^{2w+1} = (zp_0)(Z, \bar{Z}),$

$ZZ^{2w} = (zp_1)(Z, \bar{Z}),$

$Z^2 Z^{2w-1} = (zp_2)(Z, \bar{Z}),$

$\vdots$

$Z^{w+1} Z^{w-1} = (zp_{w+1})(Z, \bar{Z}).$
We first show that $\tilde{A}$ is an extension of $M(2w,s)$. We need to check $\langle \bar{z}^i z^j, \bar{z}^{i'} z^{j'} \rangle_{\tilde{A}} = \gamma_{j+k,i+l}$, for $k+l = 2w+1$, $\max\{k-s,0\} \leq i$, $0 \leq i+j \leq 2w-1$, $\max\{i-s,0\} \leq j$, $|(j+k)-(i+l)| \leq 2w$. In fact,
\[
\langle \bar{z}^i z^j, \bar{z}^{i'} z^{j'} \rangle_{\tilde{A}} = \langle z p_k(z,\bar{z}), \bar{z}^{i+1} z^{j+1} \rangle_{\tilde{A}} = \langle \bar{z}^i z^{2w-k}, \bar{z}^{i+1} z^{j+1} \rangle_A = A_{(i+1,j)(k,2w-k)}
\]

We now show that $\tilde{A}$ is a moment matrix. By Theorem 3.1, we must show that
(a) $\tilde{A}$ is self-adjoint;
(b) $\langle p,q \rangle_{\tilde{A}} = \langle \tilde{p}, \tilde{q} \rangle$ $(p,\tilde{p},q,\tilde{q} \in \mathcal{P}_{n,s})$;
(c) $\langle zp,q \rangle_{\tilde{A}} = \langle \tilde{p}, zq \rangle_{\tilde{A}}$ $(p,q \in \mathcal{P}_{n-1,s}, zq \in \mathcal{P}_{n,s})$;
(d) $\langle zp,zq \rangle_{\tilde{A}} = \langle \tilde{p}, \tilde{q} \rangle_{\tilde{A}}$ $(p,q \in \mathcal{P}_{n-1,s}, \tilde{p}, \tilde{q} \in \mathcal{P}_{n,s})$.

Indeed, (a) clear.

(b) Since $n = 2w+1$ and $p,\tilde{p},q,\tilde{q} \in \mathcal{P}_{n,s}$, the polynomials $p,q$ must be the form
\[
p(z,\bar{z}) = \sum_{(i,j) \in \Lambda_{(n,s)}} a_{ij} z^i \bar{z}^j, \quad q(z,\bar{z}) = \sum_{(i,j) \in \Lambda_{(n,s)}} b_{ij} z^i \bar{z}^j,
\]
where
\[
\Lambda_{(n,s)} = \{(i,j) : 0 \leq i+j \leq n, \max\{i-s,0\} \leq j \leq i+s, \ 0 \leq s \leq n\}
\]
so (b) is clear.

(c) We take $\tilde{z}^i z^j, \bar{z}^{i'} z^{j'}$ instead of $p,q$. Since $p,q \in \mathcal{P}_{2w,s}, \tilde{z}q \in \mathcal{P}_{2w+1,s}$, we must have $0 \leq k+l \leq 2w$, $\max\{k-s,0\} \leq i$, $0 \leq i+j \leq 2w$, $\max\{i-s,0\} < j$.

For $0 \leq k+l \leq 2w-1$, $\max\{k-s,0\} \leq i$, $0 \leq i+j \leq 2w$, $\max\{i-s,0\} < j$, we have
\[
\langle z \cdot \tilde{z}^{i+1}, \bar{z}^{i+1} z^{j+1} \rangle_{\tilde{A}} = \langle \tilde{z}^i z^{2w-k}, \bar{z}^{i+1} z^{j+1} \rangle_{\tilde{A}} = \langle \tilde{z}^i z^{2w-k}, \bar{z}^{i+1} z^{j+1} \rangle_{\tilde{A}}.
\]

For $k+l = 2w$, $\max\{k-s,0\} \leq l$, $0 \leq i+j \leq 2w-1$, $\max\{i-s,0\} < j$, we have
\[
\langle z \cdot \tilde{z}^{i+1}, \bar{z}^{i+1} z^{j+1} \rangle_{\tilde{A}} = \langle \tilde{z}^i z^{2w-k}, \bar{z}^{i+1} z^{j+1} \rangle_{\tilde{A}} = \langle \tilde{z}^i z^{2w-k}, \bar{z}^{i+1} z^{j+1} \rangle_{\tilde{A}}.
\]
For \( k + l = 2w, \max\{k - s, 0\} < l, 0 \leq i + j = 2w, \max\{i - s, 0\} < j \), we have

\[
\langle z \cdot z^k z^l, z^i z^j \rangle_A = \langle z^k z^{l+1}, z^i z^{j+1} \rangle_A \\
= \langle z^k z^{2w-k+1}, z^i z^{2w-l+1} \rangle_A \\
= \langle z p_k(z, \bar{z}), p_i(z, \bar{z}) \rangle_A \quad (p_k, p_i \in P_{2w-1, s}) \\
= \langle z p_k(z, \bar{z}), z p_i(z, \bar{z}) \rangle_A \\
= \langle z p_k(z, \bar{z}), \bar{z} p_i(z, \bar{z}) \rangle_A \\
= \langle z^k z^l, z \cdot z^i z^j \rangle_A.
\]

(d) For \( 0 \leq k + l \leq 2w - 1, \max\{k-s, 0\} < l, 0 \leq i+j \leq 2w-1, \max\{i-s, 0\} < j \), we have

\[
\langle z \cdot z^k z^l, z \cdot z^i z^j \rangle_A = \langle z^k z^{l+1}, z^i z^{j+1} \rangle_A \\
= \langle z \cdot z^k z^l, z \cdot z^i z^j \rangle_A \\
= \langle z^k z^{l+1}, z \cdot z^i z^j \rangle_A \\
= \langle z^k z^{l+1}, \bar{z} \cdot z^i z^j \rangle_A.
\]

For \( k + l = 2w, \max\{k - s, 0\} < l, 0 \leq i + j \leq 2w - 1, \max\{i - s, 0\} < j \), we have

\[
\langle z \cdot z^k z^l, z \cdot z^i z^j \rangle_A = \langle z^k z^{l+1}, z^i z^{j+1} \rangle_A \\
= \langle z p_k(z, \bar{z}), z^i z^{j+1} \rangle_A \\
= \langle z p_k(z, \bar{z}), \bar{z}^{i+1} z^j \rangle_A \\
= \langle z p_k(z, \bar{z}), \bar{z}^{i+1} z^j \rangle_A \\
= \langle \bar{z} \cdot z^k z^l, z \cdot z^i z^j \rangle_A.
\]

Similarly, we can prove the case of \( 0 \leq k + l = 2w - 1, \max\{k-s, 0\} < l, i+j = 2w, \max\{i-s, 0\} < j \).

For \( k + l = 2w, \max\{k - s, 0\} < l, i + j = 2w, \max\{i - s, 0\} < j \), we have

\[
\langle z \cdot z^k z^l, z \cdot z^i z^j \rangle_A = \langle z p_k(z, \bar{z}), z p_i(z, \bar{z}) \rangle_A \\
= \langle z p_k(z, \bar{z}), z p_i(z, \bar{z}) \rangle_A \\
= \langle \bar{z} p_k(z, \bar{z}), z p_i(z, \bar{z}) \rangle_A \\
= \langle \bar{z} p_k(z, \bar{z}), \bar{z} p_i(z, \bar{z}) \rangle_A \\
= \langle z^k z^l, z \cdot z^i z^j \rangle_A.
\]
Moreover, the matrix $M(n, s)$ also admits a flat extension of the form $M(n + 2, s)$. In fact, we have

$$Z^{2m+2} = (z^2p_0)(Z, \bar{Z}),$$
$$\bar{Z}Z^{2m+1} = (z^2p_1)(Z, \bar{Z}),$$
$$\bar{Z}^2Z^{2m} = (z^2p_2)(Z, \bar{Z}),$$
$$\vdots$$
$$\bar{Z}^mZ^{m+1} = (\bar{z}z^m)(Z, \bar{Z}).$$

Hence $M(n, s)$ admits a flat extension of the form $M(n + k, s)$ for all $k \in \mathbb{N}$. □

Now we have the following

**Theorem 3.3.** Let $n > 1$. If $\Gamma_{n,s}$ is double flat (i.e., $\text{rank } M(n, s) = \text{rank } M(n-2, s)$) and $M(n, s) \geq 0$, then $M(n, s)$ admits a unique flat extension of the form $M(n+k, s), k \in \mathbb{N}$.

**Proof.** If $n$ is even number, the result follows from Proposition 3.2. If $n \geq 3$ is odd number, then $n - 1$ is even and $M(n-1, s)$ is flat and positive, thus by Proposition 3.2, $M(n - 1, s)$ admits a unique flat extension of the form $M(n + k, s)$ for all $k \in \mathbb{N}$. □

**Theorem 3.4.** The truncated complex moment sequence $\Gamma_{n,s}$ has a rank $M(n, s)$-atomic representing measure if and only if $M(n, s) \geq 0$ and $M(n, s)$ admits a double flat extension $M(n+2, s)$, i.e.,

$$\text{rank } M(n, s) = \text{rank } M(n+2, s).$$

**Proof.** Suppose $M(n, s) \geq 0$ and $M(n, s)$ admits a double flat extension $M(n+2, s)$, i.e., $\text{rank } M(n, s) = \text{rank } M(n+2, s)$. By Theorem 3.3, $M(n+2, s)$ admits a unique flat extension of the form $M(n+3, s)$. Thus, the unique flat extension of the form $M(\infty, s)$ may be constructed by successive application of Theorem 3.3, and $\text{rank } M(\infty, s) = \text{rank } M(n, s)$. (P7) implies that $M(\infty, s)$ has a rank $M(\infty, s)$-atomic representing measure $\mu$, and $\mu$ is clearly also a rank $M(n, s)$-atomic representing measure for $\Gamma_{n,s}$.

Conversely, suppose that $\mu$ is a rank $M(n, s)$-atomic representing measure for $\Gamma_{n,s}$. Consider $M(n+2, s)[\mu]$; then $\text{rank } M(n, s) = \text{card supp } \mu \geq \text{rank } M(n+2, s)[\mu]$ (by (P6)), since $\mu$ is a representing measure for $M(n+2, s)[\mu] \geq \text{rank } M(n, s)$, and thus $M(n+2, s)[\mu]$ is a double flat extension of $M(n, s)$. □

We discuss a simple example.
Example 3.5. Let

\[
M(3, 1) = \begin{pmatrix}
1 & 0 & 0 & 1 & i & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & i & 0 & 0 & 1 \\
0 & 0 & -i & 1 & 0 & 0 & -i \\
1 & 0 & 0 & 0 & 1 & i & 0 & 0 \\
-i & 0 & 0 & 0 & -i & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & i & 0 & 0 & 0 & 1 \\
\end{pmatrix}.
\]

It is easy to check that \( \text{rank } M(3, 1) = \text{rank } M(1, 1) = 3 \). By Theorem 3.4, \( \Gamma_{3, 1} \) admits a 3-atomic representing measure. Since \( Z^2 = i \bar{Z} \) and \( \bar{Z}Z = 1 \), we obtain \( Z^3 = iI \), the three atoms are the roots of \( z^3 - i = 0 \), i.e., \( z_0 = -i, z_1 = \frac{\sqrt{3} + i}{2}, z_2 = -\frac{\sqrt{3} + i}{2} \). From

\[
\begin{pmatrix}
1 & 1 & 1 \\
z_0 & z_1 & z_2 \\
z_0^2 & z_1^2 & z_2^2 \\
\end{pmatrix} \begin{pmatrix}
\rho_0 \\
\rho_1 \\
\rho_2 \\
\end{pmatrix} = \begin{pmatrix}
\gamma_{00} \\
\gamma_{01} \\
\gamma_{02} \\
\end{pmatrix},
\]

we have \( \rho_0 = \rho_1 = \rho_2 = \frac{1}{3} \). Thus we obtain the representing measure \( \mu = \frac{1}{3}(\delta_{z_0} + \delta_{z_1} + \delta_{z_2}) \). We can check that the measure does satisfy

\[
\gamma_{ij} = \int z^i z^j d\mu \quad (0 \leq i + j \leq 6, |i - j| \leq 4).
\]

4. Singular quartic moment matrices

Let \( M(1, 0) \) be any positive quadratic moment matrix. Then it has a flat extension \( M(1, 1) \). Indeed, to obtain such \( M(1, 1) \), take \( \gamma_{02} \) among roots of the equation

\[
|z - \gamma_{10}/\gamma_{00}| = (\gamma_{00}\gamma_{11} - |\gamma_{10}|^2)/\gamma_{00}.
\]

But, in general this flat extension can not be always constructed (see Proposition 4.1). So the study of \( M(n, s) \) is worthwhile in the truncated complex moment problem.

Proposition 4.1. There exists a moment matrix \( M(2, 0) \) with a representing measure \( \mu \) such that the number of atoms is different from \( \text{rank } M(2, 0) \). This matrix \( M(2, 0) \) has an extension \( M(2, 2) \) which has a rank \( M(2, 2) \)-representing measure.

Proof. Let us consider a positive matrix

\[
M(2, 0) = \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 2 & 1 - i \\
1 & 0 & 1 + i & 2 \\
\end{pmatrix}
\]
with rank $M(2, 0) = 3$. Since $ZZ = I + (\frac{1}{2} - \frac{1}{2}i)Z^2$ and $Z^3 = \gamma_{03}I + (1 + i)Z + \frac{1}{2}\gamma_{23}Z^2$, by [6, Algorithms 2.7 and 2.10] we may define

\[
\begin{align*}
\gamma_{14} &= 2\gamma_{03}, \\
\gamma_{23} &= \gamma_{03} - i\gamma_{03}, \\
\gamma_{24} &= 2 + 2i - i\gamma_{03}, \\
\gamma_{33} &= 2\gamma_{03}\gamma_{30} + 2, \\
\gamma_{34} &= 2\gamma_{03} - 2i\gamma_{03} + 2i\gamma_{30} + \gamma_{03}^2\gamma_{30} - i\gamma_{03}^2\gamma_{30}, \\
\gamma_{35} &= 4\gamma_{03}\gamma_{30} + 4i\gamma_{03}\gamma_{30} + 2 + 2i - 2i\gamma_{03} - i\gamma_{30}^2.
\end{align*}
\]

By [6, Theorem 2.11], we need to consider the following equations

\[
3i\gamma_{30}^2 - \gamma_{30} - i\gamma_{03}\gamma_{30} + i\gamma_{03}^3\gamma_{30} = 3\gamma_{03}\gamma_{30} - i\gamma_{03}^2 - \frac{1}{2}i\gamma_{03}\gamma_{30} - \gamma_{03} - \frac{1}{2}\gamma_{03}\gamma_{30},
\]

\[
2\gamma_{03}\gamma_{30} = 2 - \frac{1}{2}(1 + i)\gamma_{03}^2.
\]

From the second equality, we have $\gamma_{03}^2 = 2k(1 - i)$ for some real $k$, and so $|\gamma_{03}|^2 = 1 - k$. Substituting them in the first equation, we obtain $7k - 1 = 2k^2$ and $k = -3$, which is impossible. Thus there is not any 3-atomic representing measure. Furthermore, we may construct $M(2)$ as a positive extension of $M(2, 0)$ with rank $M(2) = 4$ as following

\[
M(2, 2) = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 1 - i & -2i \\
1 & 0 & 0 & 1 + i & 2 & 1 - i \\
0 & 0 & 0 & 2i & 1 + i & 2
\end{pmatrix}.
\]

Observe that $ZZ = I + \frac{1}{2}(1 - i)Z^2$ and $Z^2 = -iZ^2$, which implies that $\gamma_{32} = \frac{1}{2}(1 - i)\gamma_{23}$. That is $\gamma_{23} = 0$. Therefore, $\gamma_{14} = \gamma_{05} = 0$, and so we obtain the flat extension $M(3, 3)$ of $M(2, 2)$.
Thus there is a 4-atomic representing measure \( \mu = \rho_0 \delta_{z_0} + \rho_1 \delta_{z_1} + \rho_2 \delta_{z_2} + \rho_3 \delta_{z_3} \). The four atoms are roots of \( z^4 - 2z^2 + i (-2z^2 - 2) = 0 \), i.e., \( z_0 = 1 + \frac{1}{2} \sqrt{2} + \frac{i}{2} \sqrt{2} \), \( z_1 = 1 - \frac{1}{2} \sqrt{2} - \frac{i}{2} \sqrt{2} \), \( z_2 = -1 + \frac{1}{2} \sqrt{2} + \frac{i}{2} \sqrt{2} \), and \( z_3 = -1 - \frac{1}{2} \sqrt{2} - \frac{i}{2} \sqrt{2} \). By the Vandermonde equation
\[
\begin{pmatrix}
1 & 1 & 1 & 1 \\
\sqrt[3]{2} & \sqrt[3]{2} & \sqrt[3]{2} & \sqrt[3]{2} \\
\sqrt[3]{3} & \sqrt[3]{3} & \sqrt[3]{3} & \sqrt[3]{3} \\
\sqrt[4]{2} & \sqrt[4]{2} & \sqrt[4]{2} & \sqrt[4]{2}
\end{pmatrix}
\begin{pmatrix}
\rho_0 \\
\rho_1 \\
\rho_2 \\
\rho_3
\end{pmatrix}
=
\begin{pmatrix}
\gamma_{00} \\
\gamma_{01} \\
\gamma_{02} \\
\gamma_{03}
\end{pmatrix},
\]
we obtain four densities \( \rho_0 = \rho_3 = -\frac{1}{8} \sqrt{2} + \frac{1}{4} \) and \( \rho_1 = \rho_2 = \frac{1}{4} + \frac{1}{8} \sqrt{2} \). Hence we obtain a required moment matrix.

The moment matrix \( M(2,2) \) was discussed in [1] and [2] and also \( M(2,0) \) was discussed in [5]. So we need consider \( M(2,1) \) here. First we recall
\[
M(2,1) =
\begin{pmatrix}
1 & Z & Z^2 & ZZ \\
0 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 \\
0 & 1 & 1 & 0
\end{pmatrix}.
\]
By the above discussion, if \( M(2,1) \geq 0 \) and flat, i.e., rank \( M(2,1) = \) rank \( M(1,1) \), then it admits a flat extension of \( M(3,1) \). Hence there is a rank \( M(2,1) \)-atomic representing measure. Thus we have \( \mu = \rho_0 \delta_{z_0} + \rho_1 \delta_{z_1} + \rho_2 \delta_{z_2} \). In fact, the flatness of \( M(2,1) \) means that \( Z^2 = a_1 I + b_1 Z + c_1 \bar{Z} \), and \( ZZ = a_2 I + b_2 Z + c_2 \bar{Z} \), and so we obtain \( Z^3 = (c_1 a_2 - c_2 a_1) I + (a_1 + b_2 c_1 - b_1 c_2) Z + (b_1 + c_2) Z^2 \). Hence, the atoms, \( z_0, z_1, z_2 \), are the roots of \( z^3 - (b_1 + c_2) z^2 - (a_1 + b_2 c_1 - b_1 c_2) z - (c_1 a_2 - c_2 a_1) = 0 \). And the densities \( \rho_0, \rho_1, \) and \( \rho_2 \) can be obtained by Vandermonde equation.

Example 4.2. Let
\[
M(2,1) =
\begin{pmatrix}
1 & i & -i & i & 3 \\
-i & 3 & -i & 1 + i & 1 - i \\
i & i & 3 & 2i & 1 + i \\
-i & 1 - i & -2i & 6 & -2 - 11i \\
3 & 1 + i & 1 - i & -2 + 11i & 6
\end{pmatrix}.
\]
It is easy to check that rank \( M(2,1) = \) rank \( M(1,1) = 3 \). By Theorem 3.5, \( \Gamma_{2,1} \) admits a 3-atomic representing measure. Since \( Z^2 = (-1 + 4i) I + (-\frac{1}{2} + \frac{3}{2} i) \bar{Z} + (\frac{3}{2} + \frac{3}{2} i) Z \), \( ZZ = 10 I + (\frac{3}{2} + \frac{3}{2} i) \bar{Z} + (\frac{3}{2} - \frac{3}{2} i) Z \), we obtain \( Z^3 = -3i Z^2 - (\frac{7}{2} - \frac{11}{2} i) Z + (\frac{3}{2} + \frac{11}{2} i) I \). The three atoms are the roots of \( z^3 + 3iz^2 + (\frac{3}{2} - \frac{11}{2} i) z - (\frac{3}{2} + \frac{11}{2} i) = 0 \), i.e., \( z_0 \cong -1.0128 - 4.3466 i \), \( z_1 \cong -0.57411 + \)
0.48079i, and \( z_2 \cong 1.5869 + 0.86579i \). From
\[
\begin{pmatrix}
1 & 1 & 1 \\
\overline{z}_0 & \overline{z}_1 & \overline{z}_2 \\
\overline{z}_0^2 & \overline{z}_1^2 & \overline{z}_2^2
\end{pmatrix}
\begin{pmatrix}
\rho_0 \\
\rho_1 \\
\rho_2
\end{pmatrix}
= 
\begin{pmatrix}
\gamma_{00} \\
\gamma_{01} \\
\gamma_{02}
\end{pmatrix},
\]
we have \( \rho_0 \cong 0.00413 \), \( \rho_1 \cong 0.67747 \), and \( \rho_2 \cong 0.31840 \). Thus we obtain the representing measure \( \mu = \rho_0 \delta_{z_0} + \rho_1 \delta_{z_1} + \rho_2 \delta_{z_2} \).

If \( M(2,1) \) is not flat, i.e., \( \text{rank } M(2,1) > 3 \), then we have two ideas. The one is to find double flat extension \( M(4,1) \) and the other one is to find flat extension \( M(2,2) \) first, and using the results of [7].

**Lemma 4.3.** The positivity of \( M(2,1) \) is not sufficient for the existence of representing measure.

*Proof.* In fact, we let
\[
M(2,1) = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 2 & 1 \\
1 & 0 & 0 & 1 & 2
\end{pmatrix}.
\]
Then \( M(2,1) \) is positive and the \( \text{rank } M(2,1) = 4 \). But, there is no representing measure at all. Since \( M(2,1) \) has unique positive extension \( M(2,2) \) as the following
\[
M(2,2) = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 2 & 1 & 1 \\
1 & 0 & 0 & 1 & 2 & 1 \\
0 & 1 & 0 & 1 & 1 & 2
\end{pmatrix},
\]
which has no representing measure ([7, Example 2.4]). \( \square \)

**Proposition 4.4.** Let
\[
M(2,1) = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & \gamma_{03} & 0 \\
0 & 0 & \gamma_{30} & \gamma_{22} & \gamma_{31} \\
1 & 0 & 0 & \gamma_{13} & \gamma_{22}
\end{pmatrix}.
\]
If \( M(2,1) \) is positive with \( \text{rank } M(2,1) = 4 \) and \( |\gamma_{03}| \neq 1 \), then there is 4-atomic representing measure.

*Proof.* Since \( \bar{Z}Z = A1 + BZ + C\bar{Z} + DZ^2 \), where
\[
A = 1, \quad B = 0, \quad C = -\frac{\gamma_{03}\gamma_{31}}{\gamma_{22} - \gamma_{03}\gamma_{30}}, \quad D = \frac{\gamma_{31}}{\gamma_{22} - \gamma_{03}\gamma_{30}}.
\]
By [3, Theorem 3.1] we know that $M(2,1)$ admits a 4-atomic representing measure if and only if there exists $\gamma_{23} \in \mathbb{C}$ such that $\gamma_{23} \gamma_{31} + (|\gamma_{03}|^2 - \gamma_{22}) \gamma_{32} = \gamma_{03} \gamma_{31}^2$. Since
\[
\det M(2,1) = \gamma_{22}^2 + (1 - \gamma_{22})|\gamma_{03}|^2 - \gamma_{22} - |\gamma_{13}|^2 = 0,
\]
we have that $0 \neq |\gamma_{13}|^2 - (|\gamma_{03}|^2 - \gamma_{22}) = (|\gamma_{03}|^2 - \gamma_{22})(1 - |\gamma_{03}|^2)$. Since $\gamma_{22} - |\gamma_{03}|^2 > 0$, we have our conclusion.

In case of $|\gamma_{03}| = 1$, in the proof of Lemma 4.2, we showed there is a case in which there exists no representing measure.

**Lemma 4.5** ([6, Lemma 2.13]). The equation $A|z|^2 + 2\text{Re}(Cz) = B$, $(A > 0, C \in \mathbb{C}, B \in \mathbb{R})$ has a solution if and only if $AB + |C|^2 \geq 0$.

**Proof.** In fact, we have $|z + \frac{C}{A}|^2 = \frac{AB + |C|^2}{A^2}$. □

**Proposition 4.6.** Let $M(1,0) \geq 0$. Then $M(1,0)$ has a positive flat extension $M(1,1)$, that is, $\text{rank } M(1,0) = \text{rank } M(1,1)$.

**Proof.** Since
\[
M(1,1) = \begin{pmatrix}
\gamma_{00} & \gamma_{01} & \gamma_{10} \\
\gamma_{10} & \gamma_{11} & \gamma_{20} \\
\gamma_{01} & \gamma_{02} & \gamma_{11}
\end{pmatrix},
\]
if we let $z = \gamma_{02}$, then from $\det M(1,1) = 0$, we can take $z$ as the roots of the equation $z - \frac{\gamma_{10}/\gamma_{00}}{\gamma_{10}} = (\gamma_{00}\gamma_{11} - |\gamma_{10}|^2)/\gamma_{00}$. □

**Proposition 4.7.** Let $M(2,1) \geq 0$. Then $M(2,1)$ has a flat extension $M(2,2)$. In particular, if $M(2,1)$ is singular, then has a unique flat extension $M(2,2)$; if $M(2,1)$ is nonsingular, then has infinitely many flat extension $M(2,2)$.

**Proof.** Put $z = \gamma_{04}$. Then
\[
M(2,2) = \begin{pmatrix}
\gamma_{00} & \gamma_{01} & \gamma_{10} & \gamma_{02} & \gamma_{11} & \gamma_{20} \\
\gamma_{10} & \gamma_{11} & \gamma_{20} & \gamma_{12} & \gamma_{21} & \gamma_{30} \\
\gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{03} & \gamma_{12} & \gamma_{21} \\
\gamma_{20} & \gamma_{21} & \gamma_{30} & \gamma_{22} & \gamma_{31} & \bar{z} \\
\gamma_{11} & \gamma_{12} & \gamma_{21} & \gamma_{13} & \gamma_{22} & \gamma_{31} \\
\gamma_{02} & \gamma_{03} & \gamma_{12} & \bar{z} & \gamma_{13} & \gamma_{22}
\end{pmatrix}.
\]
Let $\phi(z) = \det M(2,2) = A|z|^2 + 2\text{Re}(Cz) - B$, for some $A, B \in \mathbb{R}, C \in \mathbb{C}$. Using Mathematica ([8]), we can show that $AB + |C|^2 = (\det M(2,1))^2 \geq 0$. According to Lemma 4.4, we know that $\phi(z) = 0$ has roots, so $M(2,1)$ has a flat extension $M(2,2)$. □
Example 4.8. Let

\[ M(2, 1) = \begin{pmatrix} 1 & 0 & 0 & i & 2 \\ 0 & 2 & -i & 1 + i & 1 - i \\ 0 & i & 2 & 2i & 1 + i \\ -i & 1 - i & -2i & 9 & 2 - \frac{10 - 4\sqrt{10}i}{3} \\ 2 & 1 + i & 1 - i & 2 + \frac{10 - 4\sqrt{10}i}{3} & 9 \end{pmatrix}, \]

then it is easy to see that \( M(2, 1) \) is positive and rank \( M(2, 1) = 4 \). The flat extension \( M(2, 2) \) is the form

\[ M(2, 2) = \begin{pmatrix} 1 & 0 & 0 & i & 2 \\ 0 & 2 & -i & 1 + i & 1 - i \\ 0 & i & 2 & 2i & 1 + i \\ -i & 1 - i & -2i & 9 & 2 - \frac{10 - 4\sqrt{10}i}{3} \\ 2 & 1 + i & 1 - i & 2 + \frac{10 - 4\sqrt{10}i}{3} & 9 \end{pmatrix}. \]

Put \( z = \gamma_{04} \), then \( \det M(2, 2) = -A|z|^2 - 2\text{Re}(Cz) + B \), where \( A = 11 \), \( B = \frac{256}{27} - \frac{6475}{9} \), and \( C = 81 + \left( \frac{4}{9} - \frac{32}{27} \sqrt{10} \right)i \). Since \( AB + |C|^2 = 0 \), and \( \bar{z} = -\frac{C}{A} = -\left( \frac{4}{9} - \frac{32}{27} \sqrt{10} \right)i - \frac{81}{11} \), by Lemma 4.4, \( M(2, 1) \) has a positive flat extension \( M(2, 2) \) if and only if

\[ \gamma_{04} = \left( \frac{4}{3} - \frac{32}{27} \sqrt{10} \right)i - \frac{81}{11}. \]

On the other hand, since \( \bar{Z}Z = a'1 + b'Z + c'\bar{Z} + d'Z^2 \), and \( \bar{Z}Z = a'1 + b'Z + c'\bar{Z} + d'Z^2 \), where \( a = \frac{1}{3} + \sqrt{10} + (2 - \frac{1}{3})i \), \( b = \frac{1}{3} \sqrt{10} + (\frac{1}{3} - \frac{1}{2}i) \), \( c = (\frac{1}{3} - \frac{1}{2}i) \sqrt{10} + (\frac{1}{3} + \frac{1}{2}i) \), and \( d = \frac{4}{27} \sqrt{10} + (\frac{1}{3} + \frac{1}{2}i) \), \( a' = \frac{1}{2} + \frac{1}{2}i \), \( b' = \frac{4}{27} \sqrt{10} + (\frac{1}{3} + \frac{1}{2}i) \), \( c' = \frac{1}{2} + \frac{1}{2}i \sqrt{10} + (\frac{1}{3} + \frac{1}{2}i) \), and \( d' = \frac{1}{2} + \frac{1}{2}i \sqrt{10} + (\frac{1}{3} + \frac{1}{2}i) \). Moreover, the matrix \( M(2, 1) \) has the following flat extension of the form

\[ M(2, 2) \begin{pmatrix} 1 & 0 & 0 & i & 2 \\ 0 & 2 & -i & 1 + i & 1 - i \\ 0 & i & 2 & 2i & 1 + i \\ -i & 1 - i & -2i & 9 & 2 - \frac{10 - 4\sqrt{10}i}{3} \\ 2 & 1 + i & 1 - i & 2 + \frac{10 - 4\sqrt{10}i}{3} & 9 \end{pmatrix}. \]
By [1, Theorem 3.1] we know that \( M(2, 2) \) admits a flat extension and \( \Gamma_{2,2} \) admits a 4-atomic representing measure. The atoms are the roots of

\[
0 = \left( \frac{103}{99} - \frac{119}{792} - 1i \right) \sqrt{10} + \left( \frac{8815}{1584} - \frac{1319}{792} - 1i \right) \\
+ \left( \frac{1487}{792} - \frac{205}{88} - 1i \right) + \left( \frac{34}{99} - \frac{41}{99} - 1i \right) \sqrt{10} z \\
+ \left( \frac{1253}{792} + \frac{35}{33} i \right) + \left( \frac{5}{99} + \frac{59}{99} i \right) \sqrt{10} z^2 \\
+ \left( \frac{43}{132} + \frac{35}{22} i \right) - \left( \frac{23}{132} - \frac{7}{44} i \right) \sqrt{10} z^3 + \left( \frac{45}{176} - \frac{5}{88} i \right) \sqrt{10} z^4.
\]

In fact, the roots are

\[
z_0 \approx -1.8218 + 0.88716i, \quad z_1 \approx -0.56381 - 0.87851i, \quad z_2 \approx 1.3177 + 1.034i, \quad \text{and} \quad z_3 \approx 5.5103 - 6.1107i.
\]

and the densities are

\[
\rho_0 \approx 0.09659, \quad \rho_1 \approx 0.57237, \quad \rho_2 \approx 0.33058, \quad \rho_3 \approx 0.00046.
\]

**Example 4.9.** Let

\[
M(2, 1) = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 2 & 1 \\
1 & 0 & 0 & 1 & 2
\end{pmatrix}.
\]

Since \( \det[M(2, 1)] = 2 \), \( M(2, 1) = 1 \), \( M(2, 1) \) is positive and nonsingular. If we let \( \gamma_0 = x + yi \), then \( M(2, 1) \) has a flat extension of \( M(2, 2) \) if and only if \( x^2 + y^2 = 2x \). Thus, in particular, if we take \( x = 2, y = 0 \), then one of the flat extension is

\[
M(2, 2) = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 2 & 1 \\
1 & 0 & 0 & 1 & 2
\end{pmatrix}.
\]

It is easy to see that \( \bar{Z}^2 = Z^2 \). Let \( \gamma_{23} = s + ti \), by [3, Theorem 4.1], \( \gamma_{41} = \gamma_{23} \), and \( \Gamma_{2,2} \) admits a 5-atomic representing measure if and only if \( s = 0 \) or \( t = 0 \). In particular, if we take \( \gamma_{23} = i \), then

\[
Z^4 = 3i + 1Z + 2\bar{Z} + 2iZ^2 - 3i\bar{Z}Z, \\
\bar{Z}Z^2 = -3i + 2\bar{Z} + \bar{Z} - 2iZ^2 + 3i\bar{Z}Z, \\
Z^4 = -15i - iZ + i\bar{Z} - 9Z^2 + 17\bar{Z}Z.
\]
By some transformation, we obtain $Z^5 = -9i1 + 3Z - 12iZ^2 + 2Z^3 + 5iZ^4$. Thus, the five atoms and the densities are

$$
egin{align*}
Z_0 &= -\sqrt{3}, & \rho_0 &= 0.08333, \\
Z_1 &= \sqrt{3}, & \rho_1 &= 0.08333, \\
Z_2 &= -0.80970i, & \rho_2 &= 0.39572, \\
Z_3 &= 0.72918i, & \rho_3 &= 0.43731, \\
Z_4 &= 5.0806i, & \rho_4 &= 0.00031.
\end{align*}
$$

Therefore, the 5-atomic representing measure is $\mu = \rho_0\delta_{z_0} + \rho_1\delta_{z_1} + \rho_2\delta_{z_2} + \rho_3\delta_{z_3} + \rho_4\delta_{z_4}$.

5. Nonsingular quartic moment matrices

We introduce an algorithm for finding moment measure of the nonsingular matrix $M(2, 0)$. The case of $M(2, 1)$ and $M(2, 2)$ can be discussed similarly and we leave them to interesting readers. For brevity, we write $M(3) := M(3, 3)$ and $E(3) := M(3, 0)$ as usual.

Recall that if $M(2)$ has a flat extension $M(3)$, then there exists an associated moment measure ([1]). However, the nonsingular quartic moment problem of $M(2)$ is open still ([7]). In this section, we discuss the nonsingular quartic moment problem of $E(2)$. Because the double flatness is required on the case of $E(2)$, we assume that $E(2)$ is positive and invertible and has a double flat extension $E(4)$ for being time. Then we obtain the corresponding finite sequence

$$
\gamma : 700, 701, 710, 702, 711, 720, 712, 721, 713, 722, 731, \ldots, 744, 753, 762.
$$

Since $E(2)$ is positive and invertible, $\Delta_d > 0$ for all $d = 1, 2, 3$ and 4. In this case, there exist $c_0, c_1, c_2, c_3$ and $d_0, d_1, d_2, d_3$ such that

\begin{align*}
(5.1a) \quad & Z^3 = c_01 + c_1Z + c_2Z^2 + c_3ZZ, \\
(5.1b) \quad & ZZ^2 = d_01 + d_1Z + d_2Z^2 + d_3ZZ,
\end{align*}

where

$$
\begin{align*}
c_0 &= \frac{1}{\Delta_4} \left| \begin{array}{cccc} 703 & 701 & 702 & 711 \\ 713 & 711 & 712 & 721 \\ 723 & 721 & 722 & 731 \\ 734 & 712 & 713 & 722 \end{array} \right|, & c_1 &= \frac{1}{\Delta_4} \left| \begin{array}{cccc} 700 & 703 & 702 & 711 \\ 710 & 713 & 712 & 721 \\ 720 & 723 & 722 & 731 \\ 731 & 714 & 713 & 722 \end{array} \right|, \\
c_2 &= \frac{1}{\Delta_4} \left| \begin{array}{cccc} 700 & 701 & 703 & 711 \\ 710 & 711 & 713 & 721 \\ 720 & 721 & 723 & 731 \\ 711 & 714 & 713 & 722 \end{array} \right|, & c_3 &= \frac{1}{\Delta_4} \left| \begin{array}{cccc} 700 & 701 & 702 & 703 \\ 710 & 711 & 712 & 713 \\ 720 & 721 & 722 & 723 \\ 711 & 712 & 713 & 714 \end{array} \right|.
\end{align*}
$$
Proposition 5.1. Assume that \( E(2) \) is positive and invertible. Then \( E(2) \) has a double flat extension \( E(4) \) if and only if

\[
\begin{align*}
0 \leq c_0 \gamma_{30} + c_1 \gamma_{31} + c_2 \gamma_{32} + c_3 \gamma_{41} &= d_0 \gamma_{21} + d_1 \gamma_{22} + d_2 \gamma_{23} + d_3 \gamma_{32}, \\
0 \leq c_0 \gamma_{41} + c_1 \gamma_{42} + c_2 \gamma_{43} + c_3 \gamma_{52} &= d_0 \gamma_{32} + d_1 \gamma_{33} + d_2 \gamma_{34} + d_3 \gamma_{43}, \\
0 \leq d_0 \gamma_{32} + d_1 \gamma_{33} + d_2 \gamma_{34} + d_3 \gamma_{43} &= e_0 \gamma_{22} + e_1 \gamma_{23} + e_2 \gamma_{24} + e_3 \gamma_{33}.
\end{align*}
\]
Let

\[ E(2) = \begin{pmatrix}
  1 & 0 & 0 & a \\
  0 & a & 0 & 0 \\
  0 & 0 & b & 0 \\
  a & 0 & 0 & b
\end{pmatrix} \quad a > 0, \ b > a^2. \]

Then \( E(2) \) is positive and invertible and has a double flat extension \( E(4) \).

**Proof.** For \( E(4) \), if we let \( \gamma_{23} = 0, \gamma_{03} = x + yi \) and \( \gamma_{14} = s + ti \), then we can calculate

\[
c_0 = \frac{as - xb + i(at - yb)}{-b + a^2}, \quad c_1 = c_2 = 0, \quad c_3 = \frac{(ax - s) + i(ay + t)}{-b + a^2},
\]

\[
d_0 = d_2 = d_3 = 0, \quad \text{and} \quad d_1 = \frac{1}{a} b.
\]

Define \( \gamma_{24} = 0, \gamma_{33} = \frac{1}{a} b^2 \), so \( c_0 = c_1 = c_2 = 0, e_3 = \frac{1}{a} b \). Define \( \gamma_{34} = 0, \gamma_{25} = (s + it) \frac{b}{a} \). Then it follows from (5.4) that \((x - \frac{4a}{s})^2 + (y - \frac{4t}{s})^2 = 0\) and \(s^2 + t^2 = \frac{b^2}{a^2}\). Therefore (5.4) has a solution \( \gamma_{23} = 0 \), \( \gamma_{03} = \sqrt{abe^{i\theta}} \), and \( \gamma_{14} = \sqrt{\frac{b}{a}} e^{i\theta} \). Thus we have this corollary.

**Algorithm 5.3.** (I) Calculate \( c_i, d_i, i = 0, 1, 2, 3 \);

(II) Define \( \gamma_{24}, \gamma_{33} \) as

\[
\gamma_{24} := c_0 \gamma_{21} + c_1 \gamma_{22} + c_2 \gamma_{23} + c_3 \gamma_{24}, \quad \gamma_{33} := c_0 \gamma_{30} + c_1 \gamma_{31} + c_2 \gamma_{32} + c_3 \gamma_{33},
\]

or

\[
d_0 \gamma_{21} + d_1 \gamma_{22} + d_2 \gamma_{23} + d_3 \gamma_{24}.
\]

(III) Calculate \( e_i, i = 0, 1, 2, 3 \);

(IV) Define \( \gamma_{34}, \gamma_{25} \) as

\[
\gamma_{34} := c_0 \gamma_{31} + c_1 \gamma_{32} + c_2 \gamma_{33} + c_3 \gamma_{34},
\]

\[
\gamma_{25} := c_0 \gamma_{22} + c_1 \gamma_{23} + c_2 \gamma_{24} + c_3 \gamma_{33}.
\]

(V) Solve (5.4) with respect to \( \gamma_{03}, \gamma_{23}, \gamma_{14} \);

(VI) If (5.4) has a solution, then go to the next step;

(VII) Define \( \gamma_{04}, \gamma_{15}, \gamma_{44}, \gamma_{35}, \gamma_{26} \) as

\[
\gamma_{04} := c_0 \gamma_{01} + c_1 \gamma_{02} + c_2 \gamma_{03} + c_3 \gamma_{12},
\]

\[
\gamma_{15} := c_0 \gamma_{12} + c_1 \gamma_{13} + c_2 \gamma_{14} + c_3 \gamma_{23},
\]

\[
\gamma_{44} := c_0 \gamma_{41} + c_1 \gamma_{42} + c_2 \gamma_{43} + c_3 \gamma_{52},
\]

\[
\gamma_{35} := c_0 \gamma_{32} + c_1 \gamma_{33} + c_2 \gamma_{34} + c_3 \gamma_{43},
\]

\[
\gamma_{26} := c_0 \gamma_{23} + c_1 \gamma_{24} + c_2 \gamma_{25} + c_3 \gamma_{34}.
\]

(VIII) We obtain a double flat extension \( E(4) \) of \( E(2) \).
Example 5.4. Let us consider a quartic moment matrix

\[ E(2) = \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}. \]

Since \( \Delta_1 = 1, \Delta_2 = \frac{1}{2}, \Delta_3 = \frac{1}{4}, \) and \( \Delta_4 = \frac{1}{16}, \) it is obvious that \( E(2) \) is positive and invertible. We know \( E(2) \) has a representing measure by Corollary 5.2. Here we find concrete representing measure according to Algorithm 5.3.

(I) \( c_0 = 2\gamma_{03} - 2\gamma_{14}, c_1 = 0, c_2 = 2\gamma_{23}, c_3 = -2\gamma_{03} + 4\gamma_{14}, \) and \( d_0 = -2\gamma_{23}, d_1 = 1, d_2 = 2\gamma_{32}, d_3 = 4\gamma_{23}. \)

(II) Define \( \gamma_{24}, \gamma_{33} \) as \( \gamma_{24} := 2\gamma_{23} + ( -2\gamma_{03} + 4\gamma_{14} )\gamma_{32}, \gamma_{33} := \frac{1}{2} + 6\gamma_{32}\gamma_{23}. \)

(III) \( e_0 = -12\gamma_{32}\gamma_{23}, e_1 = 2\gamma_{32}, e_2 = 4\gamma_{32} - 4\gamma_{23}\gamma_{30} + 8\gamma_{23}\gamma_{41}, \) and \( e_3 = 24\gamma_{32}\gamma_{23} + 1. \)

(IV) Define \( \gamma_{34}, \gamma_{25} \) as

\[
\gamma_{34} := 2\gamma_{23}\left( \frac{1}{2} + 2\gamma_{32}\gamma_{23} + 4\gamma_{23}\gamma_{32} \right) \\
+ (-2\gamma_{03} + 4\gamma_{14})\left( \gamma_{32} + (-2\gamma_{03} + 4\gamma_{14})\gamma_{23} \right),
\]

\[
\gamma_{25} := \gamma_{03} - \gamma_{14} + 2\gamma_{23}\left( \frac{1}{2} + ( -2\gamma_{03} + 4\gamma_{14} )\gamma_{32} \right) \\
+ (-2\gamma_{03} + 4\gamma_{14})\left( \frac{1}{2} + 6\gamma_{32}\gamma_{23} \right).
\]

(V) By Proposition 5.1, \( E(2) \) admits a double flat extension \( E(4) \) if and only if

\[
\gamma_{30}\gamma_{03} - \gamma_{30}\gamma_{14} - \gamma_{41}\gamma_{03} + 2\gamma_{41}\gamma_{14} = \frac{1}{4} + 2\gamma_{23}\gamma_{32}.
\]

and \( F_i(\gamma_{03}, \gamma_{14}, \gamma_{23}) = 0, i = 1, 2, \) where

\[
F_1(\gamma_{03}, \gamma_{14}, \gamma_{23}) = -8\gamma_{32}\gamma_{23} + 2\gamma_{41}\gamma_{14} - 48\gamma_{32}^2\gamma_{23} + 8\gamma_{30}\gamma_{23}^3 - 16\gamma_{41}\gamma_{23}^3 \\
- \frac{1}{2} + 16\gamma_{23}\gamma_{30}\gamma_{32}\gamma_{03} - 32\gamma_{23}\gamma_{30}\gamma_{32}\gamma_{14} \\
- 32\gamma_{23}\gamma_{41}\gamma_{32}\gamma_{03} - 64\gamma_{23}\gamma_{41}\gamma_{32}\gamma_{14},
\]

and

\[
F_2(\gamma_{03}, \gamma_{14}, \gamma_{23}) = -4\gamma_{32}\gamma_{23} - 80\gamma_{32}^2\gamma_{23} - 8\gamma_{30}\gamma_{23}^3 + 16\gamma_{41}\gamma_{23}^3 \\
+ 16\gamma_{23}\gamma_{30}\gamma_{32}\gamma_{03} - 32\gamma_{23}\gamma_{30}\gamma_{32}\gamma_{14} \\
- 32\gamma_{23}\gamma_{41}\gamma_{32}\gamma_{03} + 64\gamma_{23}\gamma_{41}\gamma_{32}\gamma_{14}.
\]

If \( \gamma_{23} = 0, \) then \( F_1(\gamma_{03}, \gamma_{14}, \gamma_{23}) = 2\gamma_{41}\gamma_{14} = \frac{1}{2}, \) and \( F_2(\gamma_{03}, \gamma_{14}, \gamma_{23}) = 0. \) Thus \( |\gamma_{14}| = \frac{1}{2} \) and \( \gamma_{30}\gamma_{03} - \gamma_{30}\gamma_{14} - \gamma_{41}\gamma_{03} = -\frac{1}{2}. \) If we let \( \gamma_{14} = \frac{1}{2}, \)
then we have $\gamma_{03} = \frac{1}{2}i$. Thus we obtain a solution of (5.4), $\gamma_{23} = 0$ and $\gamma_{14} = \gamma_{03} = \frac{1}{2}i$.

(VI) Define $\gamma_{04} = 0$, $\gamma_{15} = 0$, $\gamma_{44} = \frac{1}{2}$, $\gamma_{35} = 0$, and $\gamma_{26} = 0$; (VII) Finally, we obtain a double flat extension of $E(2)$ as following

$$DF = \begin{pmatrix}
1 & 0 & 0 & \frac{1}{2}i & 0 & 0 & 0 & \frac{1}{2}i & 0 & 0 & 0 & \frac{1}{2}i & 0 & 0 & 0 & \frac{1}{2}i & 0 & 0 & 0 & \frac{1}{2}i & 0 & 0 & 0 & \frac{1}{2}i & 0 & 0 & 0 & \frac{1}{2}i & 0 & 0 & 0 & \frac{1}{2}i & 0 & 0 & 0 & \frac{1}{2}i & 0 & 0 & 0 & \frac{1}{2}i & 0 & 0 & 0 & \frac{1}{2}i & 0 & 0 & 0 & \frac{1}{2}i & 0 & 0 & 0 & \frac{1}{2}i & 0 & 0 & 0 & \frac{1}{2}i & 0 & 0 & 0 & \frac{1}{2}i & 0 & 0 & 0 & \frac{1}{2}i & 0 & 0 & 0 & \frac{1}{2}i & 0 & 0 & 0 & \frac{1}{2}i & 0 & 0 & 0 & \frac{1}{2}i & 0 & 0 & 0 & \frac{1}{2}i & 0 & 0 & 0 & \frac{1}{2}i & 0 & 0 & 0 & \frac{1}{2}i \end{pmatrix}.$$

(VIII) The four atoms are $z_0 = 0$, $z_1 = \frac{1}{2}i + \frac{1}{2}\sqrt{3}$, $z_2 = \frac{1}{2}i - \frac{1}{2}\sqrt{3}$, and $z_3 = -i$. We know that $z_0$ is the center of unit disc $D$ and $z_1$, $z_2$, $z_3$ are on the unit circle $T$. Since $\gamma_{03} = \frac{1}{2}i$, then $\rho_0 = \frac{1}{2}, \rho_1 = \rho_2 = \rho_3 = \frac{1}{6}$. Therefore, one of the representing measure is

$$\mu = \frac{1}{2} \delta_0 + \frac{1}{6} (\delta_{\frac{1}{2}i + \frac{1}{2}\sqrt{3}} + \delta_{\frac{1}{2}i - \frac{1}{2}\sqrt{3}} + \delta_{-i}).$$

We close this article as the following open problem.

**Problem 5.5.** If $E(2)$ is positive and invertible, does it have a double flat extension $E(4)$?

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