STABILITY OF THE MULTI-JENSEN EQUATION

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Abstract. Given an $m \in \mathbb{N}$ and two vector spaces $V$ and $W$, a function $f : V^m \to W$ is called multi-Jensen if it satisfies Jensen’s equation in each variable separately. In this paper we unify these $m$ Jensen equations to obtain a single functional equation for $f$ and prove its stability in the sense of Hyers-Ulam, using the so-called direct method.

1. Introduction

Throughout this paper we assume that $V$ and $W$ are vector spaces over $\mathbb{Q}$ and that $m$ is a natural number.

Definition 1. A function $f : V^m \to W$ is called multi-Jensen (or $m$-Jensen) if it satisfies Jensen’s equation in each of its $m$ arguments,

$$f(x_1, \ldots, x_{i-1}, \frac{1}{2}(x_i + y_i), x_{i+1}, \ldots, x_m) = \frac{1}{2} f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_m) + \frac{1}{2} f(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_m),$$

for all $x_1, \ldots, x_i, y_i, \ldots, x_m \in V$.

All solutions of (1) have been characterized in detail in [4]. The present paper deals with a proof of Hyers-Ulam stability of (1) without referring to any knowledge of solutions. In [1] the stability problem was treated for the bi-Jensen equation in the sense of Hyers-Ulam-Rassias. As a result the authors obtain boundedness of $f(x_1, x_2) - f(0, x_2) - F(x_1, x_2)$ and $f(x_1, x_2) - f(x_1, 0) - F'(x_1, x_2)$, $F, F'$ being bi-Jensen functions, by functions with three arguments. To get a general idea of stability concepts for functional equations, see, for instance, [3] and [2].

For our purpose it will be convenient to introduce the following notation. For $n \in \mathbb{N}_0$ the boldfaced symbol $\mathbf{n}$ shall denote the set of all natural numbers from 1 to $n$, i.e., $\mathbf{n} = \{1, \ldots, n\}$ in case $n \geq 1$ and $\mathbf{n} = \emptyset$ in case $n = 0$. For
a subset $S = \{j_1, j_2, \ldots, j_i\} \subseteq \mathbf{m}$ with $1 \leq j_1 < j_2 < \cdots < j_i \leq m$ and for $x = (x_1, \ldots, x_m) \in V^m$,

$$x_S := (0, \ldots, 0, x_{j_1}, 0, \ldots, 0, x_{j_2}, 0, \ldots, 0, x_{j_i}, 0, \ldots, 0) \in V^m$$

denotes the vector which coincides with $x$ in exactly those components, which are indexed by the elements of $S$ and whose other components are set equal zero. It follows immediately that $x_0 = 0, x_m = x$ and that $(x_S)_T = (x_T)_S = x_{S \cap T}$ for any $S, T \subseteq \mathbf{m}$. The cardinality of a set $S$ will be denoted by $|S|$.

As a first step system (1) of $m$ functional equations is unified into one functional equation characterizing multi-Jensen functions.

**Lemma 1.1.** A function $f : V^m \to W$ is multi-Jensen if and only if it satisfies

$$f\left(\frac{1}{2}(x + y)\right) = \frac{1}{2^m} \sum_{S \subseteq \mathbf{m}} f(x_S + y_{\mathbf{m}\setminus S})$$

for all $x, y \in V^m$.

**Proof.** Necessity of (2) is shown by proving

$$f\left(\frac{1}{2}(x + y)\right) = \frac{1}{2^k} \sum_{S \subseteq \mathbf{k}} f(x_S + y_{\mathbf{k}\setminus S} + \frac{1}{2}(x_{\mathbf{m}\setminus \mathbf{k}} + y_{\mathbf{m}\setminus \mathbf{k}})), \quad k = 0, 1, \ldots, m,$$

by induction on $k$, the number of equations of (1) taken into account. For $k = 0$, (3) is obviously true. Assuming that (3) holds true for any $k \in \mathbf{m} - 1$, one obtains by (1)

$$\begin{align*}
f\left(\frac{1}{2}(x + y)\right) &= \frac{1}{2^k} \sum_{S \subseteq \mathbf{k}} f(x_S + y_{\mathbf{k}\setminus S} + \frac{1}{2}(x_{\mathbf{k}+1} + y_{\mathbf{k}+1})) + \frac{1}{2}(x_{\mathbf{m}\setminus \mathbf{k}+1} + y_{\mathbf{m}\setminus \mathbf{k}+1}) \\
&= \frac{1}{2^k} \sum_{S \subseteq \mathbf{k}} \left(\frac{1}{2} f(x_S + y_{\mathbf{k}\setminus S} + x_{\mathbf{k}+1}) + y_0 + \frac{1}{2}(x_{\mathbf{m}\setminus \mathbf{k}+1} + y_{\mathbf{m}\setminus \mathbf{k}+1})\right) \\
&\quad + \frac{1}{2} f(x_S + y_{\mathbf{k}\setminus S} + x_0 + y_{\mathbf{k}+1}) + \frac{1}{2}(x_{\mathbf{m}\setminus \mathbf{k}+1} + y_{\mathbf{m}\setminus \mathbf{k}+1})) \\
&= \frac{1}{2^{k+1}} \sum_{S \subseteq \mathbf{k}+1} f(x_S + y_{\mathbf{k}+1}\setminus S + \frac{1}{2}(x_{\mathbf{m}\setminus \mathbf{k}+1} + y_{\mathbf{m}\setminus \mathbf{k}+1})).
\end{align*}$$

Sufficiency of (2) is shown by taking an $i \in \mathbf{m}$, defining $\tilde{x} := (x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_m)$ and splitting up the argument on the left hand side of the $i$-th equation of (1) by $(x_1, \ldots, x_{i-1}, \frac{1}{2}(x_i + y_i), x_{i+1}, \ldots, x_m) = \frac{1}{2}x + \frac{1}{2}\tilde{x}$. Then for any $S \subseteq \mathbf{m}$ it follows $x_S + \tilde{x}_{\mathbf{m}\setminus S} = \tilde{x}$ in case $i \in S$ and $x_S + \tilde{x}_{\mathbf{m}\setminus S} = x$ in case $i \notin S$. Hence

$$f\left(\frac{1}{2}(x + \tilde{x})\right) = \frac{1}{2^m} \sum_{S \subseteq \mathbf{m}} f(x_S + \tilde{x}_{\mathbf{m}\setminus S}) = \frac{1}{2^m} \left( \sum_{S \subseteq \mathbf{m}, i \in S} f(x) + \sum_{S \subseteq \mathbf{m}, i \notin S} f(\tilde{x}) \right)$$

$$= \frac{1}{2} f(x) + \frac{1}{2} f(\tilde{x}),$$
showing that \( f \) satisfies the \( i \)-th equation of (1). □

Equation (2) is utilized now to define functions which are approximately multi-Jensen in the following sense.

**Definition 2.** Let \((W, \| \cdot \|)\) be a Banach-space and let \( \varepsilon \geq 0 \). A function \( f : V^m \to W \) is called \( \varepsilon \)-multi-Jensen if there exists a function \( \varphi : V^m \times V^m \to W \) satisfying \( \| \varphi(x, y) \| \leq \varepsilon \) for all \( x, y \in V^m \), such that

\[
 f\left(\frac{1}{2}(x + y)\right) = \frac{1}{2^m} \sum_{S \subseteq m} f(x_S + y_{m \setminus S}) + \varphi(x, y)
\]

for all \( x, y \in V^m \).

The remainder of the paper deals with the proof of the following theorem.

**Theorem 1.2.** Let \( f : V^m \to W \) be an \( \varepsilon \)-multi-Jensen function. Then there exists a multi-Jensen function \( g : V^m \to W \), such that

\[
\| f(x) - g(x) \| \leq 2\varepsilon m
\]

for all \( x \in V^m \). Moreover, \( g \) is uniquely determined up to a suitable constant.

2. Proof of the theorem

Our proof is based on the so-called direct method. For a given function \( f : V^m \to W \) let a sequence of functions \( g_n : V^m \to W \) be defined by

\[
g_n(x) := \sum_{S \subseteq m} 2^{-|S|} f(2^n x_S), \quad x \in V^m, \quad n = 1, 2, \ldots
\]

**Lemma 2.1.** Let \( f : V^m \to W \) be \( \varepsilon \)-multi-Jensen and let \( g_n \) be defined by (5). Then \( g_n \) is \( 2^{-n}(2^m - 1)\varepsilon \)-multi-Jensen for each \( n \in \mathbb{N} \).

**Proof.** Let be \( x, y \in V^m \). Estimation of the multi-Jensen kernel of \( g_n \) yields for each \( n \in \mathbb{N} \)

\[
\| g_n\left(\frac{1}{2}(x + y)\right) - \frac{1}{2^m} \sum_{T \subseteq m} g_n(x_T + y_{m \setminus T}) \|
\]

\[
= \| \sum_{S \subseteq m} 2^{-|S|} f\left(\frac{1}{2}(2^n x_S + 2^n y_S)\right) - \frac{1}{2^m} \sum_{T \subseteq m} \sum_{S \subseteq m} 2^{-|S|} f(2^n(x_T s + 2^n(y_{m \setminus T})) s) \|
\]

\[
= \| \sum_{S \subseteq m} 2^{-|S|} \frac{1}{2^m} \sum_{T \subseteq m} f(2^n(x_S) + 2^n(y_S)_{m \setminus T}) + \sum_{S \subseteq m} 2^{-|S|} \varphi(2^n x_S, 2^n y_S)
\]

\[
- \frac{1}{2^m} \sum_{T \subseteq m} \sum_{S \subseteq m} 2^{-|S|} f(2^n(x_T s + 2^n(y_{m \setminus T})) s) \|
\]

\[
= \| \sum_{\emptyset \neq S \subseteq m} 2^{-|S|} \varphi(2^n x_S, 2^n y_S) \| \leq \sum_{i=1}^{m} \binom{m}{i} 2^{-i n} \varepsilon \leq 2^{-n}(2^m - 1)\varepsilon,
\]

where we have used \( \varphi(0, 0) = 0 \). □
Lemma 2.2. Let $M$ be a finite set, $R$ a commutative ring with unity, $\mathcal{R}$ an $R$-module and let $F : \mathcal{P}(M) \to \mathcal{R}$ be any mapping from the power set of $M$ to $\mathcal{R}$. Given coefficients $\alpha_{S,T} \in R$ for $S, T \subseteq M$, then
\[
\sum_{S \subseteq M} \sum_{T \subseteq M} \alpha_{S,T} F(S \cap T) = \sum_{U \subseteq M} \beta_U F(U),
\]
where
\[
\beta_U = \sum_{S' \subseteq M \setminus U} \sum_{T' \subseteq M \setminus (U \cup S')} \alpha_{U \cup S', U \cup T'}.
\]

Proof. For $U \subseteq M$ let $D_U := \{(S, T) \in \mathcal{P}(M) \times \mathcal{P}(M) | S \cap T = U\}$ and for $U, S' \subseteq M$ with $U \cap S' = \emptyset$ let $E_{S', U} := \{(U \cup S', U \cap T') \in \mathcal{P}(M) \times \mathcal{P}(M) | T' \subseteq M \setminus (U \cup S')\}$. Then for fixed $U \subseteq M$ and for all $S', S'' \subseteq M$ with $S' \cap U = S'' \cap U = \emptyset$ it follows $E_{S', U} \cap E_{S'', U} = \emptyset$ and $\bigcup_{S' \subseteq M \setminus U} E_{S', U} = D_U$. Therefore
\[
\beta_U = \sum_{(S, T) \in D_U} \alpha_{S,T} = \sum_{(U \cup S', U \cap T') \in \bigcup_{S' \subseteq M \setminus U} E_{S', U}} \alpha_{U \cup S', U \cup T'}.
\]

We will use this lemma several times in a form stated as in the corollary.

Corollary 2.3. Let $M, R, \mathcal{R}, F$ be as in Lemma 2.2 and let $a, b, c, d \in R$. Then
\[
\sum_{S \subseteq M} \sum_{T \subseteq M} a^{|T|} b^{|S|} c^{M-|T|} d^{M-|S|} F(S \cap T) = \sum_{U \subseteq M} (ab)^{|U|} (b + d) c + ad)^{|M|-|U|} F(U).
\]

Proof. For any $U \subseteq M$, abbreviating $m := |M|, u := |U|$, we have
\[
\beta_U = \sum_{S' \subseteq M \setminus U} \sum_{T' \subseteq M \setminus (U \cup S')} a^{u + |T'|} b^{u + |S'|} c^{m-u-|T'|} d^{m-u-|S'|}
\]
\[
= \sum_{j=0}^{m-u} \sum_{k=0}^{m-u-j} \binom{m-u}{j} \binom{m-u-j}{k} a^{u+j+k} b^{u+j} c^{m-u-k} d^{m-u-j}
\]
\[
= \sum_{j=0}^{m-u} \sum_{k=0}^{m-u-j} \binom{m-u}{k} \binom{m-u-k}{j} a^{u+k+j} b^{u+k} c^{m-u-k} d^{m-u-j}
\]
\[
= (ab)^u \sum_{k=0}^{m-u} \binom{m-u}{k} a^k d^k \sum_{j=0}^{m-u-k} \binom{m-u-k}{j} b^j d^{m-u-k-j} c^{m-u-k}
\]
\[
= (ab)^u ((b + d) c + ad)^{m-u}.
\]

Looking at (5), we need a representation of $f(2^n x_S)$ in terms of $f(x_T)$, that is, with scalar factor 1 in the argument of $f$. \[\square\]
Lemma 2.4. Let $f: V^m \to W$ be any function and denote the multi-Jensen kernel of $f$ by

$$\varphi(x, y) := f \left( \frac{1}{2} (x + y) \right) - \frac{1}{2^m} \sum_{S \subseteq m} f(x_S + y_{m \setminus S}), \quad x, y \in V^m.$$  

Then

$$f(2^n x) = \sum_{S \subseteq m} 2^{|S|} (1 - 2^n)^{|m| - |S|} f(x_S)$$  

(6)

$$- \sum_{i=1}^n \sum_{S \subseteq m} 2^{|S| (n+1-i)} (1 - 2^n+1-i)^{|m| - |S|} \varphi(2^i x_S, 0)$$

for all $n \in \mathbb{N}$ and for all $x \in V^m$.

Proof. For any $x \in V^m$ the proof is by induction on $n$. In order to establish the induction basis, let $S \subseteq m$ and consider

$$f(x_S) = f \left( \frac{1}{2} (2x_S + 0) \right) = \frac{1}{2^m} \sum_{T \subseteq m} f(2x_S \cap T) + \varphi(2x_S, 0).$$

Multiplying with $2^{|S|} (-1)^{|m| - |S|}$, taking the sum over all $S \subseteq m$ and then applying Corollary 2.3 with $a = c = 1, b = 2, d = -1$ and with $F(U) = f(2x_U)$, one obtains

$$\sum_{S \subseteq m} 2^{|S|} (-1)^{|m| - |S|} f(x_S)$$

$$= \frac{1}{2^m} \sum_{S \subseteq m} \sum_{T \subseteq m} 2^{|S|} (-1)^{|m| - |S|} f(2x_S \cap T)$$

$$+ \sum_{S \subseteq m} 2^{|S|} (-1)^{|m| - |S|} \varphi(2x_S, 0)$$

$$= \frac{1}{2^m} \sum_{U \subseteq m} 2^{|U|} (-1)^{|m| - |U|} f(2x_U) + \sum_{S \subseteq m} 2^{|S|} (-1)^{|m| - |S|} \varphi(2x_S, 0).$$

The only active term in the sum over all $U \subseteq m$ is that for $U = m$, so the value of this sum reduces to $2^m f(2x)$, hence it follows

(7)  

$$f(2x) = \sum_{S \subseteq m} 2^{|S|} (-1)^{|m| - |S|} f(x_S) - \sum_{S \subseteq m} 2^{|S|} (-1)^{|m| - |S|} \varphi(2x_S, 0),$$

which is (6) for $n = 1$. In order to establish the induction step, we use at first (7) and then as induction hypothesis (6), to obtain

$$f(2^{n+1} x)$$

$$= f \left( 2^n x \right) + \sum_{T \subseteq m} 2^{|T|} (-1)^{|m| - |T|} \left( f \left( 2^n x_T \right) - \varphi(2^{n+1} x_T, 0) \right)$$

$$= \sum_{S \subseteq m} \sum_{T \subseteq m} (-1)^{|m| - |T|} 2^{|T| + |S|} n (1 - 2^n)^{|m| - |S|} f(x_S \cap T)$$
By (6) we have

\[ - \sum_{i=1}^{n} \sum_{S \subseteq m} \sum_{T \subseteq m} (-1)^{m-|T|} 2^{T+|S|(n+1-i)} (1 - 2^{n+1-i})^{m-|S|} \varphi(2^i x_{S\cap T}, 0) \]

\[ - \sum_{T \subseteq m} 2^{|T|} (-1)^{m-|T|} \varphi(2^{n+1} x_T, 0). \]

The double sum over \( S, T \subseteq m \) containing terms with \( f \) is transformed to a single sum by Corollary 2.3 with \( a = 2, b = 2^n, c = -1, d = 1 - 2^n \), in the same way as that containing terms with \( \varphi \) with \( a = 2, b = 2^{n+1-i}, c = -1, d = 1 - 2^{n+1-i} \), resulting in

\[ f(2^{n+1} x) = \sum_{U \subseteq m} 2^{U/(n+1)} (1 - 2^{n+1})^{m-|U|} f(x_U) \]

\[ - \sum_{i=1}^{n} \sum_{U \subseteq m} 2^{U/(n+2-i)} (1 - 2^{n+2-i})^{m-|U|} \varphi(2^i x_U, 0) \]

\[ - \sum_{U \subseteq m} 2^{U/(n+2-(n+1))} (1 - 2^{n+2-(n+1)})^{m-|U|} \varphi(2^{n+1} x_U, 0) \]

\[ = \sum_{S \subseteq m} 2^{S/(n+1)} (1 - 2^{n+1})^{m-|S|} f(x_S) \]

\[ - \sum_{i=1}^{n+1} \sum_{S \subseteq m} 2^{S/(n+2-i)} (1 - 2^{n+2-i})^{m-|S|} \varphi(2^i x_S, 0). \]

Now we are in a position to estimate \( f \) against \( g_n \).

**Lemma 2.5.** Let \( f : V^m \to W \) be \( \varepsilon \)-multi-Jensen and let \( g_n \) be given by (5). Then

\[ \| g_n(x) - f(x) \| \leq 2^{-n} \sum_{U \subseteq m} 2^{-m-1-|U|} \| f(x_U) \| + 2m(1 + 2^{-n})^{m-1}(1 - 2^{-n}) \varepsilon \]

for all \( x \in V^m \) and for all \( n \in \mathbb{N} \).

**Proof.** By (6) we have

\[ g_n(x) = \sum_{S \subseteq m} 2^{-|S|} f(2^n x_S) = \sum_{S \subseteq m} \sum_{T \subseteq m} 2^{(|T|-|S|)n} (1 - 2^n)^{m-|T|} f(x_{S\cap T}) \]

\[ - \sum_{i=1}^{n} \sum_{S \subseteq m} \sum_{T \subseteq m} 2^{(|T|-|S|)n} 2^{|T|(1-i)} (1 - 2^{n+1-i})^{m-|T|} \varphi(2^i x_{S\cap T}, 0). \]

Application of Corollary 2.3 with \( a = 2^n, b = 2^{-n}, c = 1 - 2^n, d = 1 \) to the double sum containing \( f \), and with \( a = 2^{n+1-i}, b = 2^{-n}, c = 1 - 2^{n+1-i}, d = 1 \)
Let taking we obtain by the mean value theorem the norm and observing

Proof. \( (5) \)

Then Lemma 2.6.

\( \tau \)

where

\( \phi \)

The double sum in the second line may be rewritten (without \( \varepsilon \)) as

\[
\sum_{i=1}^{n} ((1 + 2^{-n})^m - (1 + 2^{-n} - 2^{1-i})^m).
\]

With \( \phi : \mathbb{R} \to \mathbb{R}, \phi(t) := t^m, t_i := 1 + 2^{-n} - 2^{1-i} \) and \( h_i := 2^{1-i}, i = 1, \ldots, n \), we obtain by the mean value theorem

\[
|\sum_{i=1}^{n} (\phi(t_i + h_i) - \phi(t_i))| = |\sum_{i=1}^{n} h_i \phi'(\tau_i)| = |\sum_{i=1}^{n} 2^{1-i} m r_i^{m-1}| \\
\leq 2m(1 + 2^{-n})^{m-1} \sum_{i=1}^{n} 2^{-1} = 2m(1 + 2^{-n})^{m-1}(1 - 2^{-n}),
\]

where \( \tau_i \) was estimated by \( 1 + 2^{-n} \) for \( i = 1, \ldots, n \). \( \square \)

**Lemma 2.6.** Let \( f : V^m \to W \) be \( \varepsilon \)-multi-Jensen and let \( g_n \) be given by (5). Then \( (g_n(x))_{n \in \mathbb{N}} \) is a Cauchy-sequence for every \( x \in V^m \).

**Proof.** Taking \( x \in V^m \) and \( n, k \in \mathbb{N}, \) we have by (8)

\[
g_{n+k}(x) - g_n(x) \\
= \sum_{U \subseteq m} (2^{-(m-|U|)(n+k)} f(x_U) - 2^{-(m-|U|)n} f(x_U)) \\
- \sum_{i=1}^{n+k} \sum_{U \subseteq m} 2^{(1-i)|U|} (1 + 2^{-(n+k)} - 2^{1-i} m - |U|) \varphi(2^i x_U, 0) \\
+ \sum_{i=1}^{n} \sum_{U \subseteq m} 2^{(1-i)|U|} (1 + 2^{-n} - 2^{1-i} m - |U|) \varphi(2^i x_U, 0) \\
= : S_f + S_{\varphi}.
\]
In the sum containing \( f \), the terms for \( U = m \) cancel each other, such that

\[
\|S_f\| = \| \sum_{U \subseteq m} 2^{-(m-|U|)n}(2^{-(m-|U|)k} - 1)f(x_U) \| \\
\leq 2^{-n} \sum_{U \subseteq m} 2^{-(m-1-|U|)n}\|f(x_U)\|.
\]

Observing \( \varphi(0,0) = 0 \), the sums containing \( \varphi \) are at first rewritten as

\[
S_\varphi = -\sum_{\emptyset \neq U \subseteq m} \left( \sum_{i=1}^{n} 2^{(1-i)|U|}((1 + 2^{-(n+k)} - 2^{1-i})^{m-|U|} \right.
\]
\[
\left. - (1 + 2^{-n} - 2^{1-i})^{m-|U|})\varphi(2^ix_U,0) + \sum_{i=n+1}^{n+k} 2^{(1-i)|U|}(1 + 2^{-(n+k)} - 2^{1-i})^{m-|U|}\varphi(2^ix_U,0) \right) .
\]

For \( U \subseteq m \) we have

\[
\left(1 + 2^{-(n+k)} - 2^{1-i})^{m-|U|} - (1 + 2^{-n} - 2^{1-i})^{m-|U|} \right)
\]
\[
\leq 2^{-n} \sum_{i=0}^{m-|U|} \left( \sum_{l=0}^{m-|U|} (2^{-kl} - 1)(1 - 2^{1-i})^{m-|U|} = 2^{-n}(2^{m-|U|} - 1),
\]
and therefore

\[
\| \sum_{i=1}^{n} 2^{(1-i)|U|}((1 + 2^{-(n+k)} - 2^{1-i})^{m-|U|} - (1 + 2^{-n} - 2^{1-i})^{m-|U|})\varphi(2^ix_U,0) \|
\]
\[
\leq 2^{-n}(2^{m-|U|} - 1)2^{|U|} \sum_{i=1}^{n} 2^{-|U|i}\varepsilon = 2^{-n}(2^{m-|U|} - 1)\frac{1 - 2^{-|U|n}}{1 - 2^{-|U|}}\varepsilon.
\]

Furthermore

\[
\| \sum_{i=n+1}^{n+k} 2^{(1-i)|U|}(1 + 2^{-(n+k)} - 2^{1-i})^{m-|U|}\varphi(2^ix_U,0) \|
\]
\[
\leq \sum_{i=1}^{k} 2^{(1-i-n)|U|}2^{m-|U|}\varepsilon = 2^{-|U|n2^{m-|U|}}\frac{1 - 2^{-|U|k}}{1 - 2^{-|U|}}\varepsilon,
\]
such that

\[
\|S_\varphi\| \leq \sum_{\emptyset \neq U \subseteq m} \left( 2^{-n}(2^{m-|U|} - 1)\frac{1 - 2^{-|U|n}}{1 - 2^{-|U|}} + 2^{-|U|n2^{m-|U|}}\frac{1 - 2^{-|U|k}}{1 - 2^{-|U|}} \right)\varepsilon
\]
\[
\leq 2^{-n}2(2^m - 1)^2\varepsilon.
\]
Finally we have all ingredients for the proof of Theorem 1.2.

Proof. By Lemma 2.6 and completeness of $W$ there exists a function $g : V^m \to W$, $g(x) := \lim_{n \to \infty} g_n(x)$. Taking the limit for $n$ to infinity in Lemma 2.1, it follows that $g$ is multi-Jensen and doing the same in Lemma 2.5, one obtains (4) for all $x \in V^m$.

In order to show uniqueness of $g$ up to a constant, assume that there is another multi-Jensen function $\tilde{g} : V^m \to W$, satisfying $\|f(x) - \tilde{g}(x)\| \leq 2m\varepsilon$ for all $x \in V^m$. Defining a sequence $\tilde{g}_n : V^m \to W$, by

$$\tilde{g}_n(x) := \sum_{S \subseteq m} 2^{-|S|} \tilde{g}(2^n x_S), \quad n = 1, 2, \ldots,$$

it follows on replacing $f$ by $\tilde{g}$ from Lemma 2.6 that $(\tilde{g}_n(x))_{n \in \mathbb{N}}$ is convergent for all $x \in V^m$ and by Lemma 2.5 (with $\varepsilon = 0$) that $\lim_{n \to \infty} \tilde{g}_n(x) = \tilde{g}(x)$ for all $x \in V^m$. Furthermore, taking in

$$\|g_n(x) - \tilde{g}_n(x)\| \leq \sum_{S \subseteq m} 2^{-|S|} \|f(2^n x_S) - \tilde{g}(2^n x_S)\|$$

$$\leq \|f(0) - \tilde{g}(0)\| + \sum_{\emptyset \neq S \subseteq m} 2^{-|S|} 2m\varepsilon,$$

the limit for $n$ to infinity, one obtains $\tilde{g} = g$ in case $f(0) = \tilde{g}(0)$. If $f(0) \neq \tilde{g}(0)$, let $\hat{g} : V^m \to W$ be defined by $\hat{g}(x) := \tilde{g}(x) - \tilde{g}(0) + f(0)$. Then $\hat{g}$ is multi-Jensen, because it differs from a multi-Jensen function only by a constant. Defining a sequence $\hat{g}_n : V^m \to W$ by

$$\hat{g}_n(x) := \sum_{S \subseteq m} 2^{-|S|} \hat{g}(2^n x_S), \quad n = 1, 2, \ldots,$$

one has because of $\hat{g}(0) = f(0)$

$$\|g_n(x) - \hat{g}_n(x)\| \leq \sum_{\emptyset \neq S \subseteq m} 2^{-|S|} \|f(2^n x_S) - \hat{g}(2^n x_S) + \hat{g}(0) - f(0)\|$$

$$\leq (2m\varepsilon + \|\hat{g}(0) - f(0)\|)2^{-n}(2^m - 1),$$

and therefore, taking the limit for $n$ to infinity, $\hat{g}(x) = g(x)$ for all $x \in V^m$. But this in turn yields $\tilde{g}(x) = g(x) + \tilde{g}(0) - f(0)$. \qed
References


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