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# LOCALLY CONVEX VECTOR TOPOLOGIES ON $\mathcal{B}(X, Y)$

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# LOCALLY CONVEX VECTOR TOPOLOGIES ON $\mathcal{B}(X,Y)$

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Abstract. In this paper, we introduce and study various locally convex vector topologies on the space of bounded linear operators between Banach spaces. We also apply these topologies to approximation properties.

## 1. Introduction and notations

In the study of operators in $\mathcal{B}(X,Y)$, the space of bounded linear operators from a Banach space $X$ into another Banach space $Y$, vector topologies on $\mathcal{B}(X,Y)$ have provided us with a tool of great importance. For instance, Grothendieck [6] used the $\tau$-topology in order to give the modern definition of the approximation property, thereby obtaining fruitful results in approximation properties. Later Kalton [7] introduced the dual weak operator topology which led him to a characterization of weak compactness of sets of compact operators. The main purpose of this paper is to study fundamental properties of locally convex vector topologies on $\mathcal{B}(X,Y)$ that are in general weaker than the operator norm topology. The main topics of our study are metrizability, completeness, and compactness of these vector topologies. We also introduce several approximation properties to which we apply our study of vector topologies.

In Section 2, we study topologies induced by spaces of linear functionals on $\mathcal{B}(X,Y)$. In Section 3, we study topologies generated by subbases on $\mathcal{B}(X,Y)$. In Section 4, simple characterizations of Banach spaces having approximation properties are established.

We now start by listing notations which are used throughout this paper.

**Notation**

- $X$, $Y$ : Banach spaces.
- $X^*$ : The dual space of $X$.
- $T^*$ : The adjoint of an operator $T$.
- $\mathcal{B}(X,Y)$ : The space of bounded linear operators from $X$ into $Y$.

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- $\mathcal{F}(X,Y)$ : The space of bounded and finite rank linear operators from $X$ into $Y$.
- $\mathcal{K}(X,Y)$ : The space of compact operators from $X$ into $Y$.
- $\mathcal{K}^*(X,Y)$ : The space of compact adjoint operators from $Y^*$ into $X^*$.
- $\mathcal{K}(X,Y,\lambda)$ : The collection of compact operators $T$ from $X$ into $Y$ satisfying $\|T\| \le \lambda$.
- $\mathcal{K}^*(X,Y,\lambda)$ : The collection of compact adjoint operators $T^*$ from $Y^*$ into $X^*$ satisfying $\|T^*\| \le \lambda$.

We similarly define $\mathcal{F}^*(X,Y)$, $\mathcal{F}(X,Y,\lambda)$, $\mathcal{F}^*(X,Y,\lambda)$, $\mathcal{B}^*(X,Y)$, $\mathcal{B}(X,Y,\lambda)$, and $\mathcal{B}^*(X,Y,\lambda)$. For convenience we denote $\mathcal{B}(X,X),\ldots$ by $\mathcal{B}(X)$, ….

## 2. Topologies induced by subspaces of $\mathcal{B}(X,Y)^\sharp$

Suppose that $\mathcal{Z}$ is a subspace of $\mathcal{B}(X,Y)^\sharp$, the vector space of all linear functionals on $\mathcal{B}(X,Y)$. Then the topology *induced* by $\mathcal{Z}$ is the smallest topology on $\mathcal{B}(X,Y)$ such that every member of $\mathcal{Z}$ is continuous. In this section we study topologies induced by subspaces of $\mathcal{B}(X,Y)^\sharp$. The following are elementary facts about these topologies. One may refer to Megginson [12, Section 2.4] for rigorous proofs.

*Remark* 2.1. Suppose that $\mathcal{Z}$ is a subspace of $\mathcal{B}(X,Y)^\sharp$ and $\mathcal{T}$ is the topology induced by $\mathcal{Z}$.

(a) Let a net $(T_\alpha)$ and $T$ be in $\mathcal{B}(X,Y)$. Then

$$T_\alpha \xrightarrow{\mathcal{T}} T \text{ if and only if } \varphi(T_\alpha) \longrightarrow \varphi(T) \text{ for each } \varphi \in \mathcal{Z}.$$

(b) $\mathcal{T}$ is a locally convex vector topology and the dual space of $\mathcal{B}(X,Y)$ with respect to $\mathcal{T}$ is $\mathcal{Z}$. If $\mathcal{Z}$ is separating, then $\mathcal{T}$ is completely regular.

(c) The elements of a basis for $\mathcal{T}$ are of the form

$$N(T;\mathcal{A},\epsilon) = \{R \in \mathcal{B}(X,Y) : |\varphi(R-T)| < \epsilon \text{ for each } \varphi \in \mathcal{A}\},$$

where $T \in \mathcal{B}(X,Y)$, $\mathcal{A}$ is a finite set in $\mathcal{Z}$, and $\epsilon > 0$.

We now introduce topologies induced by some subspaces of $\mathcal{B}(X,Y)^\sharp$. First we review the *weak* and *weak** topologies.

The *weak topology* (in short, *weak*) on $\mathcal{B}(X,Y)$ is the topology induced by $\mathcal{B}(X,Y)^*$, the dual space of $\mathcal{B}(X,Y)$ with respect to the operator norm topology on $\mathcal{B}(X,Y)$. Remark 2.1 yields that for a net $(T_\alpha)$ and $T$ in $\mathcal{B}(X,Y)$

$$T_\alpha \xrightarrow{weak} T \text{ if and only if } \varphi(T_\alpha) \longrightarrow \varphi(T) \text{ for each } \varphi \in \mathcal{B}(X,Y)^*$$

and that $(\mathcal{B}(X,Y), weak)^* = \mathcal{B}(X,Y)^*$.

The *weak** *topology* (in short, *weak**) on $\mathcal{B}(X,Y^*) = (X\hat{\otimes}_\pi Y)^*$ (see Ryan [13, p. 24]) is the topology induced by $X\hat{\otimes}_\pi Y$, the projective tensor product of $X$ and $Y$, and if $\varphi = \sum_n x_n \otimes y_n \in X\hat{\otimes}_\pi Y$ and so $\sum_n \|x_n\|\|y_n\| < \infty$, then

$\varphi(T) = \sum_n (Tx_n)y_n$ for $T \in \mathcal{B}(X,Y^*)$. Remark 2.1 yields that for a net $(T_\alpha)$ and $T$ in $\mathcal{B}(X,Y^*)$

$$T_\alpha \overset{weak^*}{\longrightarrow} T \text{ if and only if } \sum_n (T_\alpha x_n)y_n \longrightarrow \sum_n (Tx_n)y_n$$

for each $(x_n) \subset X$ and $(y_n) \subset Y$ with $\sum_n \|x_n\|\|y_n\| < \infty$ and that

$$(\mathcal{B}(X,Y^*), weak^*)^* = X\hat{\otimes}_\pi Y.$$

**Definition 2.2.** Let $\mathcal{Z}_1$ be the linear span of all linear functionals $\varphi$ on $\mathcal{B}(X,Y^*)$ of the form

$$\varphi(T) = (Tx)y$$

for $x \in X$ and $y \in Y$.

Let $\mathcal{Z}_2$ be the linear span of all linear functionals $\varphi$ on $\mathcal{B}(X,Y)$ of the form

$$\varphi(T) = y^*(Tx)$$

for $x \in X$ and $y^* \in Y^*$.

Let $\mathcal{Z}_3$ be the linear span of all linear functionals $\varphi$ on $\mathcal{B}(X,Y)$ of the form

$$\varphi(T) = \sum_n y_n^*(Tx_n)$$

for $(x_n) \subset X$ and $(y_n^*) \subset Y^*$ with $\sum_n \|x_n\|\|y_n^*\| < \infty$.

Let $\mathcal{Z}_4$ be the linear span of all linear functionals $\varphi$ on $\mathcal{B}(X,Y)$ of the form

$$\varphi(T) = x^{**}(T^*y^*)$$

for $x^{**} \in X^{**}$ and $y^* \in Y^*$.

Let $\mathcal{Z}_5$ be the linear span of all linear functionals $\varphi$ on $\mathcal{B}(X,Y)$ of the form

$$\varphi(T) = \sum_n x_n^{**}(T^*y_n^*)$$

for $(x_n^{**}) \subset X^{**}$ and $(y_n^*) \subset Y^*$ with $\sum_n \|y_n^*\|\|x_n^{**}\| < \infty$.

Then the *weak* operator topology* (in short, *weak* o*) on $\mathcal{B}(X,Y^*)$ is the topology induced by $\mathcal{Z}_1$, the *weak operator topology* (in short, *wo*) on $\mathcal{B}(X,Y)$ is the topology induced by $\mathcal{Z}_2$, the *summable weak operator topology* (in short, *swo*) on $\mathcal{B}(X,Y)$ is the topology induced by $\mathcal{Z}_3$, the *weak adjoint operator topology* (in short, *wao*) on $\mathcal{B}(X,Y)$ is the topology induced by $\mathcal{Z}_4$, and the *summable weak adjoint operator topology* (in short, *swao*) on $\mathcal{B}(X,Y)$ is the topology induced by $\mathcal{Z}_5$.

From Remark 2.1(a) we see the following: for a net $(T_\alpha)$ and $T$ in $\mathcal{B}(X,Y^*)$

$$T_\alpha \overset{weak^* o}{\longrightarrow} T \text{ if and only if } (T_\alpha x)y \longrightarrow (Tx)y \text{ for each } x \in X \text{ and } y \in Y.$$

For a net $(T_\alpha)$ and $T$ in $\mathcal{B}(X,Y)$

$$T_\alpha \overset{wo}{\longrightarrow} T \text{ if and only if } y^*(T_\alpha x) \longrightarrow y^*(Tx)$$

for each $x \in X$ and $y^* \in Y^*$;

$$T_\alpha \xrightarrow{swo} T \text{ if and only if } \sum_n y_n^*(T_\alpha x_n) \longrightarrow \sum_n y_n^*(Tx_n)$$

for each $(x_n) \subset X$ and $(y_n^*) \subset Y^*$ with $\sum_n \|x_n\|\|y_n^*\| < \infty$;

$$T_\alpha \xrightarrow{wao} T \text{ if and only if } x^{**}(T_\alpha^* y^*) \longrightarrow x^{**}(T^* y^*)$$

for each $x^{**} \in X^{**}$ and $y^* \in Y^*$; and

$$T_\alpha \xrightarrow{swao} T \text{ if and only if } \sum_n x_n^{**}(T_\alpha^* y_n^*) \longrightarrow \sum_n x_n^{**}(T^* y_n^*)$$

for each $(x_n^{**}) \subset X^{**}$ and $(y_n^*) \subset Y^*$ with $\sum_n \|y_n^*\|\|x_n^{**}\| < \infty$.

Now it is easy to check the following relations between the above topologies. Here for two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$, $\mathcal{T}_1 \geq \mathcal{T}_2$ means that $\mathcal{T}_1$ is stronger than $\mathcal{T}_2$. One may refer to the diagram in Section 3 for the relationship between our topologies.

**Proposition 2.3.** (a) *On $\mathcal{B}(X,Y)$, weak $\geq$ swao $\geq$ swo, swao $\geq$ wao, swo $\geq$ wo, wao $\geq$ wo. On $\mathcal{B}(X,Y^*)$, weak $\geq$ swao $\geq$ swo $\geq$ weak$^*$, weak$^*$ $\geq$ weak$^*$o, wao $\geq$ wo $\geq$ weak$^*$o.*

(b) *On each bounded set in $\mathcal{B}(X,Y)$ swao $=$ wao $\geq$ swo $=$ wo. On each bounded set in $\mathcal{B}(X,Y^*)$ swao $=$ wao $\geq$ swo $=$ wo $\geq$ weak$^*$ $=$ weak$^*$o.*

(c) *If $X$ is reflexive, then on $\mathcal{B}(X,Y)$ swao $=$ swo $\geq$ wao $=$ wo. Hence, if $X$ is reflexive, then on each bounded set in $\mathcal{B}(X,Y)$ swao $=$ swo $=$ wao $=$ wo.*

*If $Y$ is reflexive, then on $\mathcal{B}(X,Y^*)$ swo $=$ weak$^*$ $\geq$ wo $=$ weak$^*$o. Hence, if $X$ and $Y$ are reflexive, then on $\mathcal{B}(X,Y^*)$ swao $=$ swo $=$ weak$^*$ $\geq$ wao $=$ wo $=$ weak$^*$o.*

*Consequently, if $X$ and $Y$ are reflexive, then on each bounded set in $\mathcal{B}(X,Y^*)$ swao $=$ swo $=$ weak$^*$ $=$ wao $=$ wo $=$ weak$^*$o.*

The definition of boundedness with respect to a metric topology can be extended to vector topologies not induced by metrics. We say that a set $B$ in a topological vector space is *bounded* with respect to the topology if, for each neighborhood $U$ of 0, there is a $s_U > 0$ such that $B \subset tU$ whenever $t > s_U$. Therefore, for a vector space $\mathcal{V}$ having two vector topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ with $\mathcal{T}_1 \geq \mathcal{T}_2$, if a set $B$ in $\mathcal{V}$ is bounded with respect to $\mathcal{T}_1$, then $B$ is bounded with respect to $\mathcal{T}_2$.

If a topological vector space has a topology induced by a subspace of the vector space of all linear functionals on the vector space, then the following lemma gives a way to check the boundedness with respect to the topology.

**Lemma 2.4** ([12, Proposition 2.4.14])**.** *If $B$ is a set in a vector space equipped with a topology induced by a subspace $\mathcal{Z}$ of the vector space of all linear functionals on the vector space, then $B$ is bounded with respect to the topology if and only if $\sup_{x\in B} |f(x)|$ is finite for each $f \in \mathcal{Z}$.*

Now let's consider boundedness with respect to our vector topologies. First, we have the following proposition. Here, in (a), the 'bounded' means the 'operator norm bounded'.

**Proposition 2.5.** *Let $\mathcal{A}$ be a set in $\mathcal{B}(X,Y)$. Then the following are equivalent.*

(a) *$\mathcal{A}$ is bounded.*
(b) *$\mathcal{A}$ is weak-bounded.*
(c) *$\mathcal{A}$ is swao-bounded.*
(d) *$\mathcal{A}$ is swo-bounded.*
(e) *$\mathcal{A}$ is wao-bounded.*
(f) *$\mathcal{A}$ is wo-bounded.*

*Proof.* See Figure 1, in Section 3. Then it is enough to show (f)$\Longrightarrow$(a). The proof is a review of the proof of Kim [8, Proposition 2.1]. Now assume (f). Let $x \in X$ and consider $\{Q_Y(Tx) : T \in \mathcal{A}\}$, where $Q_Y$ is the natural map from $Y$ into $Y^{**}$. Since $\mathcal{A}$ is *wo*-bounded, by Lemma 2.4 for each $y^* \in Y^*$

$$\sup_{T\in\mathcal{A}} |Q_Y(Tx)y^*| = \sup_{T\in\mathcal{A}} |y^*Tx| < \infty.$$

By Uniform Boundedness Principle

$$\sup_{T\in\mathcal{A}} \|Tx\| = \sup_{T\in\mathcal{A}} \|Q_Y(Tx)\| < \infty.$$

Again, by Uniform Boundedness Principle

$$\sup_{T\in\mathcal{A}} \|T\| < \infty.$$

Hence $\mathcal{A}$ is bounded. $\square$

By a similar proof we have the following proposition.

**Proposition 2.6.** *Let $\mathcal{A}$ be a set in $\mathcal{B}(X,Y^*)$. Then the following are equivalent.*

(a) *$\mathcal{A}$ is bounded.*
(b) *$\mathcal{A}$ is weak-bounded.*
(c) *$\mathcal{A}$ is swao-bounded.*
(d) *$\mathcal{A}$ is swo-bounded.*
(e) *$\mathcal{A}$ is weak$^*$-bounded.*
(f) *$\mathcal{A}$ is wao-bounded.*
(g) *$\mathcal{A}$ is wo-bounded.*
(h) *$\mathcal{A}$ is weak$^*$o-bounded.*

If a vector topology is induced by a metric, then the topology has useful properties. But a vector topology is seldom metrizable. In fact, it is well known that the *weak* topology of a normed space is metrizable if and only if the space is finite-dimensional, and the *weak*$^*$ topology of the dual space of a Banach space $X$ is metrizable if and only if $X$ is finite-dimensional. Therefore the *weak* topology on $\mathcal{B}(X,Y)$ is metrizable if and only if $X$ and $Y$ are finite-dimensional, and the *weak** topology on $\mathcal{B}(X,Y^*)$ is metrizable if and only if $X$ and $Y$ are finite-dimensional; here and throughout the paper, we are assuming that $X \neq \{0\}$ and $Y \neq \{0\}$. We now have the following theorem.

**Theorem 2.7.** (a) *The swao* (*respectively, swo, wao, and wo*) *topology on* $\mathcal{B}(X,Y)$ *is metrizable if and only if* $X$ *and* $Y$ *are finite-dimensional.*
(b) *The weak* o topology on* $\mathcal{B}(X,Y^*)$ *is metrizable if and only if* $X$ *and* $Y$ *are finite-dimensional.*

*Proof.* The 'if' parts of the theorem are clear and we only show the 'only if' part for the *wo* topology; the proofs of the other cases are the same. Now assume that the *wo* topology is induced by a metric $d$, and to obtain a contradiction, suppose that $X$ or $Y$ is infinite-dimensional. Then the space of linear functionals on $\mathcal{B}(X,Y)$ inducing the *wo* is infinite-dimensional. Thus every *wo*-open set is *wo*-unbounded (see [12, Proposition 2.4.15]), hence it is unbounded. Let $B_d(0;1/n)$ be the $d$-open ball with the center 0 and radius $1/n$. Then for each $n$, there is a $T_n \in B_d(0;1/n)$ such that $\|T_n\| \geq n$. Thus $T_n \xrightarrow{wo} 0$ and $(T_n)$ is an unbounded sequence. This is a contradiction because a convergent sequence in a topological vector space is bounded with respect to the topology and so is bounded by Proposition 2.5. Hence $X$ and $Y$ are finite-dimensional. □

Bounded sets in normed spaces supply good situations for vector topologies on the spaces. In [8], the author showed the following theorem for the case of *wao* and *wo* topologies on $\mathcal{B}(X)$. Recall *swao* = *wao* and *swo* = *wo* on each bounded set in $\mathcal{B}(X,Y)$, and *weak** = *weak** *o* on each bounded set in $\mathcal{B}(X,Y^*)$. The proof of Theorem 2.8 is essentially the same as the proof in [8]. So we omit it.

**Theorem 2.8.** (a) *Suppose that* $X^{**}$ *and* $Y^*$ *are separable. Then the swao*(*wao*) *topology on each bounded set in* $\mathcal{B}(X,Y)$ *has a countable basis and is metrizable. Also* $\mathcal{B}(X,Y)$ *is swao*(*wao*)*-separable.*
(b) *Suppose that* $X$ *and* $Y^*$ *are separable. Then the swo*(*wo*) *topology on each bounded set in* $\mathcal{B}(X,Y)$ *has a countable basis and is metrizable. Also* $\mathcal{B}(X,Y)$ *is swo*(*wo*)*-separable.*
(c) *Suppose that* $X$ *and* $Y$ *are separable. Then the weak**(*weak* o*) *topology on each bounded set in* $\mathcal{B}(X,Y^*)$ *has a countable basis and is metrizable. Also* $\mathcal{B}(X,Y^*)$ *is weak**(*weak* o*)*-separable.*

Next, we consider the completeness of $\mathcal{B}(X,Y)$ and $\mathcal{B}(X,Y^*)$. Of course, these spaces are complete with respect to the operator norm topology. We say that a net $(x_\alpha)$ in a topological vector space $X$ with a vector topology $\mathcal{T}$ is *Cauchy* (with respect to $\mathcal{T}$) if, for every basic neighborhood $U$ of 0 in $\mathcal{T}$, there is an $\alpha_U$ such that $\beta,\gamma \succeq \alpha_U$ implies $x_\beta - x_\gamma \in U$. Also we say that a topological vector space $(X,\mathcal{T})$ is *complete*, or $\mathcal{T}$*-complete* if every Cauchy net in the space $X$ converges. It is well known that the *weak* topology on a normed space $X$ is complete if and only if $X$ is finite-dimensional, and the *weak*$^*$ topology on the dual space of a normed space $X$ is complete if and only if $X$ is finite-dimensional. Therefore the *weak* topology on $\mathcal{B}(X,Y)$ is complete if and only if $X$ and $Y$ are finite-dimensional, and the *weak*$^*$ topology on $\mathcal{B}(X,Y^*)$ is complete if and only if $X$ and $Y$ are finite-dimensional. Now for the *wo*, *wao*, and *weak*$^*$*o* topology, we have the following results.

**Theorem 2.9.** (a) *The wo (respectively, wao) topology on $\mathcal{B}(X,Y)$ is complete if and only if $X$ and $Y$ are finite-dimensional.*

(b) *The weak$^*$o topology on $\mathcal{B}(X,Y^*)$ is complete if and only if $X$ and $Y$ are finite-dimensional.*

*Proof.* Since the "if" parts are clear, we only consider the "only if" parts.

(a) Suppose that the *wo* topology on $\mathcal{B}(X,Y)$ is complete. To show that $X$ is finite-dimensional, we show that $X^*$ is $w^*$-complete, where $w^*$ is the *weak*$^*$ topology on $X^*$. For this let $(x^*_\alpha)$ be a $w^*$-Cauchy net in $X^*$. Choose $y_0 \in Y$ and $y^*_0 \in Y^*$ so that $1 = \|y_0\| = \|y^*_0\| = y^*_0 y_0$. Now define a net $(T_\alpha)$ in $\mathcal{B}(X,Y)$ by

$$T_\alpha x = (x^*_\alpha x) y_0.$$

Since $(x^*_\alpha)$ is $w^*$-Cauchy and

$$y^*(T_\alpha - T_\beta)x = (x^*_\alpha - x^*_\beta)(y^* y_0)x$$

for each $x \in X$ and $y^* \in Y^*$, it follows that $(T_\alpha)$ is *wo*-Cauchy. By the assumption, there is a $T \in \mathcal{B}(X,Y)$ such that $T_\alpha \xrightarrow{wo} T$. Consider $y^*_0 T \in X^*$. Then for each $x \in X$, we have

$$x^*_\alpha x = y^*_0 T_\alpha x \longrightarrow y^*_0 T x.$$

Hence $x^*_\alpha \xrightarrow{w^*} y^*_0 T$. This finishes the proof that $X^*$ is $w^*$-complete.

To show that $Y$ is finite-dimensional, we show that $Y$ is $w$-complete, where $w$ is the *weak* topology on $Y$. For this let $(y_\alpha)$ be a $w$-Cauchy net in $Y$. Choose $x_0 \in X$ and $x^*_0 \in X^*$ so that $1 = \|x_0\| = \|x^*_0\| = x^*_0 x_0$. Now define a net $(T_\alpha)$ in $\mathcal{B}(X,Y)$ by

$$T_\alpha x = (x^*_0 x) y_\alpha.$$

Similarly we can check that $(T_\alpha)$ is *wo*-Cauchy. By the assumption, there is a $T \in \mathcal{B}(X,Y)$ such that $T_\alpha \xrightarrow{wo} T$. Then, similarly as above, we can check that $y_\alpha \xrightarrow{w} T x_0$.

Now suppose that the *wao* topology on $\mathcal{B}(X,Y)$ is complete. To show that $X$ is finite-dimensional, we show that $X^*$ is $w$-complete. For this let $(x^*_\alpha)$ be a $w$-Cauchy net in $X^*$. Choose $y_0 \in Y$ and $y^*_0 \in Y^*$ so that $1 = \|y_0\| = \|y^*_0\| = y^*_0 y_0$. Now define a net $(T_\alpha)$ in $\mathcal{B}(X,Y)$ by

$$T_\alpha x = (x^*_\alpha x) y_0.$$

Since $(x^*_\alpha)$ is $w$-Cauchy and

$$x^{**}(T_\alpha - T_\beta)^* y^* = (y^* y_0) x^{**}(x^*_\alpha - x^*_\beta)$$

for each $y^* \in Y^*$ and $x^{**} \in X^*$, it follows that $(T_\alpha)$ is *wao*-Cauchy. By the assumption, there is a $T \in \mathcal{B}(X,Y)$ such that $T_\alpha \xrightarrow{wao} T$. Consider $T^*y_0^* \in X^*$. Then for each $x^{**} \in X^{**}$, we have

$$x^{**}x_\alpha^* = x^{**}T_\alpha^* y_0^* \longrightarrow x^{**}T^*y_0^*.$$

Hence $x_\alpha^* \xrightarrow{w} T^*y_0^*$. This proves that $X^*$ is $w$-complete.

To show that $Y$ is finite-dimensional, we show that $Y$ is $w$-complete. For this let $(y_\alpha)$ be a $w$-Cauchy net in $Y$. Choose $x_0 \in X$ and $x_0^* \in X^*$ so that $1 = \|x_0\| = \|x_0^*\| = x_0^*x_0$. Now define a net $(T_\alpha)$ in $\mathcal{B}(X,Y)$ by

$$T_\alpha x = x_0^*(x)y_\alpha.$$

Similarly we can check that $(T_\alpha)$ is *wao*-Cauchy. By the assumption, there is a $T \in \mathcal{B}(X,Y)$ such that $T_\alpha \xrightarrow{wao} T$. Consider $Tx_0 \in Y$. Then for each $y^* \in Y$, we have

$$y^*y_\alpha = y^*T_\alpha x_0 = Q_X(x_0)(T_\alpha^*y^*) \longrightarrow Q_X(x_0)(T^*y^*) = y^*Tx_0,$$

where $Q_X$ is the natural map from $X$ into $X^{**}$. Hence $y_\alpha \xrightarrow{w} Tx_0$. This proves that $Y$ is $w$-complete.

(b) Suppose that the *weak*$^*o$ topology on $\mathcal{B}(X,Y^*)$ is complete. To show that $X$ is finite-dimensional, we show that $X^*$ is $w^*$-complete. For this let $(x_\alpha^*)$ be a $w^*$-Cauchy net in $X^*$. Choose $y_0 \in Y$ and $y_0^* \in Y^*$ so that $1 = \|y_0\| = \|y_0^*\| = y_0^*y_0$. Now define a net $(T_\alpha)$ in $\mathcal{B}(X,Y^*)$ by

$$T_\alpha x = (x_\alpha^*x)y_0^*.$$

Since $(x_\alpha^*)$ is $w^*$-Cauchy and

$$((T_\alpha - T_\beta)x)y = (x_\alpha^* - x_\beta^*)(y_0^*y)x$$

for each $x \in X$ and $y \in Y$, it follows that $(T_\alpha)$ is *weak*$^*o$-Cauchy. By the assumption, there is a $T \in \mathcal{B}(X,Y^*)$ such that $T_\alpha \xrightarrow{weak^*o} T$. Consider $Q_Y(y_0)T \in X^*$, where $Q_Y$ is the natural map from $Y$ into $Y^{**}$. Then for each $x \in X$, we have

$$x_\alpha^*x = (T_\alpha x)y_0 \longrightarrow (Tx)y_0 = (Q_Y(y_0)T)x.$$

Hence $x_\alpha^* \xrightarrow{w^*} Q_Y(y_0)T$. This proves that $X^*$ is $w^*$-complete.

To show that $Y$ is finite-dimensional, we show that $Y^*$ is $w^*$-complete. For this let $(y_\alpha^*)$ be a $w^*$-Cauchy net in $Y^*$. Choose $x_0 \in X$ and $x_0^* \in X^*$ so that $1 = \|x_0\| = \|x_0^*\| = x_0^*x_0$. Now define a net $(T_\alpha)$ in $\mathcal{B}(X,Y^*)$ by

$$T_\alpha x = (x_0^*x)y_\alpha^*.$$

Similarly we can check that $(T_\alpha)$ is *weak*$^*o$-Cauchy. By the assumption, there is a $T \in \mathcal{B}(X,Y^*)$ such that $T_\alpha \xrightarrow{weak^*o} T$. Then we can similarly check that $y_\alpha^* \xrightarrow{w^*} Tx_0$. $\square$

Now we are concerned with the compactness in $\mathcal{B}(X, Y)$. In topological vector spaces, every compact set is bounded with respect to the topology. Recall Propositions 2.3, 2.5, and 2.6. Then we see the following.

*Remark* 2.10. (a) Suppose that $\mathcal{A}$ is a set in $\mathcal{B}(X, Y)$. Then $\mathcal{A}$ is *swao*-compact if and only if $\mathcal{A}$ is *wao*-compact, and $\mathcal{A}$ is *swo*-compact if and only if $\mathcal{A}$ is *wo*-compact.

(b) Suppose that $\mathcal{A}$ is a set in $\mathcal{B}(X, Y^*)$. Then $\mathcal{A}$ is *weak**-compact if and only if $\mathcal{A}$ is *weak** *o*-compact. Thus by the virtue of the Banach-Alaoglu theorem, every bounded set in $\mathcal{B}(X, Y^*)$ is relatively *weak** *o*-compact.

Suppose that $\mathcal{A}$ is a set in $\mathcal{B}(X, Y)$ and $x \in X$. Then we use the following notations:

$$\mathcal{A}^* = \{T^* : T \in \mathcal{A}\}, \quad \mathcal{A}x = \{Tx : T \in \mathcal{A}\}.$$

Notice that for a net $(T_\alpha)$ and $T$ in $\mathcal{B}(X, Y)$,

$$T_\alpha \xrightarrow{wao} T \text{ if and only if } T_\alpha^* \xrightarrow{wo} T^* \text{ in } \mathcal{B}(Y^*, X^*),$$

$$T_\alpha \xrightarrow{swao} T \text{ if and only if } T_\alpha^* \xrightarrow{swo} T^* \text{ in } \mathcal{B}(Y^*, X^*). \tag{2.1}$$

Now we give characterizations of *wao* and *wo*-compactness. Some parts of Proposition 2.11 and Theorem 2.13 are well known (cf. [4, Exercises VI.9.2 and VI.9.3]).

**Proposition 2.11.** *Suppose that $\mathcal{A}$ is a set in $\mathcal{B}(X, Y)$.*

(a) *The following are equivalent.*
- (*i*) *$\mathcal{A}$ is wo(swo)-compact.*
- (*ii*) *$\mathcal{A}$ is wo-closed and for each $x \in X$, $\mathcal{A}x$ is w-compact in $Y$, where $w$ is the weak topology on $Y$.*
- (*iii*) *$\mathcal{A}$ is wo-closed and for each $x \in X$, $\overline{\mathcal{A}x}^w$ is w-compact in $Y$.*

(b) *The following are equivalent.*
- (*i*) *$\mathcal{A}$ is wao(swao)-compact.*
- (*ii*) *$\mathcal{A}^*$ is wo-closed in $\mathcal{B}(Y^*, X^*)$ and for each $y^* \in Y^*$, $\mathcal{A}^*y^*$ is w-compact in $X^*$, where $w$ is the weak topology on $X^*$.*
- (*iii*) *$\mathcal{A}^*$ is wo-closed in $\mathcal{B}(Y^*, X^*)$ and for each $y^* \in Y^*$, $\overline{\mathcal{A}^*y^*}^w$ is w-compact in $X^*$.*

*Proof.* (a) We show $(i) \Longrightarrow (ii) \Longrightarrow (iii) \Longrightarrow (i)$. Since $(ii) \Longrightarrow (iii)$ is clear, we show $(i) \Longrightarrow (ii)$ and $(iii) \Longrightarrow (i)$.

$(i) \Longrightarrow (ii)$ Suppose that $\mathcal{A}$ is *wo*-compact. Then clearly $\mathcal{A}$ is *wo*-closed. Let $x \in X$ and $(T_\alpha x)$ a net in $\mathcal{A}x$. Since $\mathcal{A}$ is *wo*-compact, there is a subnet $(T_\beta)$ of $(T_\alpha)$ such that $T_\beta \xrightarrow{wo} T$ for some $T \in \mathcal{A}$. In particular, for each $y^* \in Y^*$

$$y^* T_\beta x \longrightarrow y^* T x.$$

This shows

$$T_\beta x \xrightarrow{w} Tx.$$

Hence $\mathcal{A}x$ is $w$-compact in $Y$.

$(iii) \Longrightarrow (i)$ Consider the map $\psi : (\mathcal{B}(X,Y), wo) \longrightarrow \prod_{x\in X}(Y,w)_x$ defined by

$$\psi(T) = (Tx)_{x\in X},$$

where $(Y,w)_x = (Y,w)$ for all $x \in X$. Then clearly $\psi$ is injective. Let $(T_\alpha)$ be a net and $T$ in $\mathcal{B}(X,Y)$. Then it is easy to check that

$$T_\alpha \xrightarrow{wo} T \text{ if and only if } \psi(T_\alpha) \xrightarrow{pro} \psi(T),$$

where *pro* is the product topology on $\prod_{x\in X}(Y,w)_x$. Thus $\psi : (\mathcal{B}(X,Y), wo) \longrightarrow (\psi(\mathcal{B}(X,Y)), pro)$ is a *wo*-to-*pro* homeomorphism. First we will show $\psi(\overline{\mathcal{A}}^{wo}) = \overline{\psi(\mathcal{A})}^{pro}$. To show this, it is enough to show that $\overline{\psi(\mathcal{A})}^{pro} \subset \psi(\overline{\mathcal{A}}^{wo})$. Let $(y_x)_{x\in X} \in \overline{\psi(\mathcal{A})}^{pro}$. Then there is a net $(\psi(T_\alpha))$ in $\psi(\mathcal{A})$ such that $\psi(T_\alpha) \xrightarrow{pro} (y_x)_{x\in X}$. Thus

$$T_\alpha x \xrightarrow{w} y_x$$

for each $x \in X$. Now for each $x \in X$, let $Tx = y_x$. Then it is easy to check that $T$ is a linear operator from $X$ into $Y$. Since $\overline{\mathcal{A}x}^w$ is $w$-compact in $Y$ for each $x \in X$, $\mathcal{A}x$ is $w$-bounded in $Y$ for each $x \in X$. It follows that $\mathcal{A}$ is *wo*-bounded in $\mathcal{B}(X,Y)$ and so is bounded in $\mathcal{B}(X,Y)$. Now if $\|x\| \le 1$, then we have

$$\|Tx\| \le \liminf_\alpha \|T_\alpha x\| \le \sup_\alpha \|T_\alpha x\| \le \sup_{S\in\mathcal{A}} \|S\|.$$

Therefore $T \in \mathcal{B}(X,Y)$. Hence $(y_x)_{x\in X} = (Tx)_{x\in X} = \psi(T) \in \psi(\overline{\mathcal{A}}^{wo})$ which shows $\overline{\psi(\mathcal{A})}^{pro} \subset \psi(\overline{\mathcal{A}}^{wo})$. To complete the proof, we observe

$$\psi(\mathcal{A}) = \psi(\overline{\mathcal{A}}^{wo}) = \overline{\psi(\mathcal{A})}^{pro} \subset \overline{\prod_{x\in X} \mathcal{A}x}^{pro} = \prod_{x\in X} \overline{\mathcal{A}x}^{w}.$$

By the virtue of the Tychonoff's theorem $\psi(\mathcal{A})$ is *pro*-compact. Since $\psi$ is a homeomorphism, $\mathcal{A}$ is *wo*-compact.

(b) Recall (2.1). Then it is easy to check that

(2.2) $\quad \mathcal{A}$ is wao-compact if and only if $\mathcal{A}^*$ is wo-compact in $\mathcal{B}(Y^*,X^*)$.

By (a) we complete the proof. □

**Corollary 2.12.** *Suppose that $\mathcal{A}$ is a set in $\mathcal{B}(X,Y)$ and that $\mathcal{B}^*(X,Y)$ is wo-closed in $\mathcal{B}(Y^*,X^*)$. Then the following are equivalent.*

(a) *$\mathcal{A}$ is wao(swao)-compact.*
(b) *$\mathcal{A}$ is wao-closed and for each $y^* \in Y^*$, $\mathcal{A}^*y^*$ is w-compact in $X^*$, where $w$ is the weak topology on $X^*$.*
(c) *$\mathcal{A}$ is wao-closed and for each $y^* \in Y^*$, $\overline{\mathcal{A}^*y^*}^w$ is w-compact in $X^*$.*

*Proof.* By Proposition 2.11(b) we must show that $\mathcal{A}^*$ is *wo*-closed in $\mathcal{B}(Y^*,X^*)$ if and only if $\mathcal{A}$ is *wao*-closed. But it is easy to show the following:

If $\mathcal{A}^*$ is *wo*-closed in $\mathcal{B}(Y^*,X^*)$, then $\mathcal{A}$ is *wao*-closed.

If $\mathcal{B}^*(X,Y)$ is *wo*-closed and $\mathcal{A}$ is *wao*-closed, then $\mathcal{A}^*$ is *wo*-closed in $\mathcal{B}(Y^*,X^*)$.

From these we complete the proof. □

Now new versions of the Mazur's compactness theorem in $\mathcal{B}(X,Y)$ are established.

**Theorem 2.13.** *Suppose that $\mathcal{A}$ is a set in $\mathcal{B}(X,Y)$.*

(a) *If $\mathcal{A}$ is wo(swo)-compact, then $\overline{co}^{wo}(\mathcal{A})=\overline{co}^{swo}(\mathcal{A})$ is wo(swo)-compact.*

(b) *If $\mathcal{A}$ is wao(swao)-compact, then $(\overline{co}^{wao}(\mathcal{A}))^* = (\overline{co}^{swao}(\mathcal{A}))^*$ is relatively wo(swo)-compact in $\mathcal{B}(Y^*,X^*)$.*

*Proof.* (a) From Proposition 2.11(a) it is enough to show that $\overline{co}^{wo}(\mathcal{A})x$ is relatively $w$-compact in $Y$ for each $x \in X$. First it is easy to check that for any set $\mathcal{A}$ in $\mathcal{B}(X,Y)$,

$$\overline{\mathcal{A}}^{wo} x \subset \overline{\mathcal{A}x}^{w} \tag{2.3}$$

for each $x \in X$. Now let $x \in X$. Then we have

$$\overline{co}^{wo}(\mathcal{A})x \subset \overline{co(\mathcal{A})x}^{w} = \overline{co}^{w}(\mathcal{A}x).$$

By Proposition 2.11(a) $\mathcal{A}x$ is $w$-compact in $Y$. Also $\overline{co}^{w}\big(\mathcal{A}x\big)$ is $w$-compact in $Y$ by Krein-Šmulian's weak compact theorem. Hence $\overline{co}^{wo}(\mathcal{A})x$ is relatively $w$-compact in $Y$.

(b) For any set $\mathcal{A}$ in $\mathcal{B}(X,Y)$, it is easy to check the following:

$$\big(\overline{\mathcal{A}}^{wao}\big)^* \subset \overline{\mathcal{A}^*}^{wo}. \tag{2.4}$$

Now if $\mathcal{A}$ is *wao*-compact, then by (2.2) $\mathcal{A}^*$ is *wo*-compact. By (a) $\overline{co}^{wo}(\mathcal{A}^*)$ is *wo*-compact in $\mathcal{B}(Y^*,X^*)$. From (2.4) it follows that

$$(\overline{co}^{wao}(\mathcal{A}))^* \subset \overline{co(\mathcal{A})^*}^{wo} = \overline{co}^{wo}(\mathcal{A}^*).$$

Hence $(\overline{co}^{wao}(\mathcal{A}))^*$ is relatively *wo*-compact. □

**Corollary 2.14.** *Suppose that $\mathcal{A}$ is a set in $\mathcal{B}(X,Y)$ and that $\mathcal{B}^*(X,Y)$ is wo-closed in $\mathcal{B}(Y^*,X^*)$. If $\mathcal{A}$ is wao(swao)-compact, then $\overline{co}^{wao}(\mathcal{A}) = \overline{co}^{swao}(\mathcal{A})$ is wao(swao)-compact.*

*Proof.* By (2.2) it is enough to show that $(\overline{co}^{wao}(\mathcal{A}))^*$ is *wo*-compact in $\mathcal{B}(Y^*, X^*)$. From Theorem 2.13(b) we should show that $(\overline{co}^{wao}(\mathcal{A}))^*$ is *wo*-closed. Now let a net $(T_\alpha^*)$ in $(\overline{co}^{wao}(\mathcal{A}))^*$ and $T \in \mathcal{B}(Y^*,X^*)$ with $T_\alpha^* \xrightarrow{wo} T$. Since $\mathcal{B}^*(X,Y)$ is *wo*-closed, $T \in \mathcal{B}^*(X,Y)$. So $T$ is an adjoint $S^*$. Now it follows that

$$T_\alpha \xrightarrow{wao} S.$$

Since $(T_\alpha) \subset \overline{co}^{wao}(\mathcal{A})$ and $\overline{co}^{wao}(\mathcal{A})$ is *wao*-closed, $S \in \overline{co}^{wao}(\mathcal{A})$ and so $T = S^* \in (\overline{co}^{wao}(\mathcal{A}))^*$. Hence $(\overline{co}^{wao}(\mathcal{A}))^*$ is *wo*-closed. □

In [8], the following proposition was shown for $\mathcal{B}(X)$.

**Proposition 2.15.** (a) *Every wo(wao, swo, swao)-limit point compact set in $\mathcal{B}(X,Y)$ is bounded.*

(b) *Every weak* o(wo, wao, weak*, swo, swao)-limit point compact set in $\mathcal{B}(X, Y^*)$ is bounded.*

Since the proof of Proposition 2.15 is the same as the proof in [8], we omit it.

Note that the sequential compactness implies the limit point compactness in any topological space. So by Proposition 2.3(b) and Proposition 2.15 we have the following.

*Remark* 2.16. (a) If $\mathcal{A}$ is a set in $\mathcal{B}(X,Y)$, then $\mathcal{A}$ is *swao*-sequentially (respectively, limit point) compact if and only if $\mathcal{A}$ is *wao*-sequentially (respectively, limit point) compact, and $\mathcal{A}$ is *swo*-sequentially (respectively, limit point) compact if and only if $\mathcal{A}$ is *wo*-sequentially (respectively, limit point) compact. If $\mathcal{A}$ is a set in $\mathcal{B}(X,Y^*)$, then $\mathcal{A}$ is *weak*$^*$-sequentially (respectively, limit point) compact if and only if $\mathcal{A}$ is *weak*$^*$ *o*-sequentially (respectively, limit point) compact.

(b) If $\mathcal{A}$ is a set in $\mathcal{B}(X,Y)$, then it is easy to check that $\mathcal{A}$ is $wao(swao)$-sequentially compact (respectively, limit point) if and only if $\mathcal{A}^*$ is $wo(swo)$-sequentially (respectively, limit point) compact in $\mathcal{B}(Y^*,X^*)$.

We now have the following theorem. Some parts of Theorems 2.17 and 2.19 are well known (cf. [4, Exercises VI.9.4 and VI.9.6]).

**Theorem 2.17.** *Suppose that $\mathcal{A}$ is a set in $\mathcal{B}(X,Y)$.*

(a) *If $\mathcal{A}$ is $wo(swo)$-sequentially compact, then $\overline{\mathcal{A}}^{wo}$ is $wo(swo)$-compact.*

(b) *If $\mathcal{A}$ is $wao(swao)$-sequentially compact, then $\big(\overline{\mathcal{A}}^{wao}\big)^*$ is relatively $wo$ $(swo)$-compact in $\mathcal{B}(Y^*,X^*)$.*

*Proof.* (a) By Proposition 2.11(a), it is enough to show that $\overline{\mathcal{A}}^{wo}x$ is relatively $w$-compact in $Y$ for each $x\in X$. Recall (2.3). Then

$$\overline{\mathcal{A}}^{wo}x\subset\overline{\mathcal{A}x}^{w}$$

for each $x\in X$. Therefore we should show that $\overline{\mathcal{A}x}^{w}$ is $w$-compact in $Y$ for each $x\in X$. To show this, by the virtue of the Eberlein-Šmulian theorem we only need to show that $\mathcal{A}x$ is $w$-sequentially compact in $Y$ for each $x\in X$. Now let $x\in X$ and $(T_nx)$ be a sequence in $\mathcal{A}x$. Since $\mathcal{A}$ is $wo$-sequentially compact, there is a subsequence $(T_{n_k})$ of $(T_n)$ and $T\in\mathcal{A}$ such that

$$T_{n_k}\xrightarrow{wo}T.$$

In particular $y^*T_{n_k}x\longrightarrow y^*Tx$ for each $y^*\in Y^*$. Thus $T_{n_k}x\xrightarrow{w}Tx$. Hence $\mathcal{A}x$ is $w$-sequentially compact.

(b) If $\mathcal{A}$ is $wao$-sequentially compact, then by Remark 2.16(b), $\mathcal{A}^*$ is $wo$-sequentially compact in $\mathcal{B}(Y^*,X^*)$. By (a) $\overline{\mathcal{A}^*}^{wo}$ is $wo$-compact. Hence $\big(\overline{\mathcal{A}}^{wao}\big)^*$ is relatively $wo$-compact by (2.4). □

The proof of the following corollary is essentially the same as the proof of Corollary 2.14.

**Corollary 2.18.** *Suppose that $\mathcal{A}$ is a set in $\mathcal{B}(X,Y)$ and that $\mathcal{B}^*(X,Y)$ is wo-closed in $\mathcal{B}(Y^*,X^*)$. If $\mathcal{A}$ is wao(swao)-sequentially compact, then $\overline{\mathcal{A}}^{wao}$ is wao(swao)-compact.*

The following results give various characterizations of reflexivity.

**Theorem 2.19.** *The following are equivalent.*

(a) *$Y$ is reflexive.*
(b) *$\mathcal{B}(X,Y,1)$ is wo(swo)-compact.*
(c) *$\overline{\mathcal{F}(X,Y,1)}^{wo}$ is wo(swo)-compact.*

*Proof.* We show (a)$\Longrightarrow$(b)$\Longrightarrow$(c)$\Longrightarrow$(a). But (b)$\Longrightarrow$(c) is clear. So we are left with (a)$\Longrightarrow$(b) and (c)$\Longrightarrow$(a). To show these, we will use Proposition 2.11(a).

(a)$\Longrightarrow$(b) Suppose that $Y$ is reflexive. Let $(T_\alpha)$ be a net in $\mathcal{B}(X,Y,1)$ and $T\in\mathcal{B}(X,Y)$ with $T_\alpha \xrightarrow{wo} T$. Then $T_\alpha x \xrightarrow{w} Tx$ in $Y$ for each $x\in X$. Since $\mathcal{B}(X,Y,1)x$ is convex in $Y$ for each $x\in X$,

$$Tx\in\overline{\mathcal{B}(X,Y,1)x}^{w}=\overline{\mathcal{B}(X,Y,1)x}$$

for each $x\in X$. From this, it follows that $T\in\mathcal{B}(X,Y,1)$. Hence $\mathcal{B}(X,Y,1)$ is $wo$-closed. Also, for each $x\in X$, $\overline{\mathcal{B}(X,Y,1)x}^{w}$ is $w$-compact in $Y$ because $Y$ is reflexive. Hence $\mathcal{B}(X,Y,1)$ is $wo$-compact by Proposition 2.11(a).

(c)$\Longrightarrow$(a) Suppose that $\overline{\mathcal{F}(X,Y,1)}^{wo}$ is $wo$-compact. Let $x_0\in X$ with $\|x_0\|=1$. Then $\overline{\mathcal{F}(X,Y,1)}^{wo}x_0$ is $w$-compact in $Y$. Now choose $x_0^*\in X^*$ with $\|x_0^*\|=1$ such that $x_0^*x_0=1$. Consider $T_y=x_0^*(\cdot)y\in\mathcal{F}(X,Y,1)$ for each $y\in B_Y$, where $B_Y$ is the unit ball in $Y$. Then for each $y\in B_Y$

$$y=x_0^*(x_0)y=T_yx_0\in\mathcal{F}(X,Y,1)x_0.$$

It follows that $B_Y\subset\mathcal{F}(X,Y,1)x_0$. Consequently $B_Y$ is $w$-compact in $Y$. Hence $Y$ is reflexive. □

**Theorem 2.20.** *The following are equivalent.*

(a) *$X$ is reflexive.*
(b) *$\overline{\mathcal{B}^*(X,Y,1)}^{wo}$ is wo(swo)-compact in $\mathcal{B}(Y^*,X^*)$.*
(c) *$\overline{\mathcal{F}^*(X,Y,1)}^{wo}$ is wo(swo)-compact in $\mathcal{B}(Y^*,X^*)$.*

*Proof.* To show that (a), (b), and (c) are equivalent, it is enough to show that (c) implies (a) because by Theorem 2.19 (a) implies (b) and clearly (b) implies (c). Now assume (c) and let $y_0\in Y$ with $\|y_0\|=1$. Choose $y_0^*\in Y^*$ with $\|y_0^*\|=1$ such that $y_0^*y_0=1$. Since $\overline{\mathcal{F}^*(X,Y,1)}^{wo}$ is $wo$-compact, $\overline{\mathcal{F}^*(X,Y,1)}^{wo}y_0^*$ is $w$-compact in $X^*$. Therefore to show that $X$ is reflexive, it is enough to show that $B_{X^*}\subset\overline{\mathcal{F}^*(X,Y,1)}^{wo}y_0^*$. Let $x^*\in B_{X^*}$ and consider an operator $T_{x^*}=(\cdot)y_0x^*$. Then $T_{x^*}\in\mathcal{F}^*(X,Y,1)$ and we have

$$x^*=y_0^*(y_0)x^*=T_{x^*}y_0^*\in\mathcal{F}^*(X,Y,1)y_0^*.$$

This completes the proof of reflexivity of $X$. □

**Proposition 2.21.** *The following are equivalent.*

(a) $Y$ *is reflexive.*
(b) $\mathcal{F}(Y^*, X^*) = \mathcal{F}^*(X, Y)$.
(c) $\mathcal{F}(Y^*, X^*, 1) = \mathcal{F}^*(X, Y, 1)$.

*Proof.* Note that if $Y$ is reflexive, then for every operator in $\mathcal{B}(Y^*, X^*)$ is an adjoint operator. Since (a)$\Longrightarrow$(b) and (b)$\Longrightarrow$(c) are clear, we only show (c)$\Longrightarrow$(a). Suppose that $Y$ is not reflexive. Then there is a $y_0^{**} \in Y^{**}$ with $\|y_0^{**}\| = 1$ such that $y_0^{**} \in Y^{**} \setminus Q_Y(Y)$, where $Q_Y$ is the natural map from $Y$ into $Y^{**}$. Choose $x_0 \in X$ and $x_0^* \in X^*$ so that $1 = \|x_0\| = \|x_0^*\| = x_0^* x_0$. Now consider

$$T_0 = y_0^{**}(\cdot)x_0^* \in \mathcal{F}(Y^*, X^*, 1).$$

If $T_0 \in \mathcal{F}^*(X, Y, 1)$, then $Q_X(x_0)T_0 \in Q_Y(Y)$. But $Q_X(x_0)T_0 = y_0^{**} \in Y^{**} \setminus Q_Y(Y)$. This is a contradiction. Hence $\mathcal{F}(Y^*, X^*, 1) \not\subset \mathcal{F}^*(X, Y, 1)$. $\square$

In Section 3, we introduce the $\tau$ topology which is stronger than the *wo* topology, and the following lemma comes from [11, Proposition 3.1].

**Lemma 2.22.** $\mathcal{F}(Y^*, X^*, \lambda) \subset \overline{\mathcal{F}^*(X, Y, \lambda)}^{\tau}$ *for each* $\lambda > 0$.

**Proposition 2.23.** *The following are equivalent.*

(a) $Y$ *is reflexive.*
(b) $\mathcal{B}^*(X, Y)$ *is wo-closed in* $\mathcal{B}(Y^*, X^*)$.
(c) $\mathcal{B}^*(X, Y, 1)$ *is wo-closed in* $\mathcal{B}(Y^*, X^*)$.

*Proof.* We show (a)$\Longrightarrow$(b)$\Longrightarrow$(c)$\Longrightarrow$(a).

(a)$\Longrightarrow$(b) From $\overline{\mathcal{B}^*(X, Y)}^{wo} \subset \mathcal{B}(Y^*, X^*) = \mathcal{B}^*(X, Y)$.

(b)$\Longrightarrow$(c) Let $T \in \overline{\mathcal{B}^*(X, Y, 1)}^{wo} \subset \overline{\mathcal{B}^*(X, Y)}^{wo} = \mathcal{B}^*(X, Y)$. Then $T$ is an adjoint $S^*$. Then there is a net $(T_\alpha)$ in $\mathcal{B}(X, Y, 1)$ such that

$$T_\alpha \xrightarrow{wao} S.$$

As in the proof of Theorem 2.19, we have $S \in \mathcal{B}(X, Y, 1)$. Thus $T = S^* \in \mathcal{B}^*(X, Y, 1)$. Hence $\mathcal{B}^*(X, Y, 1)$ is *wo*-closed.

(c)$\Longrightarrow$(a) By Lemma 2.22 we have

$$\mathcal{F}(Y^*, X^*, 1) \subset \overline{\mathcal{F}^*(X, Y, 1)}^{\tau} \subset \overline{\mathcal{B}^*(X, Y, 1)}^{wo} = \mathcal{B}^*(X, Y, 1).$$

It follows that $\mathcal{F}(Y^*, X^*, 1) = \mathcal{F}^*(X, Y, 1)$. Hence $Y$ is reflexive by Proposition 2.21. $\square$

From Theorem 2.20 and Proposition 2.23 we have the following.

**Corollary 2.24.** $X$ *and* $Y$ *are reflexive if and only if* $\mathcal{B}^*(X, Y, 1)$ *is* $wo(swo)$*-compact in* $\mathcal{B}(Y^*, X^*)$.

*Consequently,* $X$ *and* $Y$ *are reflexive if and only if* $\mathcal{B}(X, Y, 1)$ *is* $wao(swao)$*-compact.*

The final theorem of this section is a new version of the Day's lemma in $\mathcal{B}(X,Y)$. The proof of the theorem is the same as the proof of [8, Theorem 1.6].

**Theorem 2.25.** *If $\mathcal{A}$ is a relatively weak-compact set in $\mathcal{B}(X,Y)$ and $T \in \overline{\mathcal{A}}^{wo}$, then there is a sequence $(T_n)$ in $\mathcal{A}$ such that $T_n \xrightarrow{weak} T$.*

## 3. Topologies generated by subbases on $\mathcal{B}(X,Y)$

Suppose that $\mathcal{S}$ is a collection of sets in $\mathcal{B}(X,Y)$. Then the topology *generated* by $\mathcal{S}$ is the smallest topology on $\mathcal{B}(X,Y)$ containing $\mathcal{S}$. We call $\mathcal{S}$ a *subbasis* on $\mathcal{B}(X,Y)$. In this section we study topologies generated by subbases on $\mathcal{B}(X,Y)$. Now we formally introduce some of such topologies.

**Definition 3.1.** For $x \in X$, $\epsilon > 0$, and $T \in \mathcal{B}(X,Y)$, we put

$$N(T; x, \epsilon) = \{R \in \mathcal{B}(X,Y) : \|Rx - Tx\| < \epsilon\}.$$

Let $\mathcal{S}_1$ be the collection of all such $N(T; x, \epsilon)$'s.

For $(x_n) \subset X$ satisfying $\sum_n \|x_n\| < \infty$, $\epsilon > 0$, and $T \in \mathcal{B}(X,Y)$, we put

$$N(T; (x_n), \epsilon) = \{R \in \mathcal{B}(X,Y) : \sum_n \|Rx_n - Tx_n\| < \epsilon\}.$$

Let $\mathcal{S}_2$ be the collection of all such $N(T; (x_n), \epsilon)$'s.

For compact $K \subset X$, $\epsilon > 0$, and $T \in \mathcal{B}(X,Y)$, we put

$$N(T; K, \epsilon) = \{R \in \mathcal{B}(X,Y) : \sup_{x \in K} \|Rx - Tx\| < \epsilon\}.$$

Let $\mathcal{S}_3$ be the collection of all such $N(T; K, \epsilon)$'s.

Then the *strong operator topology* (in short, *sto*) on $\mathcal{B}(X,Y)$ is the topology generated by $\mathcal{S}_1$, the *summable strong operator topology* (in short, *ssto*) on $\mathcal{B}(X,Y)$ is the topology generated by $\mathcal{S}_2$, and the *$\tau$-topology* (in short, $\tau$) on $\mathcal{B}(X,Y)$ is the topology generated by $\mathcal{S}_3$.

From Definition 3.1 we see the following: for a net $(T_\alpha)$ and $T$ in $\mathcal{B}(X,Y)$

$$T_\alpha \xrightarrow{sto} T \text{ if and only if } \|T_\alpha x - Tx\| \longrightarrow 0$$

for each $x \in X$;

$$T_\alpha \xrightarrow{ssto} T \text{ if and only if } \sum_n \|T_\alpha x_n - Tx_n\| \longrightarrow 0$$

for each $(x_n) \subset X$ satisfying $\sum_n \|x_n\| < \infty$;

$$T_\alpha \xrightarrow{\tau} T \text{ if and only if } \sup_{x \in K} \|T_\alpha x - Tx\| \longrightarrow 0$$

for each compact $K \subset X$.

It is easy to check that above topologies are locally convex vector topologies and have $T_0$-separation axiom. Notice that every vector topology having $T_0$-separation axiom is completely regular. Hence *sto*, *ssto*, and $\tau$ are completely regular locally convex vector topologies.

We now have simple relations between them.

**Proposition 3.2.** (a) $\tau \geq ssto \geq sto$.
(b) *On each bounded set in* $\mathcal{B}(X,Y)$ $\tau = ssto = sto$.

*Proof.* (a) Since $ssto \geq sto$ is clear, we only show that $\tau \geq ssto$. Suppose $(T_\alpha)$ is a net in $\mathcal{B}(X,Y)$ and $T_\alpha \xrightarrow{\tau} 0$. Then we must show that $T_\alpha \xrightarrow{ssto} 0$. For this let $(x_n) \subset X$ satisfy $\sum_n \|x_n\| < \infty$. Then one can find a sequence $(\beta_n)$ such that $0 < \beta_n \uparrow \infty$ and $\sum_n \beta_n \|x_n\| = 1$. Put $K = \{x_n/(\beta_n\|x_n\|) : n \geq 1\} \cup \{0\}$. Then $K$ is compact. Hence we have

$$\sum_n \|T_\alpha x_n\| \leq \sup_{x \in K} \|T_\alpha x\|,$$

which proves $\sum_n \|T_\alpha x_n\| \longrightarrow 0$.

(b) Let $(T_\alpha)$ be a net and $T$ in a bounded set in $\mathcal{B}(X,Y)$. Then a simple calculation shows that

$$T_\alpha \xrightarrow{sto} T \quad \text{implies} \quad T_\alpha \xrightarrow{\tau} T.$$

Hence we have the conclusion by (a). $\square$

We now summarize simple relations between all of our topologies in the following figures.

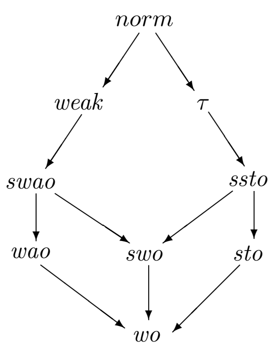


Figure 1 : On $\mathcal{B}(X,Y)$

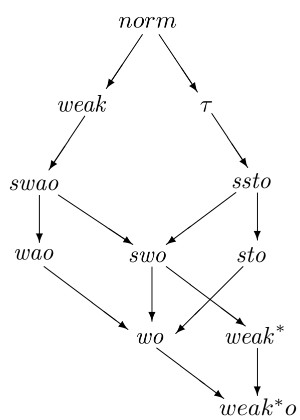


Figure 2 : On $\mathcal{B}(X,Y^*)$

$$\begin{tikzcd}
 & norm \arrow[dl] \arrow[dr] & \\
weak \arrow[d] & & \tau = ssto = sto \arrow[ddl] \\
swao = wao \arrow[dr] & & \\
 & swo = wo &
\end{tikzcd}$$

Figure 3 : On bounded sets in $\mathcal{B}(X,Y)$

$$\begin{tikzcd}
 & norm \arrow[dl] \arrow[dr] & \\
weak \arrow[d] & & \tau = ssto = sto \arrow[ddl] \\
swao = wao \arrow[dr] & & \\
 & swo = wo \arrow[d] & \\
 & weak^* = weak^*o &
\end{tikzcd}$$

Figure 4 : On bounded sets in $\mathcal{B}(X,Y^*)$

It is interesting to observe that the above diagrams are sharp for infinite-dimensional Banach spaces $X$ and $Y$. We introduce nontrivial examples.

**Example 3.3.** (1) (The $\tau$ topology is not stronger than the *wao* topology in general).

Consider the sequence $(T_n)$ in $\mathcal{B}(l_1)$ given by

$$T_n(\alpha_i) = \frac{1}{n}(\alpha_j + \cdots + \alpha_{j+n-1})_j.$$

Observe that

$$T_n^*(\beta_j) = \frac{1}{n}(\beta_1, \beta_1 + \beta_2, \ldots, \beta_1 + \cdots + \beta_n, \beta_2 + \cdots + \beta_{n+1}, \ldots).$$

Since $\|T_n\| \leq 1$ and $T_n \xrightarrow{sto} 0$, we have $T_n \xrightarrow{\tau} 0$. But if $\lambda$ is a Banach limit on $l_\infty$, then

$$\lambda T_n^*(1,1,1,\ldots) = 1$$

for all $n$, which proves that $T_n \overset{wao}{\not\longrightarrow} 0$.

(2) (The *weak* topology is not stronger than the *sto* topology in general). Let $1 < p < q < \infty$. Then by the Pitt's theorem $\mathcal{B}(l_q, l_p) = \mathcal{K}(l_q, l_p)$. Then $\mathcal{B}(l_q, l_p)$ is reflexive; see [13, Theorem 4.19]. Thus $\mathcal{B}(l_q, l_p, 1)$ is *weak*-compact. If the *weak* topology were stronger than the *sto* topology on $\mathcal{B}(l_q, l_p)$, we would have that $\mathcal{B}(l_q, l_p, 1)$ is *sto*-compact, which, in view of Theorem 3.15, is absurd because $l_p$ is obviously of infinite dimension.

(3) (The *sto* and *wao* topology is not stronger than the *weak*$^*$ topology in general). Let $W$ be the Willis space [14]. Then $W$ is a separable and reflexive Banach space satisfying

$$\mathcal{K}(W^*) \not\subset \overline{\mathcal{F}(W^*)}^{\tau} = \overline{\mathcal{F}(W^*)}^{swo} = \overline{\mathcal{F}(W^*)}^{weak^*}$$

and

$$I_{W^*} \notin \overline{\mathcal{F}(W^*)}^{\tau} = \overline{\mathcal{F}(W^*)}^{weak^*}$$

(see [3, Example 2.3], Proposition 2.3(c), and Proposition 3.6(a)), where $I_{W^*}$ is the identity in $\mathcal{B}(W^*)$. If the *sto* topology were stronger than the *weak*$^*$ topology on $\mathcal{B}(W^*)$, then by Remark 3.11

$$\mathcal{K}(W^*) \subset \overline{\mathcal{F}(W^*)}^{sto} \subset \overline{\mathcal{F}(W^*)}^{weak^*}.$$

This is a contradiction. If the *wao* topology were stronger than the *weak*$^*$ topology on $\mathcal{B}(W^*)$, then by Remark 3.11 and Proposition 2.3(c)

$$I_{W^*} \in \overline{\mathcal{F}(W^*)}^{sto} \subset \overline{\mathcal{F}(W^*)}^{wo} = \overline{\mathcal{F}(W^*)}^{wao} \subset \overline{\mathcal{F}(W^*)}^{weak^*}.$$

This is a contradiction.

In order to study more about our topologies we need the following lemmas. Grothendieck [6] showed the first equation of (a) of Lemma 3.4 and (b) is from [4, Theorem VI.1.4]. And the proof of the second equation of (a) is found to follow more or less the same lines in the proof of (b) if one is willing to use the fact that $c_0(Y)^* = l_1(Y^*)$; also, a simple proof of the equation follows from $\tau \geq ssto \geq swo$ in Figure 1.

**Lemma 3.4.** (a) $(\mathcal{B}(X,Y),\tau)^* = (\mathcal{B}(X,Y), swo)^* = (\mathcal{B}(X,Y), ssto)^*$.
(b) $(\mathcal{B}(X,Y), sto)^* = (\mathcal{B}(X,Y), wo)^*$.

**Lemma 3.5** ([12, Corollary 2.2.29])**.** *Suppose that a vector space $\mathcal{V}$ has two locally convex topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ such that the dual spaces of $\mathcal{V}$ under the two topologies are the same. If $\mathcal{C}$ is a convex set in $\mathcal{V}$, then $\overline{\mathcal{C}}^{\mathcal{T}_1} = \overline{\mathcal{C}}^{\mathcal{T}_2}$.*

See Figure 3. Then by Lemmas 3.4 and 3.5 we have the following proposition.

**Proposition 3.6.** (a) *If $\mathcal{C}$ is a convex set in $\mathcal{B}(X,Y)$, then $\overline{\mathcal{C}}^{\tau} = \overline{\mathcal{C}}^{swo} = \overline{\mathcal{C}}^{ssto}$ and $\overline{\mathcal{C}}^{sto} = \overline{\mathcal{C}}^{wo}$.*
(b) *If $\mathcal{C}$ is a bounded and convex set in $\mathcal{B}(X,Y)$, then $\overline{\mathcal{C}}^{\tau} = \overline{\mathcal{C}}^{swo} = \overline{\mathcal{C}}^{sto} = \overline{\mathcal{C}}^{ssto} = \overline{\mathcal{C}}^{wo}$.*

See Figure 1. Then the operator norm topology is the strongest topology among the vector topologies we consider and *wo* is the weakest. In Proposition 2.5 we have shown that *wo*-boundedness implies the operator norm boundedness. Hence we have the following proposition.

**Proposition 3.7.** *Let $\mathcal{A}$ be a set in $\mathcal{B}(X,Y)$. Then the following are equivalent.*
(a) *$\mathcal{A}$ is bounded.*
(b) *$\mathcal{A}$ is weak-bounded.*
(c) *$\mathcal{A}$ is swao-bounded.*
(d) *$\mathcal{A}$ is ssto-bounded.*
(e) *$\mathcal{A}$ is swo-bounded.*
(f) *$\mathcal{A}$ is $\tau$-bounded.*
(g) *$\mathcal{A}$ is wao-bounded.*
(h) *$\mathcal{A}$ is sto-bounded.*
(i) *$\mathcal{A}$ is wo-bounded.*

Similarly, by Proposition 2.6, boundedness conditions of all our topologies on $\mathcal{B}(X,Y^*)$ are equivalent.

Now we consider the metrizability on $\mathcal{B}(X,Y)$ with respect to $\tau$, *ssto*, and *sto*.

**Theorem 3.8.** *$\tau$ (respectively, ssto, sto) on $\mathcal{B}(X,Y)$ is metrizable if and only if $X$ is finite-dimensional.*

*Proof.* Suppose that $X$ is infinite-dimensional. We will show that every $\tau$, *ssto*, and *sto*-open set is unbounded. Then the proof, that $\tau$ (respectively, *ssto*, *sto*) is not metrizable, follows the same lines as in Theorem 2.7. Now let $\mathcal{U} \in \tau$ and may assume $0 \in \mathcal{U}$. Suppose, for a contradiction, that $\mathcal{U}$ is bounded, say $\|T\| \leq t$ for all $T \in \mathcal{U}$. Since $0 \in \mathcal{U}$ and $\mathcal{U}$ is $\tau$-open, there is a compact $K$ and $\epsilon > 0$ such that

$$(\dagger) \qquad T \in \mathcal{U} \quad \text{whenever} \quad \sup_{x\in K} \|Tx\| \leq \epsilon.$$

Here we may assume that $K$ is balanced and convex as well; this is due to the Mazur's compactness theorem. Now we claim that $\epsilon B_X \subset tK$, where $B_X$ is the unit ball in $X$, which implies $B_X$ is compact, hence $X$ is finite-dimensional, a contradiction. Indeed, if $\epsilon B_X \not\subset tK$, then there is a $x_0 \in B_X$ such that $\epsilon x_0 \notin tK$. By the geometric version of the Hahn -Banach theorem there is a $x_0^* \in X^*$ such that

$$\sup_{x\in tK} \mathrm{Re} x_0^* x \leq \epsilon t < \mathrm{Re} x_0^*(\epsilon x_0),$$

which implies, in view of balancedness of $K$, that $\sup_{x\in K} |x_0^* x| \leq \epsilon$ and $|x_0^* x_0| > t$. Choose $y_0 \in Y$ with $\|y_0\| = 1$ and define $T \in \mathcal{B}(X,Y)$ by

$$Tx = (x_0^* x) y_0.$$

This $T$ satisfies

$$\sup_{x\in K} \|Tx\| = \sup_{x\in K} |x_0^* x| \leq \epsilon$$

but

$$\|T\| = \|x_0^*\| \geq |x_0^* x_0| > t,$$

which contradicts ($\dagger$). We have shown that every $\tau$-open set is unbounded. By Proposition 3.2(a) every *ssto*-open or *sto*-open set is also unbounded.

To show the other part, assume that $X$ is finite-dimensional. Then it is enough to show that all topologies $\tau$, *ssto*, and *sto* coincide with the operator norm topology on $\mathcal{B}(X,Y)$. For this we show that *sto* coincides with the operator norm topology. Now let $(T_\alpha)$ be a net in $\mathcal{B}(X,Y)$ such that $T_\alpha \xrightarrow{sto} 0$, and $\epsilon > 0$. Since $X$ is finite-dimensional, there is a finite sequence $(x_1, \ldots, x_n)$ in $X$ such that

$$B_X \subset co(\{x_1, \ldots, x_n\}).$$

Now we can choose an $\alpha_0$ such that $\alpha \succeq \alpha_0$ implies

$$\|T_\alpha x_i\| \leq \epsilon$$

for all $i = 1, \ldots, n$. Thus clearly we have that $\alpha \succeq \alpha_0$ implies $\|T_\alpha\| \le \epsilon$. Hence $\|T_\alpha\| \longrightarrow 0$ which completes the proof. □

Now by the virtue of boundedness we have following Theorem 3.9, of which the proof is found in [8].

**Theorem 3.9.** (a) *Suppose that $X$ is separable. Then $\tau(ssto, sto)$ on each bounded set in $\mathcal{B}(X,Y)$ is metrizable.*

(b) *Suppose that $X$ and $Y$ are separable. Then $\tau(ssto, sto)$ on each bounded set in $\mathcal{B}(X,Y)$ has a countable basis and is metrizable. Also $\mathcal{B}(X,Y)$ is $\tau(ssto, sto)$-separable.*

For *sto*, *ssto*, and $\tau$-completeness of $\mathcal{B}(X,Y)$, we have the following theorem.

**Theorem 3.10.** (a) *$\mathcal{B}(X,Y)$ is complete with respect to both $\tau$ and ssto.*

(b) *$\mathcal{B}(X,Y)$ is sto-complete if and only if $X$ is finite-dimensional.*

*Proof.* (a) We only show that $\mathcal{B}(X,Y)$ is $\tau$-complete. The proof of *ssto*-completeness is similar. Suppose that $(T_\alpha)$ is a $\tau$-Cauchy net in $\mathcal{B}(X,Y)$. Then for each $x \in X$, $(T_\alpha x)$ converges in $Y$. Define a linear map $T : X \longrightarrow Y$ by

$$Tx = \lim_\alpha T_\alpha x$$

for each $x \in X$. We first claim $T \in \mathcal{B}(X,Y)$. Indeed, if $T$ were unbounded, then we would have a sequence $(x_n) \subset X$ such that $\|x_n\| < 1/n^2$ and $\|Tx_n\| > n$ for all $n$. Since $(T_\alpha)$ is also *ssto*-Cauchy by $\tau \ge ssto$, there is a $\alpha_0$ such that $\alpha, \beta \succeq \alpha_0$ implies

$$\sum_n \|(T_\alpha - T_\beta)x_n\| \le 1.$$

Since $T_\alpha x \longrightarrow Tx$ for each $x \in X$, we have

$$\sum_n \|(T_{\alpha_0} - T)x_n\| \le 1.$$

It follows from this that for all $n$

$$\|T_{\alpha_0}\| \ge \|T_{\alpha_0} x_n\| \ge \|Tx_n\| - \|(T_{\alpha_0} - T)x_n\| > n - 1,$$

which contradicts boundedness of $T_{\alpha_0}$.

Now to show $T_\alpha \xrightarrow{\tau} T$, let $K$ be a compact set in $X$ and $\epsilon > 0$. Since $(T_\alpha)$ is $\tau$-Cauchy, there is a $\gamma$ such that $\alpha, \beta \succeq \gamma$ implies

$$\sup_{x \in K} \|(T_\alpha - T_\beta)x\| \le \epsilon.$$

Since $T_\alpha x \longrightarrow Tx$ for each $x \in X$, it follows that $\alpha \succeq \gamma$ implies

$$\sup_{x \in K} \|(T_\alpha - T)x\| \le \epsilon.$$

Hence $T_\alpha \xrightarrow{\tau} T$ which completes the proof.

(b) If $X$ is finite-dimensional, as in the proof of (a) we can show that $\mathcal{B}(X,Y)$ is *sto*-complete. Now suppose that $X$ is infinite-dimensional. Then there is an unbounded linear operator $T$ from $X$ into $Y$. Now consider a directed set $I = \{F \subset X : F \text{ is finite}\}$ with a relation $\succeq$, where for $F$ and $G \in I$, $F \succeq G$ if and only if $F \supset G$. Since the restriction $T|_{\mathrm{span}(F)}$ to span$(F)$ of $T$ is bounded finite rank linear operator from span$(F)$ into $Y$ for each $F \in I$, by an application of the Hahn-Banach extension theorem there is a $T_F \in \mathcal{B}(X,Y)$ such that $T_F|_{\mathrm{span}(F)} = T|_{\mathrm{span}(F)}$ and $T_F(X) = T|_{\mathrm{span}(F)}(\mathrm{span}(F))$. Consider the net $(T_F)$ in $\mathcal{B}(X,Y)$. Let $x \in X$ and $\epsilon > 0$. Then $E, G \succeq \{x\}$ implies

$$\|(T_E - T_G)x\| = \|Tx - Tx\| = 0 < \epsilon.$$

It follows that $(T_F)$ is *sto*-Cauchy. Since for each $x \in X$ and $E \succeq \{x\}$ implies

$$T_E x = Tx,$$

the only possible *sto*-limit of $(T_F)$ is $T$ which is an unbounded linear operator. It follows that the *sto* topology on $\mathcal{B}(X,Y)$ is not complete. □

*Remark* 3.11. For every Banach spaces $X$ and $Y$, it is interesting to observe that

$$\mathcal{B}(X,Y) = \overline{\mathcal{F}(X,Y)}^{sto}.$$

Indeed, for every $T \in \mathcal{B}(X,Y)$, as in the proof of Theorem 3.10(b) we obtain a net $(T_\alpha)$ in $\mathcal{F}(X,Y)$ with $T_\alpha \xrightarrow{sto} T$. For dual spaces, by Lemma 4.7 in Section 4, we have the following:

$$\mathcal{B}(Y^*, X^*) = \overline{\mathcal{F}^*(X,Y)}^{sto}.$$

Next, as in Section 2, we are concerned with compactness in $\mathcal{B}(X,Y)$. By Proposition 3.2(b) $\tau$, *ssto*, and *sto*-compactness are equivalent.

First we give a characterization of *sto*-compactness. Some parts of Proposition 3.12, Theorems 3.13 and 3.14 are well known (cf. [4, Exercises VI.9.2, VI.9.3, and VI.9.4]).

**Proposition 3.12.** *Suppose that $\mathcal{A}$ is a set in $\mathcal{B}(X,Y)$. The following are equivalent.*

(a) *$\mathcal{A}$ is sto($\tau$, ssto)-compact.*
(b) *$\mathcal{A}$ is sto-closed and for each $x \in X$, $\mathcal{A}x$ is norm-compact in $Y$.*
(c) *$\mathcal{A}$ is sto-closed and for each $x \in X$, $\overline{\mathcal{A}x}$ is norm-compact in $Y$.*

*Proof.* The proof is similar to the proof of Proposition 2.11(a). Since (b)$\Longrightarrow$(c) is clear, we show that (a)$\Longrightarrow$(b) and (c)$\Longrightarrow$(a).

(a)$\Longrightarrow$(b) Suppose that $\mathcal{A}$ is *sto*-compact. Then as in the proof of Proposition 2.11(a), we see that $\mathcal{A}$ is *sto*-closed and $\mathcal{A}x$ is norm-compact in $Y$ for each $x \in X$.

(c)$\Longrightarrow$(a) Consider the map $\psi : (\mathcal{B}(X,Y), sto) \longrightarrow \prod_{x\in X} Y_x$ defined by

$$\psi(T) = (Tx)_{x\in X},$$

where $Y_x = Y$ for all $x \in X$ and $Y_x$ has the norm-topology. Then as in the proof of Proposition 2.11(a), we see that $\psi : (\mathcal{B}(X,Y), sto) \longrightarrow (\psi(\mathcal{B}(X,Y)), pro)$ is a *sto*-to-*pro* homeomorphism and $\psi(\overline{\mathcal{A}}^{sto}) = \overline{\psi(\mathcal{A})}^{pro}$. Thus we have

$$\psi(\mathcal{A}) = \psi(\overline{\mathcal{A}}^{sto}) = \overline{\psi(\mathcal{A})}^{pro} \subset \overline{\prod_{x\in X} \mathcal{A}x}^{pro} = \prod_{x\in X} \overline{\mathcal{A}x}.$$

By the virtue of the Tychonoff's theorem $\psi(\mathcal{A})$ is *pro*-compact. Since $\psi$ is a homeomorphism, $\mathcal{A}$ is *sto*-compact. □

As in Section 2, new versions of the Mazur's compactness theorem in $\mathcal{B}(X,Y)$ are established.

**Theorem 3.13.** *Suppose that $\mathcal{A}$ is a set in $\mathcal{B}(X,Y)$. If $\mathcal{A}$ is sto(ssto, $\tau$)-compact, then $\overline{co}^{sto}(\mathcal{A}) = \overline{co}^{ssto}(\mathcal{A}) = \overline{co}^{\tau}(\mathcal{A})$ is sto(ssto, $\tau$)-compact.*

*Proof.* From Proposition 3.12 it is enough to show that $\overline{co}^{sto}(\mathcal{A})x$ is relatively norm-compact in $Y$ for each $x \in X$. First it is easy to check that for any set $\mathcal{A}$ in $\mathcal{B}(X,Y)$,

$$\overline{\mathcal{A}}^{sto} x \subset \overline{\mathcal{A}x} \tag{3.1}$$

for each $x \in X$. Let $x \in X$. Then we have

$$\overline{co}^{sto}(\mathcal{A})x \subset \overline{co(\mathcal{A})x} = \overline{co}(\mathcal{A}x).$$

By Proposition 3.12 $\mathcal{A}x$ is norm-compact in $Y$. Also $\overline{co}\big(\mathcal{A}x\big)$ is norm-compact in $Y$ by the Mazur's compactness theorem. Hence $\overline{co}^{sto}(\mathcal{A})x$ is relatively norm-compact in $Y$. □

The *wo* (respectively, *weak**o) topology is the weakest topology among all our topologies on $\mathcal{B}(X,Y)$ (respectively, $\mathcal{B}(X,Y^*)$). Since every *wo* (respectively, *weak**o)-limit point compact set in $\mathcal{B}(X,Y)$ (respectively, $\mathcal{B}(X,Y^*)$) is bounded (Proposition 2.15), the limit point compactness of all our topologies on $\mathcal{B}(X,Y)$ and $\mathcal{B}(X,Y^*)$ implies boundedness. Since the sequential compactness implies the limit point compactness, by Proposition 3.2(b) *sto*, *ssto*, and $\tau$-sequential compactness (or limit point compactness) are equivalent.

Now we have the following.

**Theorem 3.14.** *Suppose that $\mathcal{A}$ is a set in $\mathcal{B}(X,Y)$. If $\mathcal{A}$ is sto(ssto, $\tau$)-sequentially compact, then $\overline{\mathcal{A}}^{sto}$ is sto(ssto, $\tau$)-compact.*

*Proof.* By Proposition 3.12, it is enough to show that $\overline{\mathcal{A}}^{sto}x$ is relatively norm-compact in $Y$ for each $x \in X$. Recall (3.1). Then

$$\overline{\mathcal{A}}^{sto} x \subset \overline{\mathcal{A}x}$$

for each $x \in X$. Therefore we must show that $\overline{\mathcal{A}x}$ is norm-compact in $Y$ for each $x \in X$. Since $\mathcal{A}$ is *sto*-sequentially compact, it is easy to show that $\mathcal{A}x$ is norm-sequentially compact in $Y$ for each $x \in X$. Hence $\overline{\mathcal{A}x} = \mathcal{A}x$ is norm-compact in $Y$ for each $x \in X$. □

The following results give characterizations of finite-dimensional spaces.

**Theorem 3.15.** *The following are equivalent.*

(a) $Y$ *is finite-dimensional.*
(b) $\mathcal{B}(X,Y,1)$ *is sto(ssto,* $\tau$*)-compact.*
(c) $\mathcal{F}(X,Y,1)$ *is sto(ssto,* $\tau$*)-compact.*

*Proof.* Recall Proposition 3.12 and the proof of Theorem 2.19. We show that (a)$\Longleftrightarrow$(b) and (a)$\Longleftrightarrow$(c).

(a)$\Longrightarrow$(b) Suppose that $Y$ is finite-dimensional. As in the proof of Theorem 2.19 we have $\mathcal{B}(X,Y,1)$ is *sto*-closed. Also for each $x \in X$, $\overline{\mathcal{B}(X,Y,1)x}$ is norm-compact in $Y$ because $Y$ is finite-dimensional. Hence $\mathcal{B}(X,Y,1)$ is *sto*-compact by Proposition 3.12.

(a)$\Longrightarrow$(c) From $\mathcal{B}(X,Y,1) = \mathcal{F}(X,Y,1)$.

(c)$\Longrightarrow$(a) Suppose that $\mathcal{F}(X,Y,1)$ is *sto*-compact. Let $x_0 \in X$ with $\|x_0\| = 1$. Then $\mathcal{F}(X,Y,1)x_0$ is norm-compact in $Y$. As in the proof of Theorem 2.19 we have

$$B_Y \subset \mathcal{F}(X,Y,1)x_0.$$

It follows that $B_Y$ is norm-compact in $Y$. Hence $Y$ is finite-dimensional.

(b)$\Longrightarrow$(a) The proof is the same as the proof of (c)$\Longrightarrow$(a). □

**Theorem 3.16.** *The following are equivalent.*

(a) $X$ *is finite-dimensional.*
(b) $\overline{\mathcal{B}^*(X,Y,1)}^{sto}$ *is sto(ssto,* $\tau$*)-compact in* $\mathcal{B}(Y^*,X^*)$.
(c) $\overline{\mathcal{F}^*(X,Y,1)}^{sto}$ *is sto(ssto,* $\tau$*)-compact in* $\mathcal{B}(Y^*,X^*)$.

*Proof.* Recall the proof of Theorem 2.20. To show that (a), (b), and (c) are equivalent, it is enough to show that (c) implies (a) because by Theorem 3.15 (a) implies (b) and clearly (b) implies (c). Now assume (c) and let $y_0 \in Y$ with $\|y_0\| = 1$. Choose $y_0^* \in Y^*$ with $\|y_0^*\| = 1$ such that $y_0^* y_0 = 1$. Since $\overline{\mathcal{F}^*(X,Y,1)}^{sto}$ is *sto*-compact, $\overline{\mathcal{F}^*(X,Y,1)}^{sto} y_0^*$ is norm-compact in $X^*$. As in the proof of Theorem 2.20, we have $B_{X^*} \subset \overline{\mathcal{F}^*(X,Y,1)}^{sto} y_0^*$. Thus $B_{X^*}$ is norm compact in $X^*$. Hence $X$ is finite-dimensional. □

The proof of the following proposition is essentially the same as Proposition 2.23.

**Proposition 3.17.** *The following are equivalent.*

(a) $Y$ *is reflexive.*
(b) $\mathcal{B}^*(X,Y)$ *is sto (respectively,* $\tau$*)-closed in* $\mathcal{B}(Y^*,X^*)$.
(c) $\mathcal{B}^*(X,Y,1)$ *is sto (respectively,* $\tau$*)-closed in* $\mathcal{B}(Y^*,X^*)$.

Then we have a corollary of Theorem 3.16.

**Corollary 3.18.** *The following are equivalent.*

(a) $X$ *is finite-dimensional and* $Y$ *is reflexive.*

(b) $\mathcal{B}^*(X,Y,1)$ *is sto(ssto,* $\tau$*)-compact.*
(c) $\mathcal{F}^*(X,Y,1)$ *is sto(ssto,* $\tau$*)-compact.*

*Proof.* (a)$\Longrightarrow$(b) and (b)$\Longrightarrow$(a) are clear. (a)$\Longrightarrow$(c) follows from $\mathcal{B}^*(X,Y,1) = \mathcal{F}^*(X,Y,1)$. Finally, assume (c). By Theorem 3.16 $X$ is finite-dimensional. Hence (a) follows from $\mathcal{B}^*(X,Y,1) = \mathcal{F}^*(X,Y,1)$. $\square$

## 4. Approximation properties

In the Banach space theory, the approximation property, which already appeared in Banach's book [1], is one of the fundamental properties. Grothendieck [6] initiated the investigation of the variants of the approximation property and the relations between them. In this section we introduce the approximation property and its recent versions, and apply some of our topologies to them.

In the study of the approximation property, one important tool is the $\tau$ topology.

**Definition 4.1.** A Banach space $X$ is said to have the *approximation property* (in short, AP) if $I_X \in \overline{\mathcal{F}(X)}^{\tau}$, where $I_X$ is the identity in $\mathcal{B}(X)$. Also $X$ is said to have the $\lambda$*-bounded approximation property* (in short, $\lambda$-BAP) if $I_X \in \overline{\mathcal{F}(X,\lambda)}^{\tau}$. In particular, if $\lambda$=1, then we say that $X$ has the *metric approximation property* (in short, MAP). A Banach space $X$ is said to have the *compact approximation property* (in short, CAP) if $I_X \in \overline{\mathcal{K}(X)}^{\tau}$. Also $X$ is said to have the $\lambda$*-bounded compact approximation property* (in short, $\lambda$-BCAP) if $I_X \in \overline{\mathcal{K}(X,\lambda)}^{\tau}$. In particular, if $\lambda$=1, then we say that $X$ has the *metric compact approximation property* (in short, MCAP).

Casazza [2] summarized various results on approximation properties, including his own results, and introduced many open problems on the approximation property and its variants.

We now introduce recent versions of the approximation property.

**Definition 4.2.** A Banach space $X$ is said to have the *weak approximation property* (in short, WAP) if $\mathcal{K}(X) \subset \overline{\mathcal{F}(X)}^{\tau}$. And $X$ is said to have the *bounded weak approximation property* (in short, BWAP) if, for every $T \in \mathcal{K}(X)$, there is a $\lambda_T > 0$ such that $T \in \overline{\mathcal{F}(X,\lambda_T)}^{\tau}$. Also $X$ is said to have the *metric weak approximation property* (in short, MWAP) if $\mathcal{K}(X,1) \subset \overline{\mathcal{F}(X,1)}^{\tau}$. A Banach space $X$ is said to have the *quasi approximation property* (in short, QAP) if $\mathcal{K}(X) = \overline{\mathcal{F}(X)}$, where the closure is the operator norm closure.

In [3], [9], [10], and [11], Choi and Kim introduced and studied the above properties.

Now by Proposition 3.6 we have simple characterizations of the approximation properties.

**Proposition 4.3.** (a) *$X$ has the AP if and only if* $I_X \in \overline{\mathcal{F}(X)}^{swo}$.
(b) *$X$ has the CAP if and only if* $I_X \in \overline{\mathcal{K}(X)}^{swo}$.

(c) *$X$ has the WAP if and only if $\mathcal{K}(X) \subset \overline{\mathcal{F}(X)}^{swo}$.*
(d) *$X$ has the $\lambda$-BAP if and only if $I_X \in \overline{\mathcal{F}(X,\lambda)}^{wo}$.*
(e) *$X$ has the $\lambda$-BCAP if and only if $I_X \in \overline{\mathcal{K}(X,\lambda)}^{wo}$.*
(f) *$X$ has the BWAP if and only if for every $T \in \mathcal{K}(X)$, there is a $\lambda_T > 0$ such that $T \in \overline{\mathcal{F}(X,\lambda_T)}^{wo}$.*
(g) *$X$ has the MWAP if and only if $\mathcal{K}(X,1) \subset \overline{\mathcal{F}(X,1)}^{wo}$.*

Grothendieck [6] showed the following characterizations of the AP.

**Lemma 4.4.** (a) *$X$ has the AP if and only if for every Banach space $Y$, $\mathcal{K}(Y,X) = \overline{\mathcal{F}(Y,X)}$, where the closure is the operator norm closure.*
(b) *$X^*$ has the AP if and only if for every Banach space $Y$, $\mathcal{K}(X,Y) = \overline{\mathcal{F}(X,Y)}$.*

Kalton [7] showed the following lemma.

**Lemma 4.5.** *Suppose that $(T_n)$ is a sequence in $\mathcal{K}(X,Y)$ and $T \in \mathcal{K}(X,Y)$. If $T_n \overset{wao}{\longrightarrow} T$, then there is a sequence $(S_n)$ of convex combinations of $\{T_n\}$ such that $\|S_n - T\| \longrightarrow 0$.*

From the definition of QAP and above lemmas we have the following.

**Proposition 4.6.** (a) *$X$ has the AP if and only if for every Banach space $Y$ and $T \in \mathcal{K}(Y,X)$, there is a sequence $(T_n)$ in $\mathcal{F}(Y,X)$ such that $T_n \overset{wao}{\longrightarrow} T$.*
(b) *$X^*$ has the AP if and only if for every Banach space $Y$ and $T \in \mathcal{K}(X,Y)$, there is a sequence $(T_n)$ in $\mathcal{F}(X,Y)$ such that $T_n \overset{wao}{\longrightarrow} T$.*
(c) *$X$ has the QAP if and only if for every $T \in \mathcal{K}(X)$, there is a sequence $(T_n)$ in $\mathcal{F}(X)$ such that $T_n \overset{wao}{\longrightarrow} T$.*

Recall Lemma 2.22. Then we see the following.

**Lemma 4.7.** *$\mathcal{F}(Y^*,X^*) \subset \overline{\mathcal{F}^*(X,Y)}^{\tau}$.*

We now have some characterizations of approximation properties for dual spaces.

**Theorem 4.8.** (a) *$X^*$ has the AP if and only if $I_X \in \overline{\mathcal{F}(X)}^{swao}$.*
(b) *$X^*$ has the $\lambda$-BAP if and only if $I_X \in \overline{\mathcal{F}(X,\lambda)}^{wao}$.*
(c) *If $X^*$ has the WAP, then $\mathcal{K}(X) \subset \overline{\mathcal{F}(X)}^{swao}$.*
(d) *If $X^*$ has the BWAP, then for every $T \in \mathcal{K}(X)$, there is a $\lambda_T > 0$ such that $T \in \overline{\mathcal{F}(X,\lambda_T)}^{wao}$.*
(e) *If $X^*$ has the MWAP, then $\mathcal{K}(X,1) \subset \overline{\mathcal{F}(X,1)}^{wao}$.*

*Proof.* We show that (a), (c), and (e). The proofs of the others are similar. To show (a), note that $I_X \in \overline{\mathcal{F}(X)}^{swao}$ if and only if $I_{X^*} \in \overline{\mathcal{F}^*(X)}^{swo}$. Now if $X^*$ has the AP, then by Lemma 4.7 $I_{X^*} \in \overline{\mathcal{F}^*(X)}^{swo}$. Hence $I_X \in \overline{\mathcal{F}(X)}^{swao}$. Also, if $I_X \in \overline{\mathcal{F}(X)}^{swao}$, then $I_{X^*} \in \overline{\mathcal{F}^*(X)}^{swo} \subset \overline{\mathcal{F}(X^*)}^{swo}$. Hence by Proposition 4.3(a) $X^*$ has the AP. To show (c), let $T \in \mathcal{K}(X)$. Then $T^* \in \mathcal{K}(X^*)$. By the assumption, Proposition 3.6, and Lemma 4.7 $T^* \in \overline{\mathcal{F}^*(X)}^{swo}$. It follows that $T \in \overline{\mathcal{F}(X)}^{swao}$. Hence $\mathcal{K}(X) \subset \overline{\mathcal{F}(X)}^{swao}$. To show (e), let $T \in \mathcal{K}(X,1)$. Then $T^* \in \mathcal{K}(X^*,1)$. By hypothesis and Lemma 2.22 $T^* \in \overline{\mathcal{F}^*(X,1)}^{wo}$. It follows that $T \in \overline{\mathcal{F}(X,1)}^{wao}$. Hence $\mathcal{K}(X,1) \subset \overline{\mathcal{F}(X,1)}^{wao}$. □

It is well known that if $X^*$ has the AP (respectively, $\lambda$-BAP), then $X$ has the AP (respectively, $\lambda$-BAP). From Theorem 4.8 (a), (b) and Proposition 4.3 (a), (d), we can also deduce these results. In [3, 9], the same results were shown for the WAP, BWAP, and MWAP. These results are also shown from Theorem 4.8 (c), (d), (e) and Proposition 4.3 (c), (f), (g).

Recall the *weak** and *weak*o* topology on $\mathcal{B}(X^*, X^*)$. Then by Lemmas 2.22 and 4.7 we have the following corollary.

**Corollary 4.9.** (a) $\overline{\mathcal{F}(X^*,\lambda)}^{weak^*o} = \overline{\mathcal{F}^*(X,\lambda)}^{weak^*o}$ *for each* $\lambda > 0$.
(b) $\overline{\mathcal{F}(X^*)}^{weak^*} = \overline{\mathcal{F}^*(X)}^{weak^*}$.

We now have some other characterizations of approximation properties. (a) and (b) of the following theorem are also in [5, Lemma 2.1].

**Theorem 4.10.** (a) *$X$ has the AP if and only if* $I_{X^*} \in \overline{\mathcal{F}(X^*)}^{weak^*}$.
(b) *$X$ has the $\lambda$-BAP if and only if* $I_{X^*} \in \overline{\mathcal{F}(X^*,\lambda)}^{weak^*o}$.
(c) *$X$ has the WAP if and only if* $\mathcal{K}^*(X) \subset \overline{\mathcal{F}(X^*)}^{weak^*}$.
(d) *$X$ has the BWAP if and only if for every* $T^* \in \mathcal{K}^*(X)$*, there is a* $\lambda_{T^*} > 0$ *such that* $T^* \in \overline{\mathcal{F}(X^*,\lambda_{T^*})}^{weak^*o}$.
(e) *$X$ has the MWAP if and only if* $\mathcal{K}^*(X,1) \subset \overline{\mathcal{F}(X^*,1)}^{weak^*o}$.

*Proof.* We show that (a), (c), and (e). The proofs of the others are similar. Notice that by Proposition 4.3(a) and Corollary 4.9(b) $X$ has the AP if and only if $I_{X^*} \in \overline{\mathcal{F}^*(X)}^{weak^*} = \overline{\mathcal{F}(X^*)}^{weak^*}$. Hence (a) follows. Also by Proposition 4.3(c) and Corollary 4.9(b) $X$ has the WAP if and only if $\mathcal{K}^*(X) \subset \overline{\mathcal{F}^*(X)}^{weak^*} = \overline{\mathcal{F}(X^*)}^{weak^*}$. Hence (c) follows. Finally by Proposition 4.3(g) and Corollary 4.9(a) $X$ has the MWAP if and only if $\mathcal{K}^*(X,1) \subset \overline{\mathcal{F}^*(X,1)}^{weak^*o} = \overline{\mathcal{F}(X^*,1)}^{weak^*o}$. Hence (e) follows. □

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