A NOTE ON DECOMPOSITION OF COMPLETE EQUIPARTITE GRAPHS INTO GREGARIOUS 6-CYCLES

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ABSTRACT. In [8], it is shown that the complete multipartite graph $K_{n(2t)}$ having $n$ partite sets of size $2t$, where $n \geq 6$ and $t \geq 1$, has a decomposition into gregarious 6-cycles if $n \equiv 0, 1, 3$ or 4 (mod 6). Here, a cycle is called gregarious if it has at most one vertex from any particular partite set. In this paper, when $n \equiv 0$ or 3 (mod 6), another method using difference set is presented. Furthermore, when $n \equiv 0$ (mod 6), the decomposition obtained in this paper is $\infty$-circular, in the sense that it is invariant under the mapping which keeps the partite set which is indexed by $\infty$ fixed and permutes the remaining partite sets cyclically.

1. Introduction

Decompositions of graphs into edge-disjoint cycles have been an active research area for many years. Especially, decompositions by cycles of a fixed length have been considered in a number of different ways. Recently, it was shown that a complete graph of odd order, or a complete graph of even order minus an 1-factor, has a decomposition into $k$-cycles if $k$ divides the number of edges (see [1], [11] and [12] as well as their references). One of the key factors for all these works was the cycle decomposition of complete bipartite graphs obtained by Sotteau ([13]). Many authors began to consider cycle decompositions with special properties ([4], [5], [9], [10]). Then, Billington and Hoffman ([2]) introduced the notion of a gregarious cycle in a tripartite graph. However, the definition of gregarious cycles has been modified in later research papers ([2], [4], [7]) for general partite graphs. Recently, Billington and Hoffman ([3]) and Cho and et el. ([7]) independently produced gregarious 4-cycle decompositions for certain complete multipartite graphs. In [8], Cho and Gould showed that $K_{n(2t)}$ has a decomposition into 6-cycles for all $t \geq 1$ if $n \equiv 0, 1, 3$ or 4 (mod 6).

In this paper, as a note to the earlier paper ([8]), the author shows another method of proof when $n \equiv 0$ or 3 (mod 6), which uses the complete difference
set of a set of numbers. Furthermore, when \( n \equiv 0 \pmod{6} \), the decomposition obtained is invariant under the mapping which keeps one partite set fixed and permutes the remaining partite sets cyclically.

First of all, we make the definition of gregarious cycles clear. We call a cycle in a multipartite graph gregarious if it has at most one vertex from any particular partite set.

For simplicity, we will call a graph \( \gamma_6\)-decomposable if it is decomposable into gregarious 6-cycles, and a decomposition into gregarious 6-cycles will be called a \( \gamma_6\)-decomposition.

Let \( K(m_1, m_2, \ldots, m_n) \) denote the complete multipartite graph with partite sets of size \( m_i, i = 1, \ldots, n \). If all sizes are the same and equal to \( m \), we denote it by \( K_{n(m)} \), and call the graph a complete equipartite graph. Thus, the graph \( K_{n(1)} \) means the complete graph \( K_n \) with \( n \)-vertices.

**Lemma 1.1** ([1], [12]). Let \( n \) be an odd integer and \( m \) be any positive integer with \( m \leq n \). Then, \( K_n \) has a decomposition into \( m \)-cycles if and only if \( m \) divides \( \frac{n(n-1)}{2} \).

**Lemma 1.2** ([5]). If \( K(m_1, m_2, \ldots, m_n) \) is decomposable into cycles, then \( m_1, m_2, \ldots, m_n \) have the same parity, and furthermore \( n \) must be odd if the parity is odd.

The following lemma is proved in [5] for decompositions into arbitrary (non-necessarily gregarious) cycles, by the standard “expanding points method”. However, exactly the same method can be applied for decompositions into gregarious cycles.

**Lemma 1.3.** If \( K(m_1, m_2, \ldots, m_n) \) is decomposable into gregarious \( k \)-cycles for an even integer \( k \), then so is \( K(m_1t, m_2t, \ldots, m_nt) \) for every integer \( t \geq 1 \).

**Proof.** Let \( \mathcal{C} \) be a decomposition of \( K(m_1, m_2, \ldots, m_n) \) into gregarious \( k \)-cycles. Expand each vertex \( v \) of \( K(m_1, m_2, \ldots, m_n) \) to a set of \( t \) vertices \( v^{(1)}, v^{(2)}, \ldots, v^{(t)} \), and make edges \( uv^{(i)}v^{(j)} \) for \( i, j = 1, 2, \ldots, t \) if \( uv \) is an edge of \( K(m_1, m_2, \ldots, m_n) \). Then, the resulting graph is \( K(m_1t, m_2t, \ldots, m_nt) \). For each \( k \)-cycle \( \langle v_1, v_2, \ldots, v_k \rangle \) in \( \mathcal{C} \), we choose \( t^2 \) gregarious \( k \)-cycles

\[
\langle v_1^{(i)}, v_2^{(j)}, v_3^{(i)}, v_4^{(j)}, \ldots, v_{k-1}^{(i)}, v_k^{(j)} \rangle \quad (1 \leq i \leq t, \ 1 \leq j \leq t)
\]

of \( K(m_1t, m_2t, \ldots, m_nt) \). Let \( \mathcal{C}^* \) be the collection of all such gregarious \( k \)-cycles obtained from each cycles in \( \mathcal{C} \), then \( \mathcal{C}^* \) is a decomposition of

\[
K(m_1t, m_2t, \ldots, m_nt)
\]

into gregarious \( k \)-cycles. \( \square \)

## 2. Decomposition of \( K_{n(m)} \) into gregarious 6-cycles

**Lemma 2.1.** \( K_n \) is \( \gamma_6 \)-decomposable if and only if \( n \equiv 0, 1, 4 \) or \( 9 \pmod{12} \). If \( n \) is one of such an integer then \( K_{n(m)} \) is \( \gamma_6 \)-decomposable for all \( m \geq 1 \).
Proof. If $K_n$ has decomposition into cycles, the degree of each vertex must be even. Thus, $n$ should be odd. Now, if $n$ is an odd number, then $6$ divides $\frac{n(n-1)}{2}$ if and only if $n \equiv 0, 1, 4$ or $9 \pmod{12}$. The conclusions follows by Lemmas 1.1 and 1.3. \qed

If $m$ is odd and $K_{n(m)}$ is $\gamma_6$-decomposable, then $n$ is also odd by Lemma 1.2. Since $6$ must divides the number $\frac{n(n-1)m^2}{2}$ of edges of $K_{n(m)}$, $12$ divides $n(n-1)m^2$. Since $m$ and $n$ are odd, we have $n \equiv 1 \pmod{4}$ and $nm \equiv 0 \pmod{3}$. If $n \equiv 0 \pmod{3}$ then $n \equiv 9 \pmod{12}$ and so $K_{n(m)}$ is $\gamma_6$-decomposable by Lemma 2.1. So it remains to settle the cases when $n \equiv 1$ or $5 \pmod{12}$ and $m \equiv 0 \pmod{3}$. The decomposition problem for these cases is not settled yet.

In [8], it is proven that, for $n \geq 6$, $K_{n(2)}$ is $\gamma_6$-decomposable if and only if $6$ divide $2n(n-1)$, the number of edges in $K_{n(2)}$ and, in such a case, $K_{n(2)}$ is also $\gamma_6$-decomposable for every positive integer $t$. The authors used a difference set method for $n \equiv 1$ or $4 \pmod{6}$. For $n \equiv 0$ or $3 \pmod{6}$, they presented $K_{n(2)}$ as a join of two graphs which are already known to have $\gamma_6$-decomposable and showed that the join is $\gamma_6$-decomposable. We will not explain the join of graphs here. Anyway, such decompositions do not have nice symmetry as the case when $n \equiv 1$ or $4 \pmod{6}$. In the following sections, we use a method using the difference set of the extended number system $\mathbb{Z}^\infty_{n-1}$ to prove the following.

Theorem 2.1. Let $n \geq 6$ and $n \equiv 0$ or $3 \pmod{6}$. There is a systematic procedure to produce a $\gamma_6$-decomposition of $K_{n(2)}$ with a nice symmetry for all $t \geq 1$ by using the complete difference set of the extended number system $\mathbb{Z}^\infty_{n-1}$.

Due to Theorem 1.3, we will consider the decomposition of $K_{n(2)}$ only in the following sections.

3. Cycles from feasible sequences of differences

In this section, we assume $n \equiv 0$ or $3 \pmod{6}$ with $n \geq 6$. Let $\mathbb{Z}^\infty_{n-1} = \{\infty, 0, 1, 2, \ldots, n-2\}$. The arithmetic in $\mathbb{Z}^\infty_{n-1}$ is done modulo $n-1$ when $\infty$ is not involved. When $\infty$ is involved, we make the convention that $a \pm \infty = \infty \pm a = \infty$ for $a = 0, 1, \ldots, n-2$ and $\infty \pm \infty = 0$. In this paper, all arithmetic is done in $\mathbb{Z}^\infty_{n-1}$.

Let the partite sets of $K_{n(2)}$ be $A_\infty = \{\infty, \infty\}$, $A_0 = \{0, \overline{0}\}$, $A_1 = \{1, \overline{1}\}$, $\ldots$, and $A_{n-2} = \{n-2, \overline{n-2}\}$. Thus, elements in $\mathbb{Z}^\infty_{n-1}$ are used as indices of the partite sets as well as the vertices.

Let $D^*_{n-1} = \{\infty, \pm 1, \ldots, \pm \frac{n-1}{2}\}$ if $n$ is odd, and $D^*_{n-1} = \{\infty, \pm 1, \ldots, \pm \frac{n-2}{2}\}$ if $n$ is even. Then, $D^*_{n-1}$ is a complete set of differences of two distinct numbers in $\mathbb{Z}^\infty_{n-1}$. A sequence $\rho = (r_1, r_2, \ldots, r_6)$ of differences in $D^*_{n-1}$ is called a feasible sequence, or an $f$-sequence for simplicity, if

(i) $\rho = (r_1, r_2, \ldots, r_6)$, where $r_i \in D^*_{n-1} \setminus \{\infty\}$ for $i = 1, 2, \ldots, 6$, $\sum_{i=1}^6 r_i = 0$, and $\sum_{i=p}^q r_i \neq 0$ for $p, q$ with $1 < p$ or $q < 6$, or
(ii) \( \rho = (r_1, r_2, r_3, r_4, \infty, \infty) \), where \( r_i \in D^*_{n-1} \setminus \{ \infty \} \) for \( i = 1, 2, 3, 4, \) and 
\[ \sum_{i=p} q r_i \neq 0 \] for \( p, q \) with \( 1 < p \) or \( q < 4 \).

Note that any proper partial sum of consecutive entries of an \( f \)-sequence is nonzero.

Let \( \rho = (r_1, r_2, \ldots, r_6) \) be an \( f \)-sequence. A sequence \( \sigma_{\rho} = (0, s_1, \ldots, s_5) \) of elements in \( Z^\infty_{n-1} \) is called the sequence of initial sums, or an \( s \)-sequence for short, of \( \rho \) if \( s_j = \sum_{i=1}^j r_i \) for \( j = 1, 2, 3, 4, 5 \). Thus, if we put \( s_0 = 0 \) then \( s_j = s_{j-1} + r_j \) if \( 1 \leq j \leq 5 \), and \( s_5 + r_6 = 0 \) or \( s_5 = \infty \) by the definition of \( f \)-sequences. For example, when \( n = 12 \), \( \sigma_{\rho_0} = (0, 3, 10, 4, 8, 5) \) for \( \rho_0 = (3, -4, 5, 4, -3, -5) \) and \( \sigma_{\rho_1} = (0, 1, 3, 4, 2, \infty) \) for \( \rho_1 = (1, 2, 1, -2, \infty, \infty) \).

Intuitively, an \( s \)-sequence represents the ordering of partite sets which a 6-cycle traverses, and the feasibility of the corresponding \( f \)-sequence guarantees that the cycle is proper and gregarious. Now, the following lemma is trivial by definitions.

**Lemma 3.1.** Let \( \sigma = (0, s_1, s_2, \ldots, s_5) \) be the sequence of initial sums of a sequence \( \rho = (r_1, r_2, \ldots, r_6) \) of differences. Then, \( \rho \) is an \( f \)-sequence if and only if \( 0, s_1, s_2, \ldots, s_5 \) are mutually distinct and \( \sum_{i=1}^6 r_i = 0 \).

Let \( \phi^+ \) and \( \phi^- \) be the mappings of \( Z^\infty_{n-1} \) into \( \bigcup_{i \in Z^\infty_{n-1}} A_i \) defined by \( \phi^+(i) = i \) and \( \phi^-(i) = 7 - i \) for all \( i \) in \( Z^\infty_{n-1} \). A flag is a sequence \( \phi^* = (\phi_0, \phi_1, \ldots, \phi_5) \) where each \( \phi_i \) is \( \phi^+ \) or \( \phi^- \) for \( i = 0, 1, \ldots, 5 \). Given a flag \( \phi^* \), we also use the same notation \( \phi^* \) to denote the mapping defined by \( \phi^*(s_0, s_1, \ldots, s_5) = (\phi_0(s_0), \phi_1(s_1), \ldots, \phi_5(s_5)) \) for every \( s \)-sequence \( (s_0, s_1, \ldots, s_5) \). Note that

\[ \phi^*(s_0, s_1, \ldots, s_5) \]

is a \( \gamma_6 \)-cycle of \( K_{n(2)} \).

Let \( \tau \) be the permutation \((0, 1, \ldots, n-2)(\infty)\) on \( Z^\infty_{n-1} \), that is, \( \tau(i) = i + 1 \) for all \( i \) in \( Z^\infty_{n-1} \). In this sense, we may call \( \tau \) a translation on \( Z^\infty_{n-1} \). We extend \( \tau \) to a mapping \( \tau_* \) on 6-cycles of \( K_{n(2)} \) by defining

\[ \tau_*(\phi_0(s_0), \phi_1(s_1), \ldots, \phi_5(s_5)) = (\phi_0(\tau(s_0)), \phi_1(\tau(s_1)), \ldots, \phi_5(\tau(s_5))) \]

\[ = (\phi_0(s_0+1), \phi_1(s_1+1), \ldots, \phi_5(s_5+1)). \]

Note that \( \tau_*(s_0, s_1, \ldots, s_5) \) is the identity mapping and, by convention, \( \tau_*(0, s_1, \ldots, s_5) \) is the identity mapping.

Given an \( s \)-sequence \( \sigma \) and a flag \( \phi^* \), we can generate the class \( \{ \tau_i^*(\phi^*(\sigma)) \mid 0 \leq i \leq n-2 \} \) containing \( n-1 \) \( \gamma_6 \)-cycles, which we call an full class generated from \( \phi^*(\sigma) \). Sometimes, when \( n \) is odd, we need to generate the class \( \{ \tau_i^*(\phi^*(\sigma)) \mid 0 \leq i \leq \frac{n-1}{2} \} \) or \( \{ \tau_i^*(\phi^*(\sigma)) \mid \frac{n-1}{2} \leq i \leq n-2 \} \) containing \( \frac{n-1}{2} \) \( \gamma_6 \)-cycles, which we call an half class generated from \( \phi^*(\sigma) \). The cycle \( \phi^*(\sigma) \) is called the starter cycle of the relevant class. For example, if \( \sigma_{\rho} = (s_0, s_1, \ldots, s_5) \) from \( \rho = (r_1, r_2, \ldots, r_6) \) and \( \phi^* = (\phi^+, \phi^-, \phi^+, \phi^+, \phi^+, \phi^-) \), then the \( \gamma_6 \)-cycles...
in the full class generated by $\phi^*(\sigma)$ are as below:

$$
\begin{align*}
(0, \overline{s_1}, \overline{s_2}, s_3, s_4, \overline{s_5}), \\
(1, \overline{s_1+1}, \overline{s_2+1}, s_3+1, s_4+1, \overline{s_5+1}), \\
(2, \overline{s_1+2}, \overline{s_2+2}, s_3+2, s_4+2, \overline{s_5+2}), \\
\vdots \\
(n-1, \overline{s_1+n-1}, \overline{s_2+n-1}, s_3+n-1, s_4+n-1, \overline{s_5+n-1}).
\end{align*}
$$

If neither $\infty$ nor $\overline{\infty}$ appears in a column of the above table, that column either has all $i$ or has all $\overline{i}$ for $i = 0, 1, \ldots, n-1$. Note that, if $q-p = r_1$ then the edge $p\overline{q}$ appears exactly once as the first edge of a cycle above. Similarly, each of the edges $pq$ with $q-p = r_2$, $pq$ with $q-p = r_3$, $pq$ with $q-p = r_4$, $pq$ with $q-p = r_5$, and $pq$ with $q-p = r_6$, appears exactly once in the above.

The above procedure is the principal method we are going to use to obtain $\gamma_6$-decompositions of $K_{n(2)}$. The main problem then is how to choose appropriate $f$-sequences and flags, which will be discussed in the next section.

An edge joining a vertex in $A_i$ and a vertex in $A_j$ is called an edge of distance $d$ if $i-j = \pm d$ for some $d$ in $D_{n-1}$ with $0 < d \leq n^{-1}$. For example, the edges $0\overline{4}$, $73$, $\overline{73}$ and $\overline{3}$ are all edges of distance $4$ in $K_{10(2)}$. An edge involving the vertex $\infty$ or $\overline{\infty}$, such as $\infty\overline{3}$ and $6\overline{\infty}$, are called an edge of infinite distance. When $n$ is odd and $d = \frac{n-1}{2}$, an edge of distance $d$ is called a diagonal edge.

---

Figure 1

---

$\cdots$ : $\Phi^1(\sigma_n) = (0, 2, 7, 3, 6, \overline{4})$

$\cdots$ : $\tau_*(\Phi^1(\sigma_n)) = (1, 3, 0, 4, 7, 5)$

$\cdots$ : $\tau^2(\Phi^1(\sigma_n)) = (2, 4, 1, 5, 6, \overline{0})$

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Figure 2

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$\cdots$ : $\Phi^1(\sigma_4) = (0, 1, 3, 4, 2, \overline{\infty})$

$\cdots$ : $\tau_*(\Phi^1(\sigma_4)) = (1, 2, 4, 5, 3, \overline{\infty})$

$\cdots$ : $\tau^2(\Phi^1(\sigma_4)) = (2, 3, 5, 6, \overline{4}, \overline{\infty})$
4. Proof of Theorem 2.1

For $i$ with $1 \leq i \leq \frac{n-2}{2} - 2$, using differences in $\{\pm i, \pm(i+1), \pm(i+2)\}$, we generate two special full classes as follow. Put $\eta = (-i, -(i+1), i+2, i+1, -i, -(i+2))$, then we have the $s$-sequence $\sigma_{\eta} = (0, i, n-2, i+1, 2i+2, i+2)$. Since $2i+2 \leq 2(\frac{n-2}{2} - 2) + 2 = n - 4$, all entries of $\sigma_{\eta}$ are mutually distinct. Thus, $\eta$ is indeed an $f$-sequence by Lemma 3.1. Now, we take two special flags $\Phi_{1}^{*} = (\phi^{+}, \phi^{-}, \phi^{+}, \phi^{-}, \phi^{-})$ and $\Phi_{2}^{*} = (\phi^{-}, \phi^{+}, \phi^{-}, \phi^{-}, \phi^{+})$, and generate two full classes $C$ and $D$ from the starter cycles $\Phi_{1}^{*}(\sigma_{\eta})$ and $\Phi_{2}^{*}(\sigma_{\eta})$, respectively. These two classes are called the standard classes generated from $\eta$. They are listed below and depicted by the picture in Figure 1.

The following Lemma is easily checked from the above table.

**Lemma 4.1.** In the cycles of the classes $C$ and $D$ above, each of the edges of the form $pq$, $p\eta$, $p\eta$ and $p\eta$ of distance $d$ appears exactly once for $d = i$, $i+1$ and $i+2$. The classes $C$ and $D$ are invariant under $\tau_{*}$.

With the differences in $\{\infty, \pm 1, \pm 2\}$, we produce two $f$-sequences $\delta_{1} = (1, 2, 1, -1, 2, \infty, \infty)$ and $\delta_{2} = (1, 2, -1, 2, \infty, \infty)$, and get the corresponding $s$-sequences $\sigma_{\delta_{1}} = (0, 1, 3, 4, 2, \infty)$ and $\sigma_{\delta_{2}} = (1, 3, 2, 4, \infty)$. Take two special flags $\Psi_{1}^{*} = (\phi^{+}, \phi^{+}, \phi^{-}, \phi^{-}, \phi^{-})$ and $\Psi_{2}^{*} = (\phi^{-}, \phi^{+}, \phi^{-}, \phi^{-}, \phi^{+})$, and generate two full classes $E_{1}$ and $E_{2}$ from starter cycles $\Psi_{1}^{*}(\sigma_{\delta_{1}})$ and $\Psi_{2}^{*}(\sigma_{\delta_{2}})$, respectively, as below. They are depicted by the picture in Figure 2.

The following lemma is easily checked from the above table.

**Lemma 4.2.** In the cycles of the classes $E_{1}$ and $E_{2}$ above, each of the edges of the form $pq$, $p\eta$, $p\eta$ and $p\eta$ of distance $d$ appears exactly once for $d = 1$, 2 and $\infty$. The classes $E_{1}$ and $E_{2}$ are invariant under $\tau_{*}$.

We divide the proof of Theorem 2.1 into two cases depending on $n$.

**Case (1).** Suppose $n \equiv 0 \pmod{6}$ and put $n = 6k \ (k \geq 1)$. Since $K_{n(2)}$ has $4 \cdot \binom{6k}{6} = 12k(6k-1)$ edges, we need to produce $2k(6k-1) = 2k(n-1)$ disjoint 6-cycles. In fact, we will produce $2k$ full classes. In this case,

$\mathcal{Z}_{n-1}^{\infty} = \{\infty, 0, 1, \ldots, 6k-1\} \quad \text{and} \quad \mathcal{D}_{n-1}^{*} = \{\infty, \pm 1, \pm 2, \ldots, \pm(3k-1)\}$. 


Starting with the differences in \( \{\infty, \pm 1, \pm 2\} \), we obtain two full classes \( E_1 \) and \( E_2 \) of Lemma 4.2. If \( k > 1 \) then the set \( D^*_n \setminus \{\infty, \pm 1, \pm 2\} = \{\pm 3, \pm 4, \ldots, \pm (3k-1)\} \) is not empty. Partition this into \( k-1 \) subsets \( \{\pm 3i, \pm (3i+1), \pm (3i+2)\} \) for \( i = 1, 2, \ldots, k-1 \). For each \( i \), we put \( \eta_i = (3i, -(3i+1), 3i+2, 3i+1, -3i, -(3i+2)) \) and generate two standard classes \( C_i \) and \( D_i \) from \( \eta_i \), as in Lemma 4.1. If \( k = 1 \), we do not have these classes. Put

\[
C^* = \left( \bigcup_{i=0}^{\infty} E_i \right) \cup \left( \bigcup_{i=0}^{k-1} C_i \right) \cup \left( \bigcup_{i=1}^{k-1} D_i \right).
\]

Then, by Lemmas 4.1 and 4.2, \( \gamma_6 \)-cycles in \( C^* \) involve each edge of \( K_{n(2)} \) exactly once. Thus, we have the following theorem.

**Theorem 4.1.** For \( n \equiv 0 \pmod{6} \), the class \( C^* \) above is a \( \gamma_6 \)-decomposition of \( K_{n(2)} \) and is \( \infty \)-circular in the sense that it is invariant under \( \tau \).

**Example 4.1.** Let \( n = 12 \). Then \( D^*_n = \{\infty, \pm 1, \pm 2, \ldots, \pm 5\} \). By the procedure above, we have \( \delta_1 = (1, 2, 1, -2, \infty, \infty) \), \( \delta_2 = (1, 2, -1, 2, \infty, \infty) \) and \( \eta_1 = (3, -4, 5, 4, -3, -5) \). The corresponding \( s \)-sequences are \( \sigma_{\delta_1} = (0, 1, 3, 4, 2, \infty) \), \( \sigma_{\delta_2} = (0, 1, 3, 2, 4, \infty) \) and \( \sigma_{\eta_1} = (0, 3, 10, 4, 8, 5) \). The \( \gamma_6 \)-decomposition is given below:

\[
\begin{align*}
(0, 1, 3, 4, 2, \infty), & \quad (5, 3, 5, 4, \infty), & \quad (0, 3, 10, 4, 8, 5), & \quad (5, 3, 5, 4, \infty), \\
(2, 3, 3, 5, 4, \infty), & \quad (3, 3, 5, 4, \infty), & \quad (2, 5, 1, 6, 1, \infty), & \quad (3, 5, 7, 5, 7, \infty), \\
(3, 4, 6, 7, 8, \infty), & \quad (3, 4, 6, 7, \infty), & \quad (3, 6, 2, 7, \infty), & \quad (3, 6, 2, 7, \infty), \\
(4, 7, 8, 9, \infty), & \quad (4, 7, 8, \infty), & \quad (4, 7, 8, \infty), & \quad (4, 7, 8, \infty), \\
(5, 6, 8, 9, \infty), & \quad (5, 6, 8, \infty), & \quad (5, 6, 8, \infty), & \quad (5, 6, 8, \infty), \\
(6, 7, 9, 10, \infty), & \quad (6, 7, 9, \infty), & \quad (6, 9, 5, 10, \infty), & \quad (6, 9, 5, 10, \infty), \\
(7, 8, 10, 11, \infty), & \quad (7, 8, 10, \infty), & \quad (7, 10, 6, 11, \infty), & \quad (7, 10, 6, 11, \infty), \\
(8, 9, 0, 1, \infty), & \quad (8, 9, 0, 1, \infty), & \quad (8, 9, 0, 1, \infty), & \quad (8, 9, 0, 1, \infty), \\
(9, 10, 1, 2, \infty), & \quad (9, 10, 1, 2, \infty), & \quad (9, 10, 1, 2, \infty), & \quad (9, 10, 1, 2, \infty), \\
(10, 0, 2, 3, \infty), & \quad (10, 2, 3, \infty), & \quad (10, 2, 9, 3, \infty), & \quad (10, 2, 9, 3, \infty).
\end{align*}
\]

**Case (2).** Suppose \( n = 9 \). We treat this case as a special case. A 6-cycle decomposition exists for \( K_9 \) by Lemma 1, and the following is an example.

\[
(0, 1, 6, 7, 3, 4), \quad (1, 2, 7, \infty, 4, 5), \quad (2, 3, \infty, 0, 5, 6), \\
(0, 2, 5, 7, 1, 3), \quad (3, 5, \infty, 1, 4, 6), \quad (4, \infty, 6, 2, 4, 7, 0),
\]

where \( \mathbb{Z}_{\infty}^2 \) is used for the vertex set of \( K_9 \). From this, we obtain the following \( \gamma_6 \)-decomposition of \( K_{9(2)} \) by the method in Lemma 1.3.

\[
\begin{align*}
(0, 1, 6, 7, 3, 4), & \quad (0, \infty, 6, 7, 3, 4), & \quad (0, \infty, 6, 7, 3, 4), & \quad (0, \infty, 6, 7, 3, 4), & \quad (0, \infty, 6, 7, 3, 4), \\
(1, 2, 7, \infty, 4, 5), & \quad (1, 2, 7, \infty, 4, 5), & \quad (1, 2, 7, \infty, 4, 5), & \quad (1, 2, 7, \infty, 4, 5), \\
(2, 3, \infty, 0, 5, 6), & \quad (2, 3, \infty, 0, 5, 6), & \quad (2, 3, \infty, 0, 5, 6), & \quad (2, 3, \infty, 0, 5, 6), \\
(0, 2, 5, 7, 1, 3), & \quad (0, 2, 5, 7, 1, 3), & \quad (0, 2, 5, 7, 1, 3), & \quad (0, 2, 5, 7, 1, 3), \\
(3, 5, \infty, 1, 4, 6), & \quad (3, 5, \infty, 1, 4, 6), & \quad (3, 5, \infty, 1, 4, 6), & \quad (3, 5, \infty, 1, 4, 6), \\
(6, \infty, 2, 4, 7, 0), & \quad (6, \infty, 2, 4, 7, 0), & \quad (6, \infty, 2, 4, 7, 0), & \quad (6, \infty, 2, 4, 7, 0).
\end{align*}
\]

**Theorem 4.2.** \( K_9(2) \) is \( \gamma_6 \)-decomposable.

**Case (3).** Suppose \( n \equiv 3 \pmod{6} \) with \( n \geq 15 \) and put \( n = 6k+3 \) \( (k \geq 2) \). Since \( K_{n(2)} \) has \( 4 \cdot \binom{6k+3}{2} = 2(6k+3)(6k+2) = 6(2k+1)(n-1) \) edges, we need
to produce \((2k+1)(n-1)\) disjoint 6-cycles. In fact, we will produce \(2k-1\) full classes and 4 half classes. We have
\[ Z_{n-1}^{\infty} = \{ \infty, 0, 1, \ldots, 6k+1 \} \quad \text{and} \quad D_{n-1}^\infty = \{ \infty, \pm 1, \pm 2, \ldots, \pm (3k+1) \} \]
Note that \(3k+1 = -(3k+1)\) in \(Z_{n-1}^{\infty}\). With the differences in \(\{ \pm(3k-3), \pm(3k-2), \pm(3k-1), \pm 3k, \pm(3k+1) \}\), we produce two \(f\)-sequences
\[ \rho_1 = (3k-3, -(3k-2), 3k-1, 3k+1, -(3k-1), -(3k)) \quad \text{and} \quad \rho_2 = (3k-3, 3k-2, -3k, -(3k-3), 3k, -(3k-2)) \]
Then, we have
\[ \sigma_{\rho_1} = (0, 3k-3, 6k+1, 3k-2, 6k-1, 3k) \quad \text{and} \quad \sigma_{\rho_2} = (0, 3k-3, 6k-5, 3k-5, 6k, 3k-2) \]
We take four flags
\[ \phi_1^* = (\phi^-, \phi^-, \phi^+, \phi^-, \phi^-, \phi^+) \quad \phi_2^* = (\phi^-, \phi^-, \phi^+, \phi^-, \phi^+, \phi^+) \]
\[ \phi_3^* = (\phi^-, \phi^+, \phi^+, \phi^-, \phi^-, \phi^-) \quad \phi_4^* = (\phi^-, \phi^+, \phi^-, \phi^-, \phi^+, \phi^+) \]
Since \(\sigma_{\rho_1}\) has diagonal edges, we need to generate half classes from it, instead of full classes, to avoid double appearances of edges of the form \(pq\) or \(\overline{p}q\). We generate the following 4 half classes from \(\sigma_{\rho_1}\) and the above four flags.
\[ F_1 = \{ \tau_4^*(\phi_1^*(\sigma_{\rho_1})) \mid 0 \leq i \leq \frac{n-1}{2} - 1 \} \quad F_2 = \{ \tau_4^*(\phi_1^*(\sigma_{\rho_1})) \mid \frac{n-1}{2} \leq i \leq n-2 \} \]
\[ F_3 = \{ \tau_4^*(\phi_2^*(\sigma_{\rho_1})) \mid 0 \leq i \leq \frac{n-1}{2} - 1 \} \quad F_4 = \{ \tau_4^*(\phi_2^*(\sigma_{\rho_1})) \mid \frac{n-1}{2} \leq i \leq n-2 \} \]
The \(\gamma_6\)-cycles in these classes are as below. Note that \(6k+2 = 0\) and \(\frac{n-1}{2} - 1 = 3k\).

\(F_1\) : \(\emptyset, 3k-3, 6k+1, 3k-2, 5k-1, 3k, 6k\)  \(F_2\) : \(\emptyset, 3k-3, 5k-1, 3k-2, 6k-1, 3k\)
\(\emptyset, 3k-2, 5k-1, 3k, 6k+1, 3k+1\)  \(\emptyset, 3k-2, 5k, 3k-1, 6k, 3k+1\)
\(\emptyset, 3k-2, 5k, 3k, 6k+1, 3k+1\)  \(\emptyset, 3k-1, 5k-1, 3k-2, \ldots\)
\(\emptyset, 3k-1, 5k-2, 3k-3, \ldots\)

\(F_3\) : \(\emptyset, 3k-3, 6k-1, 3k-2, 5k-1, 3k, 6k\)  \(F_4\) : \(\emptyset, 3k-3, 5k-1, 3k-2, 6k-1, 3k\)
\(\emptyset, 3k-2, 5k-1, 3k, 6k+1, 3k+1\)  \(\emptyset, 3k-2, 5k, 3k-1, 6k, 3k+1\)
\(\emptyset, 3k-2, 5k, 3k, 6k+1, 3k+1\)  \(\emptyset, 3k-1, 5k-1, 3k-2, \ldots\)
\(\emptyset, 3k-1, 5k-2, 3k-3, \ldots\)

Then, in the \(\gamma_6\)-cycles of \(F_1, F_2, F_3\) and \(F_4\), each of the following edges appears exactly once. Note that an edge \(pq\) with \(q-p = r\) and an edge \(\overline{q}p\) with \(p-q = -r\) are the same edge.

(i) Diagonal edges, i.e., edges of distance \(3k+1\). Edges \(pq, \overline{p}q, p\overline{q}, \overline{p}q\) with \(q-p = 3k+1\) appear as the fourth edges in \(F_3, F_1, F_2, F_4\), respectively.

(ii) Edges of distance \(3k-1\). Edges \(pq\) with \(q-p = 3k-1\) appear as the third edges in \(F_2\) or as the fifth edges in \(F_4\) in the form \(pq\). Edges \(\overline{p}q\) appear as the third edges in \(F_4\) or as the fifth edges in \(F_2\) in the form \(\overline{q}p\). Edges \(p\overline{q}\) appear as the third edges in \(F_1\) or as the fifth edges in
Edges of the form $p \overline{q}$. Edges $q \overline{p}$ appear as the third edges in $F_3$ as the fifth edges in $F_1$ in the form $q \overline{p}$.

(iii) Edges $p \overline{q}$ and $p \overline{q}$ with $q - p = 3k$. Edges $p \overline{q}$ appear in the form $q \overline{p}$ as the last edges in $F_2$ or $F_3$. Edges $p \overline{q}$ appear in the form $q \overline{p}$ as the last edges in $F_1$ or $F_4$.

(iv) Edges $p \overline{q}$ and $p \overline{q}$ with $q - p = 3k - 2$. Edges $p \overline{q}$ appear in the form $q \overline{p}$ as the second edges in $F_1$ or $F_2$. Edges $p \overline{q}$ appear in the form $q \overline{p}$ as the second edges in $F_3$ or $F_4$.

(v) Edges $p \overline{q}$ and $p \overline{q}$ with $q - p = 3k - 3$. Edges $p \overline{q}$ appear as the first edges in $F_3$ or $F_4$. Edges $p \overline{q}$ appear as the first edges in $F_1$ or $F_2$.

Next, we take another flag $\phi_5^* = (\phi^+, \phi^-, \phi^-, \phi^+, \phi^+)$. and generate the full class

$$F_3 = \{ \tau_1^i(\phi_5^*(\sigma_{p2})) \mid 0 \leq i \leq n - 2 \}$$

from $\sigma_{p2}$ and $\phi_5^*$. The $\gamma_6$-cycles in this class are as below:

$$(F_3) \quad \{ \quad 0, \ 3k - 3, \ 6k - 5, \ 3k - 5, \ 6k - 5k - 2 \},$$

$$\{ 1, \ 3k - 2, \ 6k - 1, \ 3k - 4, \ 6k - 1, \ 3k - 1 \},$$

$$\{ 2, \ 3k - 7, \ 6k - 3, \ 3k - 3, \ 6k - 3k - 2 \},$$

$$\{ 3k, \ 6k - 3, \ 6k - 7, \ 6k - 5, \ 3k - 2, \ 6k - 2 \},$$

$$\{ 3k + 1, \ 6k - 2, \ 6k - 6, \ 3k - 4, \ 3k - 1, \ 6k - 1 \},$$

$$\{ 6k - 5, \ 6k - 7, \ 6k - 7, \ 6k - 2, \ 3k - 4 \},$$

$$\{ 6k + 1, \ 3k - 4, \ 6k - 6, \ 3k - 6, \ 6k - 1, \ 3k - 3 \}.$$
and generate two standard classes $C_i$ and $D_i$ from $\eta_i$, as in Lemma 4.1. Then, in the $\gamma_0$-cycles in $C_i$ and $D_i$, every edge of distance $3i, 3i+1$ and $3i+2$ appears exactly once. If $k = 2$, we do not have these classes.

Finally, put $C^* = \bigcup_{i=1}^{3E} E_i \bigcup \bigcup_{i=1}^{3D} F_i \bigcup \bigcup_{i=1}^{3C} C_i \bigcup \bigcup_{i=1}^{3D} D_i$. Then, $\gamma_0$-cycles in $C^*$ involve each edge of every distance in $K_{n(2)}$ exactly once. Thus, we have the following theorem.

**Theorem 4.3.** For $n \equiv 3 \pmod{6}$ with $n \geq 15$, the class $C^*$ above is a $\gamma_0$-decomposition of $K_{n(2)}$.

**Example 4.2.** Let $n = 15$. Using hexadecimal digits, let $\mathbb{Z}_{14}^{\infty} = \{\infty, 0, 1, \ldots, 9, a, b, c, d\}$. We have $D_{14} = \{\pm \infty, \pm 1, \ldots, \pm 7\}$. Starting with $f$-sequences $\rho_1 = (3, -4, 5, 7, -5, -6)$ and $\rho_2 = (3, 4, -6, -3, 6, -4)$, we obtain $s$-sequences $\sigma_{\rho_1} = (0, 3, d, 4, b, 6)$ and $\sigma_{\rho_2} = (0, 3, 7, 1, c, 4)$. We generate classes $F_1, F_2, F_3, F_4, F_5$ as in Case (3). From $f$-sequences $\delta_1 = (1, 2, 1, -1, 2, \infty, \infty)$ and $\delta_2 = (1, 2, -1, 2, \infty, \infty)$, we obtain $s$-sequences $\sigma_{\delta_1} = (0, 1, 3, 4, 2, \infty)$ and $\sigma_{\delta_2} = (0, 1, 3, 2, 3, \infty)$. We generate two full classes $E_1$ and $E_2$ as in Lemma 4.2. These classes constitute a $\gamma_0$-decomposition of $K_{15(2)}$ and are listed below.

$$(5, 7, d, 3, 5, 6), (5, 3, 7, 4, 6, 5), (0, 7, 1, e, 4), (0, 1, 3, 7, d, \infty), (5, 1, 7, 3, 4, \infty).$$

$$(7, 0, 0, \eta, 7), (\eta, 4, 5, 5, c, 7), (1, 7, 2, e, 5, 7), (1, 2, 4, 3, 7, \infty), (7, 2, 3, 5, 7, \infty).$$

$$(2, 5, 1, 7, 3, 8), (3, 5, 7, 6, 7, 8), (2, 3, 5, 7, 3, 6), (2, 3, 5, 7, 6, 6).$$

$$(5, 3, 7, 3, 5, 9), (6, 3, 7, 7, 5, 9), (3, 3, 8, 4, 1, 7), (3, 4, 6, 7, 7, \infty).$$

$$(7, 3, 2, 1, a, 4), (7, 7, 5, 8, 1, 4), (4, 7, 3, 5, 2, 8), (4, 5, 7, 3, 2, \infty), (4, 5, 7, 3, 8, \infty).$$

$$(5, 3, 5, 3, 6, b), (5, 8, 5, 9, 2, 4), (5, 8, 5, 9, 3, 9), (5, 6, 5, 9, 3, \infty), (5, 6, 5, 9, 7, \infty).$$

$$(5, 8, 5, 5, c, 3), (7, 9, 5, 8, a, 3), (7, 9, 5, 8, a, \infty).$$

$$(7, 5, 6, 4, 7), (7, 5, 6, 4, 9), (7, 5, 8, 5, b), (7, 8, 5, 3, \infty), (7, 8, 5, b, \infty).$$

$$(8, 5, 7, 9, \infty), (8, 5, 7, a, 5, \infty), (8, 5, 7, 9, 5, 0), (8, 5, 7, 9, 6, c), (8, 9, 5, 8, b, \infty), (8, 9, 5, 8, b, \infty).$$

$$(5, \eta, 8, a, 5, \eta), (\eta, 8, 9, 8, 6, 1), (9, \eta, 8, 9, 6, 1), (5, \eta, 8, 9, \infty, \infty).$$

$$(\eta, 9, 0, \eta, 7), (\eta, d, 5, 5, 7, 2), (9, a, c, 6, 8, 0), (a, b, d, 5, 5, \infty).$$

$$(5, \eta, a, 1, 8, \eta), (5, 0, \eta, 7, 8, 3), (6, 8, \eta, 1, 9, 1), (b, c, d, 1, 2, \infty), (b, c, d, 1, 2, \infty).$$

$$(7, 0, 3, \eta, a, 5), (7, 2, 5, a, 5), (4, d, 0, 2, 3, 7, \infty).$$

$$(7, 0, 3, \eta, 3, \infty).$$

**Proof of Theorem 2.1.** By Theorems 4.1 - 4.3 and Theorem 1.3. □

**References**


A NOTE ON DECOMPOSITION OF COMPLETE EQUIPARTITE GRAPHS


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