CONTINUITY FOR MULTILINEAR INTEGRAL OPERATORS 
ON BESOV SPACES

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Abstract. The continuity for the multilinear operators associated to some non-convolution type integral operators on Besov spaces are obtained. The operators include Littlewood-Paley operators, Marcinkiewicz operators and Bochner-Riesz operator.

1. Introduction

As the development of the singular integral operators, their commutators and multilinear operators have been well studied (see [1-7]). From [2], [7], [13], [15] and [18], we know that the commutators and multilinear operators generated by the singular integral operators and the Lipschitz functions are bounded on the Triebel-Lizorkin and Lebesgue spaces. The purpose of this paper is to introduce some multilinear operators associated to certain non-convolution type integral operators and prove the continuity properties for the multilinear operators on the Besov spaces. The operators include Littlewood-Paley operators, Marcinkiewicz operators and Bochner-Riesz operator.

2. Preliminaries and Theorem

In this paper, we will study a class of multilinear operators associated to some non-convolution type integral operators as following.

Let $m_j$ be the positive integers $(j = 1, \ldots, l)$, $m_1 + \cdots + m_l = m$ and $A_j$ be the functions on $\mathbb{R}^n$ $(j = 1, \ldots, l)$. Set

$$R_{m_j+1}(A_j; x, y) = A_j(x) - \sum_{|\gamma| \leq m_j} \frac{1}{\gamma!} D^\gamma A_j(y)(x - y)^\gamma.$$ 

Let $F_t(x, y)$ define on $\mathbb{R}^n \times \mathbb{R}^n \times [0, +\infty)$. Set

$$F_t(f)(x) = \int_{\mathbb{R}^n} F_t(x, y) f(y) dy.$$
and
\[ F_t^A(f)(x) = \int_{\mathbb{R}^n} \prod_{j=1}^m Q_{m_j+1}(A_j; x, y) F_t(x, y) f(y) dy \]
for every bounded and compactly supported function \( f \). Let \( H \) be the Banach space \( H = \{ h : ||h|| < \infty \} \) such that, for each fixed \( x \in \mathbb{R}^n \), \( F_t(f)(x) \) and \( F_t^A(f)(x) \) may be viewed as a mapping from \([0, +\infty)\) to \( H \). Then, the multilinear operator associated to \( F_t \) is defined by
\[ T^A(f)(x) = ||F_t^A(f)(x)||, \]
where \( F_t \) satisfies: for fixed \( \varepsilon > 0 \) and \( 0 \leq \delta < n - 1 \),
\[ ||F_t(x, y)|| \leq C|x - y|^{-n+\delta} \]
and
\[ ||F_t^1(y, x) - F_t^1(z, x)|| \leq C|y - z|^\varepsilon |x - z|^{-n-\varepsilon+\delta} \]
if \( 2|y - z| \leq |x - z| \). We define \( T(f)(x) = ||F_t(f)(x)|| \).

Note that when \( m = 0 \), \( T^A \) is just the multilinear commutator of \( T \) and \( A \) (see [8-13], [21]). While when \( m > 0 \), it is non-trivial generalizations of the commutators. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [2-6]). The purpose of this paper is to study the boundedness properties for the multilinear operator \( T^A \) on Besov spaces. In Section 4, some applications of Theorem in this paper are given.

First, let us introduce some notations. Throughout this paper, \( Q \) will denote a cube of \( \mathbb{R}^n \) with sides parallel to the axes. For a locally integrable function \( f \), the sharp function of \( f \) is defined by
\[ f^\#(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy, \]
where, and in what follows, \( f_Q = |Q|^{-1} \int_Q f(x) dx \). It is well-known that (see [14], [15])
\[ f^\#(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y) - c| dy. \]

For \( \beta \geq 0 \), the Besov space \( \dot{\lambda}_\beta \) is the space of functions \( f \) such that
\[ ||f||_{\dot{\lambda}_\beta} = \sup_{x, h \in \mathbb{R}^n \atop h \neq 0} \left| \Delta_h^{[\beta+1]} f(x) \right| / ||h||^\beta < \infty, \]
where \( \Delta_h^k \) denotes the \( k \)-th difference operator (see [18]).

**Definition 1.** Let \( 0 < p, q \leq \infty, \alpha \in \mathbb{R} \). For \( k \in \mathbb{Z} \), set \( B_k = \{ x \in \mathbb{R}^n : |x| \leq 2^k \} \) and \( C_k = B_k \setminus B_{k-1} \). Denote by \( \chi_k \) the characteristic function of \( C_k \) and \( \chi_0 \) the characteristic function of \( B_0 \).

1. The homogeneous Herz space is defined by
\[ \dot{K}^p_q = \{ f \in L^q_{loc}(\mathbb{R}^n \setminus \{0\}) : ||f||_{\dot{K}^p_q} < \infty \}, \]
Theorem 2. \(0 \leq \eta < n\) and \(j\) bounded from \(L_{\alpha, q}(R^n)\) to \(L^\infty(R^n)\) for any \(n/(\beta + \delta) \leq p \leq n/\delta\).

Theorem 2. Let \(0 < \beta < \min(1/\varepsilon/l, 1/q_2 = 1/q_1 - (l\beta + \delta)/n, \max(-n/q_2 - 1, -n/q_2 - \varepsilon) < \alpha \leq -n/q_2\) and \(D\gamma A_j \in \hat{\Lambda}_\beta\) for all \(\gamma\) with \(|\gamma| = m_j\) and \(j = 1, \ldots, l\). Suppose that \(T^A\) is bounded from \(L^\infty(R^n)\) to \(L^\infty(R^n)\) for any \(0 \leq \eta < n\), \(1 < r < n/\eta\) and \(1/r - 1/s = \eta/n\). Then \(T^A\) is bounded from \(K_{q_1}^{\alpha, p}(R^n)\) to \(CL_{\alpha, q}(R^n)\).

Remark. Theorem 4 also hold for the nonhomogeneous Herz type Hardy space.

3. Proofs of Theorems

To prove the theorems, we need the following lemmas.

Lemma 1 (see [18]). For \(0 < \beta < 1, 1 \leq p \leq \infty\), we have
\[
\|b\|_{\hat{\Lambda}_\beta} \approx \sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |b(x) - b_Q| dx \approx \sup_Q \frac{1}{|Q|^{1/2}} \left( \frac{1}{|Q|} \int_Q |b(x) - b_Q|^p dx \right)^{1/p} 
\]
\[
\approx \sup_Q \inf_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |b(x) - c| dx \approx \sup_Q \inf_Q \frac{1}{|Q|^{1/2}} \left( \frac{1}{|Q|} \int_Q |b(x) - c|^p dx \right)^{1/p}.
\]
Lemma 2 (see [5]). Let \( A \) be a function on \( \mathbb{R}^n \) and \( D^\gamma A \in L^q(\mathbb{R}^n) \) for \( |\gamma| = m \) and some \( q > n \). Then
\[
|R_m(A; x, y)| \leq C|x - y|^m \sum_{|\gamma| = m} \left( \frac{1}{Q(x, y)} \int_{Q(x, y)} |D^\gamma A(z)|^q dz \right)^{1/q},
\]
where \( Q(x, y) \) is the cube centered at \( x \) and having side length \( 5\sqrt{n}|x - y| \).

Lemma 3 (see [17]). For \( \alpha < 0, 0 < q < \infty \), we have
\[
||f||_{K^\alpha_q} \approx \sup_{\mu \in \mathbb{Z}} 2^{\mu\alpha} ||f \chi_{B_\mu}||_{L^q}.
\]

Proof of Theorem 1. Without loss of generality, we may assume \( l = 2 \). By Lemma 1, it is only to prove that there exists a constant \( C_0 \) such that
\[
\frac{1}{|Q|^{1+(2\beta+\delta)/n-1/p}} \int_Q |T^A(f)(x) - C_0| dx \leq C||f||_{L^p}.
\]
Fix a cube \( Q = Q(x_0, d) \). Let \( \tilde{Q} = 5\sqrt{n}Q \) and
\[
\tilde{A}_j(x) = A_j(x) - \sum_{|\gamma| = m} \frac{1}{|\gamma|!} (D^\gamma A_j)_{\tilde{Q}} x^\gamma.
\]
Then \( R_m(A_j; x, y) = R_m(\tilde{A}_j; x, y) \) and \( D^\gamma \tilde{A}_j = D^\gamma A_j - (D^n A_j)_{\tilde{Q}} \) for \( |\gamma| = m_j \).

We write, for \( f_1 = f \chi_{\tilde{Q}} \) and \( f_2 = f \chi_{\mathbb{R}^n \setminus \tilde{Q}} \),
\[
F^A_i(f)(x) = \int_{\mathbb{R}^n} \prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j; x, y)_{\tilde{Q}} f_i(x, y) f(y) dy
\]

\[
= \int_{\mathbb{R}^n} \prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j; x, y)_{\tilde{Q}} f_i(x, y) f_2(y) dy
\]

\[
+ \int_{\mathbb{R}^n} \prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, y)_{\tilde{Q}} f_i(x, y) f_1(y) dy
\]

\[
- \sum_{|\gamma_1| = m_1} \frac{1}{\gamma_1!} \int_{\mathbb{R}^n} R_{m_2}(\tilde{A}_2; x, y)(x - y)^{\gamma_1} D^{\gamma_1} \tilde{A}_1(x) f_i(x, y) f_1(y) dy
\]

\[
- \sum_{|\gamma_2| = m_2} \frac{1}{\gamma_2!} \int_{\mathbb{R}^n} R_{m_1}(\tilde{A}_1; x, y)(x - y)^{\gamma_2} D^{\gamma_2} \tilde{A}_2(x) f_i(x, y) f_1(y) dy
\]

\[
+ \sum_{|\gamma_1| = m_1} \frac{1}{\gamma_1!\gamma_2!} \int_{\mathbb{R}^n} (x - y)^{\gamma_1 + \gamma_2} D^{\gamma_1} \tilde{A}_1(x) D^{\gamma_2} \tilde{A}_2(y) f_i(x, y) f_1(y) dy,
\]
then

\[
\left| T^\Lambda(f)(x) - T^\Lambda(f_2)(x_0) \right| = \left| |F^\Lambda_t(f)(x)| - |F^\Lambda_t(f_2)(x_0)| \right|
\]

\[
\leq \left| |F^\Lambda_t(f)(x)| - |F^\Lambda_t(f_2)(x_0)| \right|
\]

\[
\leq \left\| \int_{R^n} \prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, y) \left( \sum_{|\gamma_j|=m_j} \frac{1}{\gamma_j!} \int_{R^n} \frac{R_{m_j}(\tilde{A}_j; x, y)(x-y)^{\gamma_j}}{|x-y|^m} D^{\gamma_1} \tilde{A}_1(y) F_1(x, y) f_1(y) dy \right) \right\| dx
\]

\[
+ \left\| \sum_{|\gamma_1|=m_1} \frac{1}{\gamma_1!} \int_{R^n} \frac{R_{m_1}(\tilde{A}_1; x, y)(x-y)^{\gamma_1}}{|x-y|^m} D^{\gamma_1} \tilde{A}_1(y) F_1(x, y) f_1(y) dy \right\| dx
\]

\[
+ \left\| \sum_{|\gamma_2|=m_2} \frac{1}{\gamma_2!} \int_{R^n} \frac{(x-y)^{\gamma_1+\gamma_2} D^{\gamma_1} \tilde{A}_1(y) D^{\gamma_2} \tilde{A}_2(y) F_1(x, y) f_1(y) dy}{|x-y|^m} \right\| dx
\]

Thus

\[
\frac{1}{|Q|^{1+(2\beta+\delta)/n-1/p}} \int_Q \left| T^\Lambda(f)(x) - T^\Lambda(f_2)(x_0) \right| dx
\]

\[
\leq \frac{1}{|Q|^{1+(2\beta+\delta)/n-1/p}} \int_Q \left\| \int_{R^n} \prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, y) \left( \sum_{|\gamma_j|=m_j} \frac{1}{\gamma_j!} \int_{R^n} \frac{R_{m_j}(\tilde{A}_j; x, y)(x-y)^{\gamma_j}}{|x-y|^m} D^{\gamma_1} \tilde{A}_1(y) F_1(x, y) f_1(y) dy \right) \right\| dx
\]

\[
+ \frac{1}{|Q|^{1+(2\beta+\delta)/n-1/p}} \int_Q \left\| \sum_{|\gamma_1|=m_1} \frac{1}{\gamma_1!} \int_{R^n} \frac{R_{m_1}(\tilde{A}_1; x, y)(x-y)^{\gamma_1}}{|x-y|^m} D^{\gamma_1} \tilde{A}_1(y) F_1(x, y) f_1(y) dy \right\| dx
\]

\[
+ \frac{1}{|Q|^{1+(2\beta+\delta)/n-1/p}} \int_Q \left\| \sum_{|\gamma_2|=m_2} \frac{1}{\gamma_2!} \int_{R^n} \frac{(x-y)^{\gamma_1+\gamma_2} D^{\gamma_1} \tilde{A}_1(y) D^{\gamma_2} \tilde{A}_2(y) F_1(x, y) f_1(y) dy}{|x-y|^m} \right\| dx
\]
\[
\times \int_Q \left| \sum_{|\gamma_1| = m_1} \int_{R^n} \frac{(x - y)^{\gamma_1 + \gamma_2} D^{\gamma_1} A_1(y) D^{\gamma_2} A_2(y)}{|x - y|^n} F_i(x, y) f_1(y) dy \right| dx \\
+ \frac{1}{|Q|^{1+\delta/n-1/p}} \int_Q \left| T^{\tilde{A}}(f_2)(x) - T^{\tilde{A}}(f_2)(x_0) \right| dx
\]

\[:= I_1 + I_2 + I_3 + I_4 + I_5.\]

Now, let us estimate \( I_1, I_2, I_3, I_4 \) and \( I_5 \), respectively. First, by Lemma 5 and Lemma 2, we get, for \( x \in Q \) and \( y \in Q \),

\[
|R_m(\tilde{A}; x, y)| \\
\leq C|x - y|^m \sum_{|\gamma| = m} \sup_{x \in Q} |D^{\gamma} A_j(x) - (D^{\gamma} A_j)_Q| \\
\leq C|x - y|^m |Q|^{2/n} \sum_{|\gamma| = m} \|D^{\gamma} A_j\|_{L^p},
\]

thus, by the \((L^r, L^s)\)-boundedness of \( T \) with \( 1 < r < p \leq n/\delta \) and \( 1/s = 1/r - \delta/n \), we obtain, using Hölder’s inequality,

\[
I_1 \leq C \prod_{j=1}^2 \left( \sum_{|\gamma_j| = m_j} \|D^{\gamma_j} A_j\|_{L^p} \right) \frac{1}{|Q|^{1+\delta/n-1/p}} \int_Q |T(f_1)(x)| dx \\
\leq C \prod_{j=1}^2 \left( \sum_{|\gamma_j| = m_j} \|D^{\gamma_j} A_j\|_{L^p} \right) \frac{|Q|^{1-1/s}}{|Q|^{1+\delta/n-1/p}} \left( \int_Q |T(f_1)(x)|^s dx \right)^{1/s} \\
\leq C \prod_{j=1}^2 \left( \sum_{|\gamma_j| = m_j} \|D^{\gamma_j} A_j\|_{L^p} \right) \frac{|Q|^{1-1/s}}{|Q|^{1+\delta/n-1/p}} \left( \int_Q |f(x)|^r dx \right)^{1/r} \\
\leq C \prod_{j=1}^2 \left( \sum_{|\gamma_j| = m_j} \|D^{\gamma_j} A_j\|_{L^p} \right) \frac{|Q|^{1-1/s}}{|Q|^{1+\delta/n-1/p}} \|f\|_{L^p} |Q|^{1/r-1/p} \\
\leq C \prod_{j=1}^2 \left( \sum_{|\gamma_j| = m_j} \|D^{\gamma_j} A_j\|_{L^p} \right) \|f\|_{L^p}.
\]

For \( I_2 \), using Hölder’s inequality, we get, for \( 1 < r < p \leq n/\delta \) and \( 1/s = 1/r - \delta/n \),

\[
I_2 \leq C \sum_{|\gamma_2| = m_2} \|D^{\gamma_2} A_2\|_{L^p} \sum_{|\gamma_1| = m_1} \frac{1}{|Q|^{1+\delta/\gamma_1/n-1/p}} \int_Q |T(D^{\gamma_1} A_1 f_1)(x)| dx \\
\leq C \sum_{|\gamma_2| = m_2} \|D^{\gamma_2} A_2\|_{L^p}
\]
\[ \times \sum_{|\gamma| = m_1} \frac{|Q|^{1-1/s}}{|Q|^{1+(3+\delta)/n-1/p}} \left( \int_Q |T(D^{\gamma_1} A - (D^{\gamma_1} A)_{Q}) f_1(x)|^s dx \right)^{1/s} \]

\[ \leq C \sum_{|\gamma_2| = m_2} \|D^{\gamma_2} A_2\|_{\lambda, \beta} \]

\[ \times \sum_{|\gamma_1| = m_1} \frac{|Q|^{1-1/s}}{|Q|^{1+(3+\delta)/n-1/p}} \left( \int_{R^n} |(D^{\gamma_1} A(x) - (D^{\gamma_1} A)_{Q}) f_1(x)|^r dx \right)^{1/r} \]

\[ \leq C \prod_{j=1}^2 \left( \sum_{|\gamma_j| = m_j} \|D^{\gamma_j} A_j\|_{\lambda, \alpha} \right) \frac{|Q|^{1-1/s}}{|Q|^{1+(3+\delta)/n-1/p}} \left( \int_Q |f(x)|^r dx \right)^{1/r} \]

\[ \leq C \prod_{j=1}^2 \left( \sum_{|\gamma_j| = m_j} \|D^{\gamma_j} A_j\|_{\lambda, \alpha} \right) \|f\|_{L^r} |Q|^{1/r-1/p} \]

\[ \leq C \prod_{j=1}^2 \left( \sum_{|\gamma_j| = m_j} \|D^{\gamma_j} A_j\|_{\lambda, \alpha} \right) \|f\|_{L^p}. \]

For \( I_3 \), similar to the proof of \( I_2 \), we get

\[ I_3 \leq C \prod_{j=1}^2 \left( \sum_{|\gamma_j| = m_j} \|D^{\gamma_j} A_j\|_{\lambda, \alpha} \right) \|f\|_{L^p}. \]

Similarly, for \( I_4 \), we obtain, for \( 1 < r < p \leq n/\delta \) and \( 1/s = 1/r - \delta/n \),

\[ I_4 \]

\[ \leq C \sum_{|\gamma_1| = m_1} \frac{1}{|Q|^{1+(3+\delta)/n-1/p}} \int_Q |T(D^{\gamma_1} \tilde{A}_1 D^{\gamma_2} \tilde{A}_2 f_1)(x)| dx \]

\[ \leq C \sum_{|\gamma_2| = m_2} \frac{|Q|^{1-1/s}}{|Q|^{1+(3+\delta)/n-1/p}} \left( \int_{R^n} |T(D^{\gamma_1} \tilde{A}_1 D^{\gamma_2} \tilde{A}_2 f_1(x)|^s dx \right)^{1/s} \]

\[ \leq C \sum_{|\gamma_1| = m_1} \frac{|Q|^{1-1/s}}{|Q|^{1+(3+\delta)/n-1/p}} \left( \int_{R^n} |D^{\gamma_1} \tilde{A}_1(x) D^{\gamma_2} \tilde{A}_2 f_1(x)|^r dx \right)^{1/r} \]

\[ \leq C \prod_{j=1}^2 \left( \sum_{|\gamma_j| = m_j} \|D^{\gamma_j} A_j\|_{\lambda, \alpha} \right) \frac{|Q|^{1-1/s}}{|Q|^{1+(3+\delta)/n-1/p}} \left( \int_Q |f(x)|^r dx \right)^{1/r} \]
\[ \leq C \prod_{j=1}^{2} \left( \sum_{|\gamma_j|=m_j} \|D^{\gamma_j} A_j\|_{\lambda, \beta} \right) \frac{|Q|^{1-\delta}}{|Q|^{1+d/n-1/p}} \|f\|_{L^p} |Q|^{1/r-1/p} \]

For \( I_5 \), we write

\[
F_t^A(f_2)(x) - F_t^A(f_2)(x_0)
= \int_{R^n} \left( F_t(x, y) - F_t(x_0, y) \right) \prod_{j=1}^{2} R_m, (\tilde{A}_j; x, y) f_2(y) dy
+ \int_{R^n} \left( R_m, (\tilde{A}_1; x, y) - R_m, (\tilde{A}_1; x_0, y) \right) \frac{R_m, (\tilde{A}_2; x, y)}{|x_0 - y|^m} F_t(x_0, y) f_2(y) dy
+ \int_{R^n} \left( R_m, (\tilde{A}_2; x, y) - R_m, (\tilde{A}_2; x_0, y) \right) \frac{R_m, (\tilde{A}_1; x_0, y)}{|x_0 - y|^m} F_t(x_0, y) f_2(y) dy
- \sum_{|\gamma_1|=m_1} \frac{1}{\gamma_1!} \int_{R^n} \left[ R_m, (\tilde{A}_2; x, y)(x-y)^{\gamma_1} \frac{F_t(x, y)}{|x-y|^m} - \frac{R_m, (\tilde{A}_2; x_0, y)(x-y)^{\gamma_1}}{|x_0-y|^m} F_t(x_0, y) \right] \times D^{\gamma_1} \tilde{A}_1(y) f_2(y) dy
- \sum_{|\gamma_2|=m_2} \frac{1}{\gamma_2!} \int_{R^n} \left[ R_m, (\tilde{A}_1; x, y)(x-y)^{\gamma_2} \frac{F_t(x, y)}{|x-y|^m} - \frac{R_m, (\tilde{A}_1; x_0, y)(x_0-y)^{\gamma_2}}{|x_0-y|^m} F_t(x_0, y) \right] \times D^{\gamma_2} \tilde{A}_2(y) f_2(y) dy
+ \sum_{|\gamma_1|=m_1} \frac{1}{\gamma_1! \gamma_2!} \int_{R^n} \left[ (x-y)^{\gamma_1+\gamma_2} \frac{F_t(x, y)}{|x-y|^m} - \frac{(x_0-y)^{\gamma_1+\gamma_2}}{|x_0-y|^m} F_t(x_0, y) \right] \times D^{\gamma_1} \tilde{A}_1(y) D^{\gamma_2} \tilde{A}_2(y) f_2(y) dy
= I_5^{(1)} + I_5^{(2)} + I_5^{(3)} + I_5^{(4)} + I_5^{(5)} + I_5^{(6)}.
\]

By the following inequality, for \( b \in \lambda, \beta \),

\[ |b(x) - b_Q| \leq \frac{1}{|Q|} \int_Q \|b\|_{\lambda, \beta} |x-y|^\beta dy \leq \|b\|_{\lambda, \beta} (|x-x_0| + d)^\beta, \]
we get
\[ |R_{m_j}(\tilde{A}_j; x, y)| \leq \sum_{|\gamma|=m_j} ||D^{\gamma}A_j||_{\lambda_\alpha} (|x - y| + d)^{m_j + \beta}. \]

Note that \(|x - y| \sim |x_0 - y|\) for \(x \in Q\) and \(y \in \mathbb{R}^n \setminus \tilde{Q}\), we obtain, by the condition of \(F_k\),
\[
||f_{1(1)}|| \leq C \int_{\mathbb{R}^n \setminus \tilde{Q}} \left( \frac{|x - x_0|}{|x_0 - y|^{m+n+1-\delta}} + \frac{|x - x_0|^{\varepsilon}}{|x_0 - y|^{m+n+\varepsilon-\sigma}} \right)^2 \prod_{j=1}^{2} |R_{m_j}(\tilde{A}_j; x, y)||f(y)|dy
\]
\[
\leq C \sum_{j=1}^{2} \left( \sum_{|\gamma_j|=m_j} ||D^{\gamma_j}A_j||_{\lambda_\alpha} \right) \times \sum_{k=0}^{\infty} \left( \frac{d}{(2k\varepsilon)^{n+1-\delta-2\beta}} + \frac{d^{\varepsilon}}{(2k\varepsilon)^{n+\varepsilon-\delta-2\beta}} \right) \int_{2^k \tilde{Q}} |f(y)|dy
\]
\[
\leq C \sum_{j=1}^{2} \left( \sum_{|\gamma_j|=m_j} ||D^{\gamma_j}A_j||_{\lambda_\alpha} \right) \times \sum_{k=0}^{\infty} \left( 2^{k(\delta+2\beta-1-n/p)} + 2^{k(\delta+2\beta-\varepsilon-n/p)} \right) ||f||_{L^p}
\]
\[
\leq C \sum_{j=1}^{2} \left( \sum_{|\gamma_j|=m_j} ||D^{\gamma_j}A_j||_{\lambda_\alpha} \right) \times \sum_{k=0}^{\infty} \left( 2^{k(\delta+2\beta-1-n/p)} + 2^{k(\delta+2\beta-\varepsilon-n/p)} \right) ||f||_{L^p}.
\]

For \(f_{2(1)}^{(2)}\), by the formula (see [5]):
\[
R_{m_j}(\tilde{A}_j; x, y) - R_{m_j}(\tilde{A}_j; x_0, y) = \sum_{|\eta|<m_j} \frac{1}{\eta!} R_{m_j-|\eta|}(D^{\eta}A_j; x, x_0)(x - y)^n
\]
and Lemma 5, we get
\[
||f_{2(1)}^{(2)}|| \leq C \sum_{j=1}^{2} \left( \sum_{|\gamma_j|=m_j} ||D^{\gamma_j}A_j||_{\lambda_\alpha} \right) \sum_{k=0}^{\infty} \int_{2^k \tilde{Q} \setminus 2^{k+1} \tilde{Q}} \frac{|x - x_0|}{|x_0 - y|^{n+1-\delta-2\beta}} |f(y)|dy
\]
$\leq C \prod_{j=1}^{2} \left( \sum_{|\gamma|=m_j} \|D^\gamma A_j\|_{\lambda, \delta} \right) |Q|^{\delta/n + 2\beta/n - 1/p} \|f\|_{L^p}.$

Similarly,

$\|I_5^{(3)}\| \leq C \prod_{j=1}^{2} \left( \sum_{|\gamma|=m_j} \|D^\gamma A_j\|_{\lambda, \delta} \right) |Q|^{\delta/n + 2\beta/n - 1/p} \|f\|_{L^p}.$

For $I_5^{(4)}$, similar to the estimates of $I_5^{(1)}$ and $I_5^{(2)}$, we obtain,

$\|I_5^{(4)}\| \leq C \sum_{|\gamma|=m_1} \int_{R^n \setminus \tilde{Q}} \left\| \frac{(x - y)^{\gamma_1} F_i(x, y)}{|x - y|^m} - \frac{(x_0 - y)^{\gamma_1} F_i(x_0, y)}{|x_0 - y|^m} \right\| dy \times \|R_{m_2}(\tilde{A}_2; x, y)||D^{\gamma_2} \tilde{A}_1(y)||f(y)|dy$

$+ C \sum_{|\gamma|=m_2} \int_{R^n \setminus \tilde{Q}} |R_{m_2}(\tilde{A}_2; x, y) - R_{m_2}(\tilde{A}_2; x_0, y)|$

$\times \left\| \frac{(x_0 - y)^{\gamma_2} F_i(x_0, y)}{|x_0 - y|^m} \right\| \|D^{\gamma_2} \tilde{A}_1(y)||f(y)|dy$

$\leq C \prod_{j=1}^{2} \left( \sum_{|\gamma|=m_j} \|D^\gamma A_j\|_{\lambda, \delta} \right) |Q|^{\delta/n + 2\beta/n - 1/p}$

$\times \sum_{k=0}^{\infty} \left( 2^{k(\delta + 2\beta - 1 - n/p)} + 2^{k(\delta + 2\beta - \varepsilon - n/p)} \right) \|f\|_{L^p}$

$\leq C \prod_{j=1}^{2} \left( \sum_{|\gamma|=m_j} \|D^\gamma A_j\|_{\lambda, \delta} \right) |Q|^{\delta/n + 2\beta/n - 1/p} \|f\|_{L^p}.$

Similarly,

$\|I_5^{(5)}\| \leq C \prod_{j=1}^{2} \left( \sum_{|\gamma|=m_j} \|D^\gamma A_j\|_{\lambda, \delta} \right) |Q|^{\delta/n + 2\beta/n - 1/p} \|f\|_{L^p}.$

For $I_5^{(6)}$, we get

$\|I_5^{(6)}\|$

$\leq C \sum_{|\gamma|=m_1,|\gamma_2|=m_2} \int_{R^n \setminus \tilde{Q}} \left\| \frac{(x - y)^{\gamma_1+\gamma_2} F_i(x, y)}{|x - y|^m} - \frac{(x_0 - y)^{\gamma_1+\gamma_2} F_i(x_0, y)}{|x_0 - y|^m} \right\|$

$\times \|D^{\gamma_2} \tilde{A}_1(y)||D^{\gamma_2} \tilde{A}_2(y)||f(y)|dy$
\[ \leq C \sum_{|\gamma| = m_1, |\gamma| = m_2} \sum_{k=0}^{\infty} \int_{2^k \mathbb{Q} \setminus 2^{k+1} \mathbb{Q}} \left( \frac{|x-x_0|}{|x_0-y|^{n+1-\delta}} + \frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon-\delta}} \right) |D_{\gamma_1} \tilde{A}_1(y)||D_{\gamma_2} \tilde{A}_2(y)||f(y)|dy \]
\[ \leq C \prod_{j=1}^{2} \left( \sum_{|\gamma| = m_j} ||D_{\gamma_j} A_j||_{\Lambda_\delta} \right) \]
\[ \times \sum_{k=0}^{\infty} \int_{2^{k+1} \mathbb{Q} \setminus 2^k \mathbb{Q}} \left( \frac{|x-x_0|}{|x_0-y|^{n+1-2\delta}} + \frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon-2\delta}} \right) |f(y)|dy \]
\[ \leq C \prod_{j=1}^{2} \left( \sum_{|\gamma| = m_j} ||D_{\gamma_j} A_j||_{\Lambda_\delta} \right) |Q|^{\delta/n+2\beta/n-1/p} ||f||_{L^p}. \]

Thus
\[ |T^\Lambda (f_2)(x) - T^\Lambda (f_2)(x_0)| \]
\[ \leq C \prod_{j=1}^{2} \left( \sum_{|\gamma| = m_j} ||D_{\gamma_j} A_j||_{\Lambda_\delta} \right) |Q|^{\delta/n+2\beta/n-1/p} ||f||_{L^p}. \]

and
\[ I_5 \leq C \prod_{j=1}^{2} \left( \sum_{|\gamma| = m_j} ||D_{\gamma_j} A_j||_{\Lambda_\delta} \right) ||f||_{L^p}. \]

This completes the proof of the theorem. \(\square\)

**Proof of Theorem 2.** Without loss of generality, we may assume \(l = 2\). Fix a ball \(B = B(0,d)\), there exists \(\mu_0 \in \mathbb{Z}\) such that \(2^\mu_0 - 1 \leq d < 2^\mu_0\). Let \(\tilde{A}_j(x) = A_j(x) - \sum_{|\gamma| = m} \frac{1}{\gamma!} (D^\gamma A_j)_{B_{\rho_0}} x^\gamma\), then \(R_m (A_j; x, y) = R_m (\tilde{A}_j; x, y)\) and \(D^\gamma \tilde{A}_j = D^\gamma A_j - (D^\gamma A_j)_{B_{\rho_0}}\) for \(|\gamma| = m_j\). We choose \(x_0\) such that \(2d < |x_0| < 3d\). It is only to prove that
\[ 2^{\mu_0(n+n/q_2)} \left( \frac{1}{2^{\mu_0 n}} \int_{|x| \leq 2^\rho_0} |T^A(f)(x) - T^\Lambda (f_2)(x_0)|^{q_2} dx \right)^{1/q_2} \leq C ||f||_{K_{\delta_1}^{\infty}}. \]

We write, for \(f_1 = f \chi_{4B_{\rho_0}}\) and \(f_2 = f \chi_{R^n \setminus 4B_{\rho_0}}\),
\[ \left| T^A(f)(x) - T^\Lambda (f_2)(x_0) \right| = \left| ||F^A_t (f)(x)|| - ||F^\Lambda_t (f_2)(x_0)|| \right| \]
\[ \leq \left| ||F^A_t (f)(x)|| - ||F^\Lambda_t (f_2)(x_0)|| \right| + \left| T^A(f_1)(x) \right| - \left| T^\Lambda (f_2)(x) - T^\Lambda (f_2)(x_0) \right|. \]
then

\[
2^{\mu_0(\alpha+\eta/q_2)} \left( \frac{1}{2^{\mu_0 n}} \int_{|x| \leq 2^{\mu_0}} \left| T^A(f)(x) - T^\Lambda(f_2)(x_0) \right|^q dx \right)^{1/q_2} \\
\leq 2^{\mu_0(\alpha+\eta/q_2)} \left( \frac{1}{2^{\mu_0 n}} \int_{|x| \leq 2^{\mu_0}} \left| T^\Lambda(f_1)(x) \right|^q dx \right)^{1/q_2} \\
+ 2^{\mu_0(\alpha+\eta/q_2)} \left( \frac{1}{2^{\mu_0 n}} \int_{|x| \leq 2^{\mu_0}} \left| T^\Lambda(f_2)(x) - T^\Lambda(f_2)(x_0) \right|^q dx \right)^{1/q_2} \\
:= J_1 + J_2.
\]

For \(J_1\), by the \((L^{q_1}, L^{q_2})\)-boundedness of \(T^A\) and Lemma 3, we get

\[
J_1 \leq C 2^{\mu_0(\alpha+\eta/q_2)} 2^{-\mu_0 n/q_2} \left( \int_{R^n} |f_1(x)|^{q_1} dx \right)^{1/q_1} \\
\leq C 2^{\mu_0(\alpha)} \|f_X f_0\|_{L^{q_1}} \\
\leq C \|f\|_{K^{\alpha, \infty}}.
\]

For \(J_2\), similar to the estimates of Theorem 1, we obtain, by Hölder’s inequality and recall that \(\max(-\eta/q_2 - 1, -\eta/q_2 - \varepsilon) < \alpha, 1/q_2 = 1/q_1 = (2\beta + \delta)/n\),

\[
|T^\Lambda(f_2)(x) - T^\Lambda(f_2)(x_0)| \\
\leq C \prod_{j=1}^{2} \left( \sum_{|\gamma_j| = m_j} \|D^{\gamma_j} A_j\|_{\Lambda, \rho} \right) \\
\times \sum_{k=1}^{\infty} \left( \frac{2^{\mu_0}}{2(\mu_0 + k)(n + 1 - \delta - 2\beta)} + \frac{2^{\mu_0}}{2(\mu_0 + k)(n + \varepsilon - 2\delta)} \right) \int_{C_{\mu_0 + k}} |f(y)| dy \\
\leq C \prod_{j=1}^{2} \left( \sum_{|\gamma_j| = m_j} \|D^{\gamma_j} A_j\|_{\Lambda, \rho} \right) \\
\times \sum_{k=1}^{\infty} \left( \frac{2^{\mu_0}}{2(\mu_0 + k)(n + 1 - \delta - 2\beta)} + \frac{2^{\mu_0}}{2(\mu_0 + k)(n + \varepsilon - 2\delta)} \right) \|f_X f_0 + k\|_{L^{q_1}} 2^{(\mu_0 + k) n(1 - \frac{\alpha}{n})} \\
\leq C \prod_{j=1}^{2} \left( \sum_{|\gamma_j| = m_j} \|D^{\gamma_j} A_j\|_{\Lambda, \rho} \right) \\
\times \sum_{k=1}^{\infty} \left( \frac{2^{\mu_0}}{2(\mu_0 + k)(1 - \delta - 2\beta)} + \frac{2^{\mu_0}}{2(\mu_0 + k)(\varepsilon - 2\delta)} \right) \frac{2^{(\mu_0 + k)(-\eta/q_1)}}{2^{\alpha(\mu_0 + k)}} \|f\|_{K^{\alpha, \infty}}.
\]
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\[ \leq C \prod_{j=1}^{2} \left( \sum_{|\gamma_j| = m_j} \| D^{\gamma_j} A_j \|_{\lambda, \beta} \right) \]
\[ \times \sum_{k=1}^{\infty} \left( 2^{k(\delta + 2 \beta - 1 - \alpha - n/q_1)} + 2^{k(\delta + 2 \beta - \varepsilon - \alpha - n/q_1)} \right) 2^{\mu_0(\delta + 2 \beta - \alpha - n/q_1)} \| f \|_{K_{\alpha 1}^\beta} \]
\[ \leq C \prod_{j=1}^{2} \left( \sum_{|\gamma_j| = m_j} \| D^{\gamma_j} A_j \|_{\lambda, \beta} \right) 2^{\mu_0(\delta + 2 \beta - \alpha - n/q_1)} \| f \|_{K_{\alpha 1}^\beta} \]

thus
\[ J_2 \leq C \prod_{j=1}^{2} \left( \sum_{|\gamma_j| = m_j} \| D^{\gamma_j} A_j \|_{\lambda, \beta} \right) 2^{\mu_0(\alpha + n/q_2)} 2^{\mu_0(\delta + 2 \beta - \alpha - n/q_1)} \| f \|_{K_{\alpha 1}^\beta} \]
\[ \leq C \prod_{j=1}^{2} \left( \sum_{|\gamma_j| = m_j} \| D^{\gamma_j} A_j \|_{\lambda, \beta} \right) \| f \|_{K_{\alpha 1}^\beta}. \]

This completes the proof of the theorem. \( \square \)

4. Applications

Now we give some applications of results in this paper.

Application 1. Littlewood-Paley operators.

Fixed \( 0 \leq \delta < n - 1, \varepsilon > 0 \) and \( \mu > 1 \). Let \( \psi \) be a fixed function which satisfies:

1. \( \int_{R^n} \psi(x)dx = 0 \),
2. \( |\psi(x)| \leq C(1 + |x|)^{-(n+1)} \),
3. \( |\psi(x + y) - \psi(x)| \leq C|y|^\delta (1 + |x|)^{-(n+1+\varepsilon)} \) when \( 2|y| < |x| \).

We denote \( \Gamma(x) = \{(y, t) \in R^{n+1} : |x - y| < t\} \) and the characteristic function of \( \Gamma(x) \) by \( \chi_{\Gamma(x)} \). The Littlewood-Paley multilinear operators are defined by

\[ g^A_{\mu}(f)(x) = \left( \int_0^{\infty} |F_t^A(f)(x)|^2 \frac{dt}{t} \right)^{1/2} \],
\[ S^A_{\psi}(f)(x) = \left[ \int_{\Gamma(x)} |F_t^A(f)(x, y)|^2 \frac{dydt}{t^{n+1}} \right]^{1/2} \]

and

\[ g^A_{\mu}(f)(x) = \left[ \int_{R^{n+1}} \left( \frac{t}{t + |x - y|} \right)^{n\mu} |F_t^A(f)(x, y)|^2 \frac{dydt}{t^{n+1}} \right]^{1/2} \],

where

\[ F_t^A(f)(x) = \int_{R^n} \prod_{j=1}^{l} R_{m_j+1}^{t}(A_j; x, y) \frac{\psi_t(x - y)f(y)dy}{|x - y|^m}. \]
\[ F_t^A(f)(x, y) = \int_{\mathbb{R}^n} \prod_{j=1}^n R_{m_j+1}(A_j; x, z) f(z) \psi_t(y - z) dz \]

and \( \psi_t(x) = t^{-n+\delta} \psi(x/t) \) for \( t > 0 \). Set \( F_t(f)(y) = f * \psi_t(y) \). We also define that
\[
g_\psi(f)(x) = \left( \int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t} \right)^{1/2},
\]
\[
S_\psi(f)(x) = \left( \int \int_{\Gamma(x)} |F_t(f)(y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2},
\]
and
\[
g_\mu(f)(x) = \left( \int \int_{\mathbb{R}^n_+} \left( \frac{t}{t + |x - y|} \right)^{\nu/2} |F_t(f)(y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2},
\]
which are the Littlewood-Paley operators (see [20]). Let \( H \) be the space
\[
H = \left\{ h : ||h|| = \left( \int_0^\infty |h(t)|^2 dt/t \right)^{1/2} < \infty \right\}
\]
or
\[
H = \left\{ h : ||h|| = \left( \int \int_{\mathbb{R}^n_+} |h(y, t)|^2 dydt/t^{n+1} \right)^{1/2} < \infty \right\},
\]
then, for each fixed \( x \in \mathbb{R}^n \), \( F_t^A(f)(x) \) and \( F_t^A(f)(x, y) \) may be viewed as the mapping from \([0, +\infty)\) to \( H \), and it is clear that
\[
g_\psi^A(f)(x) = ||F_t^A(f)(x)||, \quad g_\psi(f)(x) = ||F_t(f)(x)||,
\]
\[
S_\psi^A(f)(x) = ||\chi_{\Gamma(x)} F_t^A(f)(x, y)||, \quad S_\psi(f)(x) = ||\chi_{\Gamma(x)} F_t(f)(y)||
\]
and
\[
g_\mu^A(f)(x) = \left\| \left( \frac{t}{t + |x - y|} \right)^{\nu/2} F_t^A(f)(x, y) \right\|,
\]
\[
g_\mu(f)(x) = \left\| \left( \frac{t}{t + |x - y|} \right)^{\nu/2} F_t(f)(y) \right\|.
\]
It is easily to see that \( g_\psi^A, S_\psi^A \) and \( g_\mu^A \) satisfy the conditions of Theorem 1 and 2(see [8-12]), thus Theorem 1 and 2 hold for \( g_\psi^A, S_\psi^A \) and \( g_\mu^A \).

**Application 2.** Marcinkiewicz operators.

Fixed \( 0 \leq \delta < n - 1, \lambda > 1 \) and \( 0 < \gamma \leq 1 \). Let \( \Omega \) be homogeneous of degree zero on \( \mathbb{R}^n \) with \( \int_{S^{n-1}} \Omega(x')d\sigma(x') = 0 \). Assume that \( \Omega \in \text{Lip}_\gamma(S^{n-1}) \). The Marcinkiewicz multilinear operators are defined by
\[
p_\Omega^A(f)(x) = \left( \int_0^\infty |F_t^A(f)(x)|^2 \frac{dt}{t^\gamma} \right)^{1/2},
\]
\[ \mu^A_S(f)(x) = \left[ \int \int_{\Gamma(x)} |F^A_t(f)(x, y)|^2 \frac{dydt}{t^{n+3}} \right]^{1/2} \]

and

\[ \mu^A(f)(x) = \left[ \int \int_{R^{n+1}} \left( \frac{t}{t + |x - y|} \right)^{n\lambda} |F^A_t(f)(x, y)|^2 \frac{dydt}{t^{n+3}} \right]^{1/2}, \]

where

\[ F^A_t(f)(x) = \int_{|x-y| \leq t} \frac{\prod_{j=1}^l R_{m_j+1}(A_j; x, y)}{|x-y|^m} \frac{\Omega(x - y)}{|x-y|^{n\lambda} + 3} f(y)dy \]

and

\[ F^A_t(f)(x, y) = \int_{|y-z| \leq t} \frac{\prod_{j=1}^l R_{m_j+1}(A_j; y, z)}{|y-z|^m} \frac{\Omega(y - z)}{|y-z|^{n\lambda} + 3} f(z)dz. \]

Set

\[ F_t(f)(x) = \int_{|x-y| \leq t} \frac{\Omega(x - y)}{|x-y|^{n\lambda} + 3} f(y)dy; \]

We also define that

\[ \mu_\Omega(f)(x) = \left( \int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2}, \]

\[ \mu_\Omega(f)(x) = \left( \int \int_{\Gamma(x)} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}} \right)^{1/2} \]

and

\[ \mu_\lambda(f)(x) = \left( \int \int_{R^{n+1}} \left( \frac{t}{t + |x - y|} \right)^{n\lambda} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}} \right)^{1/2}, \]

which are the Marcinkiewicz operators (see [21]). Let \( H \) be the space

\[ H = \left\{ h : ||h|| = \left( \int_0^\infty |h(t)|^2 \frac{dt}{t^3} \right)^{1/2} < \infty \right\} \]

or

\[ H = \left\{ h : ||h|| = \left( \int \int_{R^{n+1}} |h(y, t)|^2 \frac{dydt}{t^{n+3}} \right)^{1/2} < \infty \right\}. \]

Then, it is clear that

\[ \mu^A_\Omega(f)(x) = ||F^A_t(f)(x)||, \quad \mu_\Omega(f)(x) = ||F_t(f)(x)||, \]

\[ \mu^A_\Omega(f)(x) = ||\chi_{\Gamma(x)} F^A_t(f)(x, y)||, \quad \mu_\Omega(f)(x) = ||\chi_{\Gamma(x)} F_t(f)(y)|| \]

and

\[ \mu^A_\lambda(f)(x) = \left\| \left( \frac{t}{t + |x - y|} \right)^{n\lambda/2} F^A_t(f)(x, y) \right\|. \]
\[ \mu_\lambda(f)(x) = \left| \left( \frac{t}{t + |x - y|} \right)^{\lambda/2} F_t(f)(y) \right|. \]

It is easily to see that \( \mu^A_{12}, \mu^A_S \) and \( \mu^A_\lambda \) satisfy the conditions of Theorem 1 and 2 (see [8-10], [13]), thus Theorem 1 and 2 hold for \( \mu^A_{12}, \mu^A_S \) and \( \mu^A_\lambda \).

**Application 3.** Bochner-Riesz operators.

Let \( \delta > (n-1)/2 \), \( B_\delta^A(f)(\xi) = (1 - t^2|\xi|^2)^{\delta/4} \hat{f}(\xi) \) and \( B_\delta^A(z) = t^{-n} B_\delta^A(z/t) \) for \( t > 0 \). Set

\[ F^A_{\delta,t}(f)(x) = \int_{\mathbb{R}^n} \prod_{j=1}^m R_{m_j+1}(A_j; x, y) \frac{B_\delta^A(x - y)f(y)dy}{|x - y|^m}. \]

The maximal Bochner-Riesz multilinear operator are defined by

\[ B^A_{\delta,\ast}(f)(x) = \sup_{t>0} |B^A_{\delta,t}(f)(x)|. \]

We also define that

\[ B^*_\delta(f)(x) = \sup_{t>0} |B^*_\delta(t)(f)(x)| \]

which is the maximal Bochner-Riesz operator (see [14]). Let \( H \) be the space \( H = \{ h : ||h|| = \sup_{t>0} |h(t)| < \infty \} \), then

\[ B^A_{\delta,\ast}(f)(x) = ||B^A_{\delta,\ast}(f)(x)||, \quad B^*_\delta(f)(x) = ||B^*_\delta(f)(x)||. \]

It is easily to see that \( B^A_{\delta,\ast} \) satisfies the conditions of Theorem 1 and 2, thus Theorem 1 and 2 hold for \( B^A_{\delta,\ast} \).

**References**


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