SPECTRAL LOCALIZING SYSTEMS THAT ARE $t$-SPLITTING MULTIPlicative SETS OF IDEALS

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SPECTRAL LOCALIZING SYSTEMS THAT ARE 
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Abstract. Let $D$ be an integral domain with quotient field $K$, $\Lambda$ a nonempty set of height-one maximal $t$-ideals of $D$, $F(\Lambda) = \{ I \subseteq D | I$ is an ideal of $D$ such that $I \nsubseteq P$ for all $P \in \Lambda \}$, and $D_{F(\Lambda)} = \{ x \in K | xA \subseteq D$ for some $A \in F(\Lambda) \}$. In this paper, we prove that if each $P \in \Lambda$ is the radical of a finite type $v$-ideal (resp., a principal ideal), then $D_{F(\Lambda)}$ is a weakly Krull domain (resp., generalized weakly factorial domain) if and only if the intersection $D_{F(\Lambda)} = \cap_{P \in \Lambda} D_P$ has finite character, if and only if $F(\Lambda)$ is a $t$-splitting set of ideals, if and only if $F(\Lambda)$ is $v$-finite.

1. Introduction

Throughout this paper $D$ will be an integral domain with quotient field $K$ and an ideal means an integral ideal. A nonempty set $S$ of ideals of $D$ is said to be multiplicative if $S$ is multiplicatively closed, i.e., if $A, B \in S$ implies $AB \in S$. Let $S$ be a multiplicative set of ideals of $D$. The following overring of $D$

$$D_S = \{ x \in K | xA \subseteq D$ for some $A \in S \}$$

is called the $S$-transform of $D$ or the generalized ring of fractions of $D$ with respect to $S$ (cf. [5]). Let Sat$(S)$ be the set of ideals $C$ of $D$ such that $A \subseteq C$ for some $A \in S$ and $S^\perp = \{ B \subseteq D | B$ is an ideal of $D$ such that $(B + J)_t = D$ for all $J \in S \}$. If $S = \text{Sat}(S)$, then $S$ is called saturated. We say that $S$ is finitely generated if every ideal $I \in S$ contains a finitely generated ideal which is still in $S$, while $S$ is $v$-finite if each $t$-ideal $A \in \text{Sat}(S)$ contains a finitely generated ideal $J$ such that $J_A \in \text{Sat}(S)$. Clearly, each finitely generated multiplicative set of ideals is $v$-finite, but the converse does not hold (see [11, p.124]). If $\Lambda$ is a nonempty set of nonzero prime ideals of $D$, we define

$$F(\Lambda) = \{ A \subseteq D | A$ is an ideal of $D$ such that $A \nsubseteq P$ for all $P \in \Lambda \}.$$
Then $\mathcal{F}(\Lambda)$, called a spectral localizing system, is a saturated multiplicative system of ideals of $D$ and $D_{\mathcal{F}(\Lambda)} = \bigcap_{P \in \Lambda} D_P$ [10, Proposition 5.1.4]. If $P$ is a prime ideal of $D$, we denote $\mathcal{F}(\{P\})$ by $\mathcal{F}(P)$. It is obvious that $\mathcal{F}(\Lambda) = \bigcap_{P \in \Lambda} \mathcal{F}(P)$.

A multiplicative subset $N$ of $D$ is called a $t$-splitting set if for each $0 \neq d \in D$, we have $dD = (AB)_t$ for some integral ideals $A$ and $B$ of $D$, where $A_t \cap sD = sA_t$ for all $s \in N$ and $B_t \cap N \neq \emptyset$ (see [1, 7]). Anderson-Anderson-Zafrullah introduced the concept of $t$-splitting sets and proved that the ring $D + XD_N[X]$ is a PVMD if and only if $D$ is a PVMD and $N$ is a $t$-splitting set [1, Theorem 2.5]. (Recall that $D$ is a Prufer $v$-multiplication domain (PVMD) if each nonzero finitely generated ideal of $D$ is $t$-invertible.) Chang-Dumitrescu-Zafrullah further studied $t$-splitting sets [7] and extended the notion of $t$-splitting sets to multiplicative sets of ideals as follows [8]; $S$ is a $t$-splitting set of ideals if every nonzero principal ideal $dD$ of $D$ can be written as $dD = (AB)_t$ with $A \in \text{Sat}(S)$ and $B \in S^*$. Clearly, if $S$ is a $t$-splitting set of ideals, then $S^*$ is also a $t$-splitting set of ideals [8, Proposition 2]. It is proved that $S$ is a $t$-splitting set of ideals if and only if $S$ is $v$-finite and $dD_S \cap D$ is $t$-invertible for each $0 \neq d \in D$ [8, Proposition 5]. Also, a multiplicative subset $N$ of $D$ is a $t$-splitting set if and only if $N = \{sD|s \in N\}$ is a $t$-splitting set of ideals (cf. [1, Corollary 2.3]).

Let $\Lambda$ be a nonempty set of height-one maximal $t$-ideals of $D$. The purpose of this paper is to study when $\mathcal{F}(\Lambda)$ is a $t$-splitting set of ideals. In particular, we show that if each $P \in \Lambda$ is the radical of a finite type $v$-ideal (resp., principal ideal), then $D_{\mathcal{F}(\Lambda)}$ is a weakly Krull domain (resp., generalized weakly factorial domain) if and only if the intersection $D_{\mathcal{F}(\Lambda)} = \bigcap_{P \in \Lambda} D_P$ has finite character, if and only if $\cap_n P_1 \cdots P_n = (0)$ for each infinite sequence $(P_n)$ of distinct elements of $\Lambda$, if and only if $\mathcal{F}(\Lambda)$ is a $t$-splitting set of ideals, if and only if $t\text{-Max}(D_{\mathcal{F}(\Lambda)}) = \{P_{\mathcal{F}(\Lambda)}|P \in \Lambda\}$, if and only if $\mathcal{F}(\Lambda)$ is finitely generated, and if and only if $\mathcal{F}(\Lambda)$ is $v$-finite.

We first review some notation and definitions. Let $\mathcal{F}(D)$ be the set of nonzero fractional ideals of $D$. For each $I \in \mathcal{F}(D)$, let $I^{-1} = \{x \in K|xI \subseteq D\}$, $I_v = (I^{-1})^{-1}$, and $I_t = \bigcup\{J_u|J \subseteq I$ is a nonzero finitely generated fractional ideal of $D\}$. Obviously, if $I \in \mathcal{F}(D)$ is finitely generated, then $I_v = I_t$. An $I \in \mathcal{F}(D)$ is called a divisorial ideal (resp., $t$-ideal) if $I_v = I$ (resp., $I_t = I$). A $t$-ideal $I$ is called a finite type $v$-ideal if $I = (x_1, \ldots, x_n)_e$ for some $0 \neq (x_1, \ldots, x_n) \subseteq I$. An $I \in \mathcal{F}(D)$ is said to be $t$-invertible if $(II_t)^{-1} = D$. It is known that if $I$ is $t$-invertible, then $I_t$ is a finite type $v$-ideal. Let $t\text{-Max}(D)$ be the set of ideals maximal among proper integral $t$-ideals of $D$. It is well known that each ideal $P \in t\text{-Max}(D)$ is a prime ideal, $t\text{-Max}(D) \neq \emptyset$ if $D$ is not a field, and $D = \bigcap_{P \in t\text{-Max}(D)} D_P$. We say that an ideal $P \in t\text{-Max}(D)$ is a maximal $t$-ideal and that $D$ has a $t$-dimension one, denoted by $t\text{-dim}(D) = 1$, if each maximal $t$-ideal of $D$ has height-one. Let $X^1(D)$ be the set of height-one prime ideals of $D$; so $t\text{-dim}(D) = 1 \iff t\text{-Max}(D) = X^1(D)$. Examples of integral domains of $t$-dimension one include (weakly) Krull domains and one-dimensional integral domains. For more on the $v$- and the $t$-operation, the reader may consult [12, Sections 32 and 34].
Let $\mathcal{S}$ be a multiplicative set of ideals of $D$. If $I$ is a fractional ideal of $D$, then $I_\mathcal{S} = \{ x \in K | xA \subseteq I \text{ for some } A \in \mathcal{S} \}$ is a fractional ideal of $D_\mathcal{S}$. In particular, if $I$ is a prime ideal of $D$, then $I_\mathcal{S}$ is a prime ideal of $D_\mathcal{S}$. We call $\mathcal{S}^\perp$ the $t$-complement of $\mathcal{S}$. Let $A, B_1, B_2, C$ be ideals of $D$ such that $A \in \mathcal{S}$, $B_i \in \mathcal{S}^\perp$, and $B_1 \subseteq C$. Then $D = (A + B_1)_1 \subseteq (A + C)_1 \subseteq D$, and hence $C \in \mathcal{S}^\perp$. Also, $D = (A + B_1)_1(A + B_2)_1 \subseteq ((A + B_1)_1(A + B_2)_1)_1 \subseteq (A + B_1 B_2)_1 \subseteq D$; so $(A + B_1 B_2)_1 = D$. Thus $\mathcal{S}^\perp$ is a saturated multiplicative set of ideals. Also, $\text{Sat}(\mathcal{S})$ is a saturated multiplicative set of ideals. It is known that $D_\mathcal{S} = D_{\text{Sat}(\mathcal{S})}$ and $D = D_\mathcal{S} \cap D_{\mathcal{S}^\perp}$ [8, Lemma 7].

A nonempty family $\mathcal{F}$ of ideals of $D$ is called a localizing system if

(i) $I \in \mathcal{F}, J$ an ideal of $D, I \subseteq J \Rightarrow J \in \mathcal{F}$;

(ii) $I \in \mathcal{F}, J$ an ideal of $D, (J :_D iD) \in \mathcal{F}$ for all $i \in I \Rightarrow J \in \mathcal{F}$.

It can be easily shown that a localizing system is a saturated multiplicative set of ideals [10, Proposition 5.1.1] and that if $\Lambda$ is a nonempty set of prime ideals of $D$, then $\mathcal{F}(\Lambda)$ is a localizing system [10, Proposition 5.1.4]. A localizing system $\mathcal{F}$ is said to be spectral if $\mathcal{F} = \mathcal{F}(\Lambda)$ for some nonempty set $\Lambda$ of prime ideals of $D$. The reader is referred to the papers [1, 7, 8] for $t$-splitting sets. For more on multiplicative sets of ideals, generalized ring of fractions of $D$, and localizing systems, see, for example, [5], [10, Section 5.1], or [11].

2. Weakly Krull domains

Let $R$ be a commutative ring with identity, and let $I$ be an ideal of $R$. Then there exist only a finite number of prime ideals of $R$ minimal over $I$ under one of the following conditions;

(1) ([16, Theorem 88]) $R$ satisfies the ascending chain condition on radical ideals.

(2) ([13, Theorem 1.6] or [6, Theorem 2.1]) Every prime ideal of $R$ minimal over $I$ is the radical of a finitely generated ideal.

As the $t$-operation analog, El Baghdadi showed that if $D$ satisfies the ascending chain conditions on radical $t$-ideals, then each $t$-ideal of $D$ has a finite number of minimal prime ideals [9, Lemma 3.8]. The following lemma is a generalization of El Baghdadi’s result. The proof is similar to the proofs of [6, Theorem 2.1] and [9, Lemma 3.8], and hence omitted.

Lemma 2.1. Let $I$ be a proper integral $t$-ideal of $D$. If every prime ideal of $D$ minimal over $I$ is the radical of a finite type $v$-ideal, then $I$ has only a finite number of minimal prime ideals.

Lemma 2.2. Let $\Lambda$ be a nonempty subset of $t\text{-Max}(D)$ and $\Sigma = t\text{-Max}(D) \setminus \Lambda$.

(1) $\mathcal{F}(\Lambda)^\perp = \mathcal{F}(\Sigma)$.

(2) If $\Lambda \subseteq X^1(D)$ and $\mathcal{F}(\Lambda)$ is $v$-finite, then $t\text{-Max}(D_{\mathcal{F}(\Lambda)}) = \{ P_{\mathcal{F}(\Lambda)} | P \in \Lambda \}$.
Proof. (1) \((\subseteq)\) Let \(A \in F(\Lambda)\). If \(Q \in \Sigma\), then \(Q \not\subseteq P\) for all \(P \in \Lambda\), and hence \(Q \notin F(\Lambda)\). So \((A + Q)_t = D\), and since \(Q \in t\text{-}\text{Max}(D)\), we have \(A \not\subseteq Q\). Thus \(A \in F(\Sigma)\). \((\supseteq)\) Conversely, assume that \(B\) is an ideal of \(D\) such that \(B \notin F(\Lambda)\). Then \((B + C)_t \subseteq D\) for some \(C \in F(\Lambda)\), and since \(C \not\subseteq P\) for all \(P \in \Lambda\), there exists a maximal \(t\text{-}i\)deal \(Q \in \Sigma\) such that \(B \subseteq (B + C)_t \subseteq Q\); hence \(B \notin F(\Sigma)\). Thus \(F(\Sigma) \subseteq F(\Lambda)\).

(2) \((\subseteq)\) Let \(Q\) be a maximal \(t\text{-}i\)deal of \(D_{F(\Lambda)}\) and \(P = Q \cap D\). Then \(P\) is a prime \(t\text{-}i\)deal of \(D\) \([11, \text{Proposition 1.3}]\). If \(P \notin \Lambda\), then \(P \in F(\Lambda)\) (note that each prime ideal in \(\Lambda\) has height-one), and since \(F(\Lambda)\) is \(v\)-finite, there exists a finite type \(v\text{-}i\)deal \(I\) of \(D\) such that \(I \in F(\Lambda)\) and \(I \subseteq P\); so \(Q \supseteq (ID_{F(\Lambda)})_v = (I_{F(\Lambda)})_v = (D_{F(\Lambda)})_v = D_{F(\Lambda)}\) \([11, \text{Propositions 1.1(a) and 1.2(b)}]\). This contradiction shows that \(P \in \Lambda\), and thus \(Q = P_{F(\Lambda)}\) \([5, \text{Theorem 1.1(2)}]\) since \(P = Q \cap D\) implies that \(AD_{F(\Lambda)} \not\subseteq Q\) for all \(A \in F(\Lambda)\). \((\supseteq)\) Let \(P \in \Lambda\). Since \((D_{F(\Lambda)})_{F(\Lambda)} = D_P\) \([5, \text{Theorem 1.1}]\), we have \(ht(P_{F(\Lambda)}) = htP = 1\), and hence \(P_{F(\Lambda)}\) is a prime \(t\text{-}i\)deal of \(D_{F(\Lambda)}\) (cf. \([11, \text{Proposition 1.6(a)}]\)). Thus \(P_{F(\Lambda)}\) is a maximal \(t\text{-}i\)deal of \(D_{F(\Lambda)}\) (see the proof of the “\((\subseteq)\) case”). \(\square\)

An integral domain \(D\) is called a weakly Krull domain if \(D = \cap_{P \in X^1(D)} D_P\) and this intersection has finite character. One can easily show that \(D\) is a weakly Krull domain if and only if \(t\text{-}dim(D) = 1\) and for each \(P \in X^1(D)\), \(P = \sqrt{(a, b)}\) for some \(a, b \in D\) (cf. \([4, \text{Theorem 2.6}]\)). Let \(D\) be a weakly Krull domain, and let \(\Lambda\) be a nonempty set of prime \(t\text{-}i\)deals of \(D\). Then \(F(\Lambda)\) is finitely generated \([11, \text{Lemma 1.16}]\), and hence \(t\text{-}\text{Max}(D_{F(\Lambda)}) = \{P_{F(\Lambda)}| P \in \Lambda\}\) by Lemma 2.2(2) (cf. \([11, \text{Proposition 1.17}]\)). We next give the main result of this paper.

**Theorem 2.3.** Let \(\Lambda\) be a nonempty set of height-one maximal \(t\text{-}i\)deals of \(D\) such that each \(P \in \Lambda\) is the radical of a finite type \(v\text{-}i\)deal. Then the following statements are equivalent.

1. \(D_{F(\Lambda)}\) is a weakly Krull domain.
2. The intersection \(D_{F(\Lambda)} = \cap_{P \in \Lambda} D_P\) has finite character.
3. \(\cap_n P_1 \cdots P_n = (0)\) for each infinite sequence \((P_n)\) of distinct elements of \(\Lambda\).
4. \(F(\Lambda)\) is a \(t\text{-}s\)plitting set of ideals.
5. \(t\text{-}\text{Max}(D_{F(\Lambda)}) = \{P_{F(\Lambda)}| P \in \Lambda\}\).
6. \(F(\Lambda)\) is \(v\)-finite.
7. \(F(\Lambda)\) is \(v\)-finite.

Proof. (1) \(\Rightarrow\) (3) This follows directly from the fact that \(D_P = (D_{F(\Lambda)})_{P_{F(\Lambda)}}\) for all \(P \in \Lambda\) \([5, \text{Theorem 1.1(4)}]\). (2) \(\Rightarrow\) (1) This appears in \([11, \text{Lemma 2.5}]\). For (2) \(\Rightarrow\) (6), see \([11, \text{Lemma 1.16}]\).

(3) \(\Rightarrow\) (4) Suppose that \(\cap_n P_1 \cdots P_n = (0)\) for each infinite sequence \((P_n)\) of distinct elements of \(\Lambda\). Let \(0 \neq d \in D\). By assumption, the number of prime ideals in \(\Lambda\) containing \(d\) is finite, say \(P_1, \ldots, P_n\). Let \(A_i = dD_{P_i} \cap D\) and \(A = (A_1 \cdots A_n)\).
We first show that each $A_i$, and hence $A$, is $t$-invertible. Note that since each $P_i$ is of height-one, $dD_{P_i}$ is $P_iD_{P_i}$-primary, and hence $A_i$ is $P_i$-primary. Also, note that $A_i$ is $t$-locally principal since $P_i$ is a maximal $t$-ideal. Hence it suffices to show that each $A_i$ is of finite type [15, Corollary 2.7]. Let $I_i$ be a finitely generated ideal of $D$ such that $\sqrt{(I_i)}_t = P_i$. Since $I_i$ is finitely generated, there is a positive integer $m$ such that $I_i^mD_{P_i} = (I_iD_{P_i})^m \subseteq dD_{P_i}$; hence $(I_i^mD_{P_i})_t \subseteq ((I_i^mD_{P_i})_1)_t \subseteq dD_{P_i}$ (cf. [11, Proposition 1.3] for the second equality). Replacing $I_i$ with $I_i^m$, we may assume that $I_iD_{P_i} \subseteq dD_{P_i}$.

Let $J_i = (d, I_i)_t$. Then $J_i$ is a finite type $v$-ideal and a $P_i$-primary ideal [4, Lemma 2.1]. Hence $(A_i)_Q = D_Q = (J_i)_Q$ for any $Q \in t\text{-Max}(D)$, and hence $(A_i)_t = (d, I_i)_tD_{P_i} = ((d, I_i)_tD_{P_i})_t = ((d, I_i)_tD_{P_i})_1 = (J_iD_{P_i})_1 \supseteq J_iD_{P_i} \supseteq dD_{P_i}$. Thus $A_i = J_i [15, Proposition 2.8(3)]$.

Now, let $B = dA^{-1}$; then $dD = (AB)_t$. We next show that $A \in F(\Lambda)^\perp$ and $B \in F(\Lambda)$, which means that $F(\Lambda)$ is a $t$-splitting set of ideals. Note that each $A_i$ is $P_i$-primary, $d \in A_i$, and $P_i$ is a maximal $t$-ideal of $D$. So $A = (A_1 \cdots A_n)_t = A_1 \cap \cdots \cap A_n$, and thus $d \in A$, $A \subseteq D$, and $B \subseteq D$. If $C \in F(\Lambda)$, then $(A + C)_t = D$ since $C \nsubseteq P$ for all $P \in \Lambda$ and $A \nsubseteq Q$ for all $Q \in t\text{-Max}(D) \setminus \Lambda$ (for $A \subseteq Q \Rightarrow A_i \subseteq Q$ for some $i \Rightarrow P_i = \sqrt{A_i} \subseteq Q \Rightarrow P_i \in \Lambda$).

Hence $A \in F(\Lambda)^\perp$. Next, assume that $B \not\in F(\Lambda)$. Then $B \subseteq P$ for some $P \in \Lambda$, and since $d \in B$, we have $d \in P$; hence $P = P_i$ for some $i$. Hence $B = dA^{-1} \subseteq P_i \Rightarrow dAA^{-1} \subseteq P_iA \Rightarrow d(AA^{-1})_t = dD \subseteq (P_iA)_t$ since $A$ is $t$-invertible by the above paragraph. Let $\lambda, \mu 
exists (P_iA)_t$, since $A$ is $t$-invertible by the above paragraph $A_iD_{P_i} = dD_{P_i} \subseteq (P_iA)_tD_{P_i} \subseteq ((P_iA)_tD_{P_i})_t = (P_iD_{P_i}A_D)_{P_i} = (P_iD_{P_i}A_D)_{P_i} = (P_iD_{P_i}D_{P_i})_t = (P_iD_{P_i})_1 \Rightarrow D_{P_i} \subseteq (P_iD_{P_i})_1 \Rightarrow D_{P_i} \subseteq (P_iD_{P_i})_1 \Rightarrow D_{P_i} \subseteq (P_iD_{P_i})_1$.

But since $\text{ht}(P_iD_{P_i}) = \text{ht}P_i = 1$, we have $D_{P_i} \subseteq (P_iD_{P_i})_1 = P_iD_{P_i} \subseteq D_{P_i}$, a contradiction. Thus $B \not\in F(\Lambda)$.

(4) $\Rightarrow$ (5) Let $\Sigma = t\text{-Max}(D) \setminus \Lambda$. Then $F(\Lambda)^\perp = F(\Sigma)$ by Lemma 2.2(1), and hence $t\text{-Max}(D) \cap F(\Lambda)^\perp = \Lambda$. Therefore, $t\text{-Max}(D_{F(\Lambda)}) = \{P_{F(\Lambda)} | P \in \Lambda\}$ by the remark before [8, Corollary 15].

(5) $\Rightarrow$ (2) For any $P \in \Lambda$, let $I$ be a finite type $v$-ideal such that $\sqrt{I} = P$. Since $\text{ht}P = 1$, we have $(P_{F(\Lambda)})_t = P_{F(\Lambda)}$ [11, Proposition 1.6(a)]; so $(ID_{F(\Lambda)})_t \subseteq (PD_{F(\Lambda)})_t \subseteq (P_{F(\Lambda)})_t \subseteq D_{F(\Lambda)}$. Let $Q$ be a prime ideal of $D_{F(\Lambda)}$ minimal over $(ID_{F(\Lambda)})_t$. Since $I \subseteq ID_{F(\Lambda)} \cap D \subseteq Q \cap D$ and $\sqrt{Q} = P$, we have $P \subseteq Q \cap D$, and hence $P = Q \cap D$ since $P$ is a maximal $t$-ideal and $Q \cap D$ is a $t$-ideal [11, Proposition 1.3]. In particular, $P = Q \cap D$ implies that $AD_{F(\Lambda)} \nexists Q$ for all $A \in F(\Lambda)$, and so $(Q \cap D)_{F(\Lambda)} \subseteq D_{F(\Lambda)}$. Therefore, $P_{F(\Lambda)} = \sqrt{(IA_{F(\Lambda)})_t}$, and since $(Q \cap D)_{F(\Lambda)} = (JD_{F(\Lambda)})_t$ for any nonzero finitely generated ideal $J$ of $D$ [1, Proposition 1.2(b)], $P_{F(\Lambda)}$ is the radical of a finite type $v$-ideal. Note that $(D_{F(\Lambda)})_{P_{F(\Lambda)}} = D_{P}$ for all $P \in \Lambda$ [5, Theorem 1.1]. Thus the intersection $D_{F(\Lambda)} = \cap_{P \in \Lambda} D_{P}$ has finite character by Lemma 2.1.

(6) $\Rightarrow$ (7) Clear. (7) $\Rightarrow$ (5) See Lemma 2.2(2).

**Corollary 2.4.** Let $\Lambda$ be a nonempty set of $t$-invertible height-one prime ideals of $D$. Then the following statements are equivalent.

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*Corollary 2.4.* Let $\Lambda$ be a nonempty set of $t$-invertible height-one prime ideals of $D$. Then the following statements are equivalent.

---
(1) $D_{\mathcal{F}(\Lambda)}$ is a Krull domain.
(2) $D_{\mathcal{F}(\Lambda)}$ is a weakly Krull domain.
(3) The intersection $D_{\mathcal{F}(\Lambda)} = \cap_{P \in \Lambda} D_P$ has finite character.
(4) $\cap_{P \in \Lambda} \cap_{1 \leq i \leq n} P_i = (0)$ for each infinite sequence $(P_n)$ of distinct elements of $\Lambda$.
(5) $\mathcal{F}(\Lambda)$ is a t-splitting set of ideals.
(6) $t\text{-Max}(D_{\mathcal{F}(\Lambda)}) = \{P_{\mathcal{F}(\Lambda)} | P \in \Lambda\}$.
(7) $\mathcal{F}(\Lambda)$ is finitely generated.
(8) $\mathcal{F}(\Lambda)$ is v-finite.

Proof. (1) $\Rightarrow$ (2) is clear and (3) $\Rightarrow$ (1) appears in [11, Theorem 2.9]. The other implications are immediate consequences of Theorem 2.3 since $t$-invertible prime $t$-ideals are maximal $t$-ideals [14, Proposition 1.3] and of finite type. □

An integral domain $D$ is said to be of $t$-finite character if each nonzero nonunit of $D$ is contained in only a finite number of maximal $t$-ideals of $D$, i.e., if the intersection $D = \cap_{P \in \text{Max}(D)} D_P$ has finite character. It is clear that a weakly Krull domain is of $t$-finite character.

Corollary 2.5. Let $\Lambda$ be a nonempty set of height-one maximal $t$-ideals of $D$. If $D$ is of $t$-finite character, then $\mathcal{F}(\Lambda)$ is a t-splitting set of ideals.

Proof. First, note that $\mathcal{F}(\Lambda)$ is finitely generated [11, Proposition 1.17]. Next, let $P \in \Lambda$, and choose a nonzero element $x \in P$. Since $D$ is of $t$-finite character, there are only finitely many maximal $t$-ideals of $D$ containing $x$. So we can choose an $y \in P$ such that $P = \sqrt{(x,y)}$ since ht$P = 1$ (cf. [16, Theorem 83]). Hence $P = \sqrt{(x,y)}$. Thus $\mathcal{F}(\Lambda)$ is a t-splitting set of ideals by Theorem 2.3. □

Note that the integral domain $\mathbb{Z} + X\mathbb{Q}[X]$ does not have $t$-finite character, even though $\mathcal{F}(\Lambda)$ is finitely generated for each nonempty subset $\Lambda$ of prime $t$-ideals (see [10, Example 8.4.7] or [11, p.129]). Our next result shows that if $t\text{-dim}(D) = 1$, then $D$ has $t$-finite character if and only if $\mathcal{F}(\Lambda)$ is finitely generated for all nonempty subsets $\Lambda$ of maximal $t$-ideals of $D$.

Corollary 2.6. The following statements are equivalent.

(1) $D$ is a weakly Krull domain.
(2) $\mathcal{F}(\Lambda)$ is t-splitting for every nonempty subset $\Lambda$ of prime $t$-ideals of $D$.
(3) $t\text{-dim}(D) = 1$ and $\mathcal{F}(\Lambda)$ is finitely generated for every nonempty subset $\Lambda$ of prime $t$-ideals of $D$.

Proof. (1) $\Rightarrow$ (2) and (3) Suppose that $D$ is a weakly Krull domain, and let $\Lambda$ be a nonempty set of prime $t$-ideals of $D$. Then $t\text{-dim}(D) = 1$, and hence each prime $t$-ideal of $D$ is a height-one maximal $t$-ideal. Thus $\mathcal{F}(\Lambda)$ is finitely generated [11, Proposition 1.17]. Also, since a weakly Krull domain is of $t$-finite character, $\mathcal{F}(\Lambda)$ is a $t$-splitting set of ideals by Corollary 2.5.
First, recall that $t \in Q$. Corollary 2.6 for a Mori domain. $D$ is finitely generated by Corollary 2.6. Our next result is a restatement of the well-known property of a UFD; an ideal of $D$ is called a weakly Krull domain [1, p.8].

Let $y \in aD$ such that $y \notin P$. Then $P \notin Q$ for all $Q \in \Lambda$, and hence $P \in F(\Lambda)$; so there is a finitely generated ideal $I$ of $D$ such that $I \subseteq P$ and $I \notin Q$ for all $Q \in \Lambda$. So $P = \sqrt{t}$ since $t$-dim$(D) = 1$, and thus the intersection $D = \cap_{P \in X^1(D)} DP$ has finite character by Lemma 2.1. □

An integral domain $D$ is called a Mori domain if $D$ satisfies the ascending chain condition on integral divisorial ideals of $D$; equivalently, if each $t$-ideal of $D$ is a finite type $v$-ideal. It is well known, and easily verified, that a Mori domain (and hence Noetherian domain) has $t$-finite character. So if $D$ is a Mori domain with $t$-dim$(D) = 1$, then every spectral localizing system of $D$ is finitely generated by Corollary 2.6. Our next result is a restatement of Corollary 2.6 for a Mori domain.

**Corollary 2.7.** The following statements are equivalent for a Mori domain $D$.

1. $D$ is a weakly Krull domain.
2. $t$-dim$(D) = 1$.
3. $F(\Lambda)$ is $t$-splitting for every nonempty subset $\Lambda$ of prime $t$-ideals of $D$.

3. **Generalized weakly factorial domains**

A nonzero element $x \in D$ is said to be primary if $xD$ is a primary ideal, while $D$ is called a generalized weakly factorial domain (GWFD) if each nonzero prime ideal of $D$ contains a primary element (see [4]). This concept is a generalization of the well-known property of a UFD; $D$ is a UFD if and only if each nonzero prime ideal of $D$ contains a principal prime [16, Theorem 5]. It is known that $D$ is a GWFD if and only if $t$-dim$(D) = 1$ and for each $P \in X^1(D)$, $P = \sqrt{aD}$ for some $a \in D$ [4, Theorem 2.2]; so a GWFD is a weakly Krull domain. We next give the GWFD analog of Theorem 2.3. To do this, we need a lemma.

**Lemma 3.1.** Let $\Lambda$ be a nonempty set of maximal $t$-ideals of $D$, and let $P \in \Lambda$. If $P = \sqrt{aD}$, then $aD_{F(\Lambda)}$ is $P_{F(\Lambda)}$-primary and $P_{F(\Lambda)}$ is a maximal $t$-ideal of $D_{F(\Lambda)}$.

Proof. First, recall that $aD$ is $P$-primary [4, Lemma 2.1] and $(aD)_{F(\Lambda)} = \cap_{P \in \Lambda} aD_P = a(\cap_{P \in \Lambda} D_P) = aD_{F(\Lambda)}$ (see [11, p.120] for the first equality). Let $b \in D_{F(\Lambda)}$ such that $ab \in D$. Then there is an $I \in F(\Lambda)$ such that $bI \subseteq D$; so $abI \subseteq aD$. Since $I \in F(\Lambda)$ and $P \in \Lambda$, we have $I \notin P$, and since $aD$ is $P$-primary, $ab \in aD$ and $b \in D$. Hence $aD_{F(\Lambda)} \cap D \subseteq aD$, and thus $aD_{F(\Lambda)} \cap D = aD$.

Let $xy \in aD_{F(\Lambda)}$, where $x, y \in D_{F(\Lambda)}$ with $y \notin P_{F(\Lambda)}$. Then there are $I, J \in F(\Lambda)$ such that $xI \subseteq D$ and $yJ \subseteq D$; hence $(xI)(yJ) \subseteq aD_{F(\Lambda)} \cap D = aD$. Since $y \notin P_{F(\Lambda)}$ and $J \notin P = P_{F(\Lambda)} \cap D$, we have $yJ \notin P$, and thus $xI \subseteq aD$; so
Let $F \cap D$ for each nonzero $P$ that for and $\iff (8)$, see Theorem 2.3. (3) nonempty set of prime $\subseteq D$. Then the map $F$ is torsion [2, Theorem 3.4]. And that $\sqrt{aD}$ if and only if $D$-primary ideal, and hence $P_{\sqrt{aD}}$ is a prime $D$-ideal [11, Proposition 1.3]. Also, since $aD \subseteq Q \cap D$, we have $P = \sqrt{aD} \subseteq Q \cap D$. Hence the maximality of $P$ implies that $P = Q \cap D$, and thus $Q = P_{\sqrt{aD}}$ [5, Theorem 1.1(2)]. This implies that $\sqrt{aD}_{F(\Lambda)} = P_{\sqrt{aD}}$. □

The following theorem is the GWFD analog of Theorem 2.3.

**Theorem 3.2.** Let $\Lambda$ be a nonempty set of height-one maximal $t$-ideals of $D$ such that each $P \in \Lambda$ is the radical of a principal ideal. Then the following statements are equivalent.

1. $D_{\Lambda}$ is a GWFD.
2. $D_{\Lambda}$ is a weakly Krull domain.
3. The intersection $D_{\Lambda} = \cap_{P \in \Lambda}D_P$ has finite character.
4. $\cap_{n}P_1 \cdots P_n = (0)$ for each infinite sequence $(P_n)$ of elements of $\Lambda$.
5. $F(\Lambda)$ is a $t$-splitting set of ideals.
6. $t$-Max$(D_{\Lambda}) = \{P_{\Lambda}|P \in \Lambda\}$.
7. $F(\Lambda)$ is finitely generated.
8. $F(\Lambda)$ is $v$-finite.

**Proof.** (1) $\Rightarrow$ (2) [4, Corollary 2.3]. For (2) $\iff$ (3) $\iff$ (4) $\iff$ (5) $\iff$ (6) $\iff$ (7) $\iff$ (8), see Theorem 2.3. (3) $\Rightarrow$ (1) Note that $D_{\Lambda}$ is a weakly Krull domain and $X^1(D_{\Lambda}) = t$-Max$(D_{\Lambda}) = \{P_{\Lambda}|P \in \Lambda\}$ by Theorem 2.3. Also, note that for $P \in \Lambda$, if $P = \sqrt{aD}$, then $P_{\Lambda} = \sqrt{aD}_{F(\Lambda)}$ by Lemma 3.1. Thus $D_{\Lambda}$ is a GWFD [4, Theorem 2.2]. □

Let $T(D)$ be the group of $t$-invertible fractional $t$-ideals of $D$ under the $t$-multiplication $I \ast J = (IJ)_t$, and let $\text{Prin}(D)$ be its subgroup of nonzero principal fractional ideals of $D$. Then $C_l(D) = T(D)/\text{Prin}(D)$, called the class group of $D$, is an abelian group. Recall that $D$ is a weakly factorial domain (WFD) if each nonzero element of $D$ can be written as a product of primary elements and that $D$ is an almost weakly factorial domain (AWFD) if for each nonzero $d \in D$, there exists a natural number $n = n(d)$ such that $d^n$ can be written as a product of primary elements. It is well known that $D$ is a WFD if and only if $D$ is a weakly Krull domain and $C_l(D) = 0$ [3, Theorem] and that $D$ is an AWFD if and only if $D$ is a weakly Krull domain and $C_l(D)$ is torsion [2, Theorem 3.4].

Let $S$ be a $t$-splitting set of ideals of $D$ and $S^\perp$ the $t$-complement of $S$. Then the map $\alpha : C_l(D) \rightarrow C_l(D_S) \oplus C_l(D_{S^\perp})$ defined by $\alpha([I]) = ([ID_S]_t, [(ID_{S^\perp})]_t)$ is a group epimorphism [8, Remark 13], and thus the homomorphism $\beta : C_l(D) \rightarrow C_l(D_S)$ defined by $\beta([I]) = [(ID_S)_t]$ is surjective. Let $\Lambda$ be a nonempty set of prime $t$-ideals of $D$. Then $\cap_{P \in \Lambda}D_P$ is called a subintersection of $D$. 

\[ x \in (aD)_{\Lambda} = aD_{\Lambda}. \] Thus if we show that $\sqrt{aD_{\Lambda}} = P_{\Lambda}$, then $aD_{\Lambda}$ is a $P_{\Lambda}$-primary ideal, and hence $P_{\Lambda}$ is a maximal $t$-ideal [4, Lemma 2.1]. Let $Q$ be a prime ideal of $D_{\Lambda}$ minimal over $aD_{\Lambda}$. Then $Q$, and hence $Q \cap D$, is a prime $t$-ideal [11, Proposition 1.3]. Also, since $aD \subseteq Q \cap D$, we have $P = \sqrt{aD} \subseteq Q \cap D$. Hence the maximality of $P$ implies that $P = Q \cap D$, and thus $Q = P_{\sqrt{aD}}$ [5, Theorem 1.1(2)]. This implies that $\sqrt{aD}_{F(\Lambda)} = P_{\sqrt{aD}}$. □
Corollary 3.3. Any subintersection of a GWFD (resp., AWFD, WFD) is a GWFD (resp., AWFD, WFD).

Proof. Recall that a GWFD, an AWFD, and a WFD are weakly Krull domains. Let $D$ be a weakly Krull domain, and let $R$ be a subintersection of $D$. Then $R = \bigcap_{P \in \Lambda} D_P$ for some $\emptyset \neq \Lambda \subseteq t\text{-Max}(D)$, and hence $R = D_{\mathcal{F}(\Lambda)}$ [10, Proposition 5.1.4].

If $D$ is a GWFD, then $t\text{-dim}(D) = 1$, each prime ideal $P \in \Lambda$ is the radical of a principal ideal [4, Theorem 2.2], and $\mathcal{F}(\Lambda)$ is a $t$-splitting set of ideals (Corollary 2.6). Thus $R = D_{\mathcal{F}(\Lambda)}$, a GWFD by Theorem 3.2. Next, assume that $D$ is a WFD (resp., AWFD). Since WFDs and AWFDs are both GWFDs, $R = D_{\mathcal{F}(\Lambda)}$ is a GWFD. Also, since the homomorphism $\beta : Cl(D) \rightarrow Cl(D_{\mathcal{F}(\Lambda)})$ defined by $\beta([I]) = ([ID_{\mathcal{F}(\Lambda)}]_t)$ is surjective (see the remark before Corollary 3.3), $Cl(R) = 0$ if $Cl(D) = 0$ and $Cl(R)$ is torsion if $Cl(D)$ is torsion. Therefore, if $D$ is a WFD (resp., AWFD), then $R$ is a WFD (resp., AWFD). □

We end this paper with an example which shows that $\mathcal{F}(\Lambda)$ need not be a $t$-splitting set of ideals for a nonempty set $\Lambda$ of height-one principal prime ideals (and hence maximal $t$-ideals).

Example 3.4. Let $D$ be the ring of entire functions, $\mathbb{C}$ the field of complex numbers, and $\Lambda = \{M_z = (X - z)D | z \in \mathbb{C}\}$. Then $\Lambda \subseteq t\text{-Max}(D) \cap X^1(D)$, $D = \bigcap_{M \in \Lambda} DM$, [17, p.267], and $D$ is a Bezout domain with $\text{dim}(D) = \infty$ (and hence $t\text{-dim}(D) = \infty$) [10, Proposition 8.1.1]. Hence $D$ is not a GWFD, and thus $\mathcal{F}(\Lambda)$ is not a $t$-splitting set of ideals by Theorem 3.2. The ring of entire functions also serves as a counterexample of the following generalization of [13, Theorem 1.6] that if each minimal prime ideal of the ideal $I$ is the radical of a finitely generated ideal, then $I$ has only finitely many minimal prime ideals.

References


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