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SOME NECESSARY AND SUFFICIENT CONDITIONS FOR A FRÉCHET-URYSOHN SPACE TO BE SEQUENTIALLY COMPACT

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Abstract. In this paper, we introduce a new property of a topological space which is weaker than sequential compactness and give some necessary and sufficient conditions for a Fréchet-Urysohn space with the property to be sequentially compact.

1. Introduction and preliminaries

Let X be a topological space. For each subset A of X, we use $\overline{A}$ for the closure of A in X. Let $\mathbb{N}$ denote the set of all natural numbers, $(x_n)_{n\in\mathbb{N}}$ a sequence of points of a set, and $\{x_n : n \in \mathbb{N}\}$ the range of $(x_n)_{n\in\mathbb{N}}$. We recall that X is *Fréchet-Urysohn* [1] (also called *Fréchet* [5] or *closure sequential* [11]) if for each subset A of X and each $x \in \overline{A}$, there exists a sequence of points of A which converges to x. X has *unique sequential limits* [5] (also X is called a *US* space [11]) if every sequence of points of X may converge to at most one limit. It is well-known that every space with unique sequential limits is T_1, every Hausdorff space has unique sequential limits, every first countable space is Fréchet-Urysohn, and every first countable space with unique sequential limits is Hausdorff, but the converses need not be true in general (see [1], [5], [6] and [11]). In particular, S. P. Franklin [5, 6.2. Example] showed that there is a countable, compact and Fréchet-Urysohn space with unique sequential limits which is not Hausdorff.

Consider the following property of a topological space X:

(♣) *For each sequence $(x_n)_{n\in\mathbb{N}}$ of points of X which does not have any convergent subsequence and each double sequence $(x_{nm})_{n,m\in\mathbb{N}}$ of points of X such that for each $n \in \mathbb{N}$, $(x_{nm})_{m\in\mathbb{N}}$ converges to x_n in X, there exists a sequence $(z_n)_{n\in\mathbb{N}}$ of points of $\{x_{nm} : n, m \in \mathbb{N}\}$ which does not have any convergent subsequence.*

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Remark 1.1. (1) It is obvious that a sequentially compact space satisfies (♣).

(2) For a Fréchet-Urysohn space X with unique sequential limits, (♣) is equivalent to the following property:

For each sequence $(x_n)_{n\in\mathbb{N}}$ of points of X which does not have any convergent subsequence and each subset A of X with $\{x_n : n \in \mathbb{N}\} \subset \overline{A}$, there exists a sequence $(z_n)_{n\in\mathbb{N}}$ of points of A which does not have any convergent subsequence.

We begin by showing some examples of topological spaces related to (♣).

Example 1.2. The real line $\mathbb{R}$ with the usual topology satisfies (♣). Suppose that $\mathbb{R}$ does not satisfy (♣). Then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of points of $\mathbb{R}$ which does not have any convergent subsequence and for each $n \in \mathbb{N}$, there exists a sequence $(x_{nm})_{m\in\mathbb{N}}$ of points of $\mathbb{R}$ such that $(x_{nm})_{m\in\mathbb{N}}$ converges to x_n and each sequence $(z_n)_{n\in\mathbb{N}}$ of points of $\{x_{nm} : n, m \in \mathbb{N}\}$ has a convergent subsequence. It is clear that $\{x_n : n \in \mathbb{N}\} \subset \overline{\{x_{nm} : n, m \in \mathbb{N}\}}$ and $(x_n)_{n\in\mathbb{N}}$ is unbounded (i.e., $\{x_n : n \in \mathbb{N}\}$ is unbounded). Note that the closure of a bounded subset of $\mathbb{R}$ is bounded. Hence, $\{x_{nm} : n, m \in \mathbb{N}\}$ is unbounded. Thus we have easily that there exists a sequence $(z_n)_{n\in\mathbb{N}}$ of points of $\{x_{nm} : n, m \in \mathbb{N}\}$ such that $(z_n)_{n\in\mathbb{N}}$ is unbounded and $(z_n)_{n\in\mathbb{N}}$ does not have any bounded subsequence. Hence $(z_n)_{n\in\mathbb{N}}$ does not have any convergent subsequence, which is a contradiction.

Example 1.3. Let $\mathbb{R}$ be the set of all real numbers and $X = \mathbb{R} \cup \{z\}$ with $z \notin \mathbb{R}$. We define a topology τ on X by for each $x \in \mathbb{R}$, $\{x\} \in \tau$ and $z \in U \in \tau$ if and only if $\mathbb{R} - U$ is countable. Then the space X is Hausdorff, but not Fréchet-Urysohn (see [6, Example 1.1(2)]). Note that since every countable subset of $\mathbb{R}$ is closed in X, there does not exist a countable subset C of $\mathbb{R}$ such that $z \in \overline{C}$ and if a sequence $(x_n)_{n\in\mathbb{N}}$ of points of X is convergent, then it is only a eventually constant sequence. It follows directly that X satisfies (♣), but not sequentially compact.

Example 1.4. Let $\mathbb{R}$ be the set of all real numbers, $X = \mathbb{R}$ and $A = \{\frac{1}{n} : n \in \mathbb{N}\}$. We define a topology τ on X by letting $O \in \tau$ if $O = U - B$, where B is a subset of A and U is an open subset of the real line $\mathbb{R}$ with the usual topology (called the Smirnov topology on X). Then the space X is Hausdorff, but not regular, and hence not metrizable (see [9, p. 86, 64]). It is not difficult to show that X is a first countable space, and hence Fréchet-Urysohn. We now show that X does not satisfy (♣). Clearly the sequence $(\frac{1}{n})_{n\in\mathbb{N}}$ does not have any convergent subsequence in X. For each $n \in \mathbb{N}$, we construct a sequence $(x_{nm})_{m\in\mathbb{N}}$ of points of X as follows:

$x_{n1} = \frac{1}{2}(\frac{1}{n+1} + \frac{1}{n})$ and

$x_{nm} = \frac{1}{2^m}(\frac{1}{n+1} + \frac{1}{n}) + \frac{1}{n}(\frac{1}{2^{m-1}} + \frac{1}{2^{m-2}} + \cdots + \frac{1}{2})$ for each $2 \leqq m$.

Then we have that for each $n \in \mathbb{N}$, $(x_{nm})_{m\in\mathbb{N}}$ converges to $\frac{1}{n}$ in X, for each $m \in \mathbb{N}$, $(x_{nm})_{n\in\mathbb{N}}$ converges to 0 in X and moreover, for each strictly increasing function $\phi : \mathbb{N} \to \mathbb{N}$ and each increasing function $\psi : \mathbb{N} \to \mathbb{N}$, the cross sequence $(x_{\phi(n)\psi(n)})_{n\in\mathbb{N}}$ of the double sequence $(x_{nm})_{n,m\in\mathbb{N}}$ converges to 0 in X. To prove that X does not satisfy (♣), it is enough to show that each sequence of points of $\{x_{nm} : n, m \in \mathbb{N}\}$ has a convergent subsequence. Now let $(z_n)_{n\in\mathbb{N}}$ be a sequence of points of $\{x_{nm} : n, m \in \mathbb{N}\}$. If $(z_n)_{n\in\mathbb{N}}$ has a constant subsequence, then it is trivial. Hence we assume that $(z_n)_{n\in\mathbb{N}}$ does not have any constant subsequence and $z_n \neq z_m$ if $n \neq m$. Then we see that there are functions α and β from $\mathbb{N}$ into $\mathbb{N}$ itself such that $z_n = x_{\alpha(n)\beta(n)}$ for each $n \in \mathbb{N}$. Clearly $\{\alpha(n) : n \in \mathbb{N}\}$ is infinite or $\{\beta(n) : n \in \mathbb{N}\}$ is infinite. We divide the proof into three cases.

case 1: $\{\alpha(n) : n \in \mathbb{N}\}$ and $\{\beta(n) : n \in \mathbb{N}\}$ are infinite. Then there exists a subsequence $(z_{\phi(n)})_{n\in\mathbb{N}}$ of $(z_n)_{n\in\mathbb{N}}$ such that

$$\alpha(\phi(1)) < \alpha(\phi(2)) < \cdots < \alpha(\phi(n)) < \cdots$$

and

$$\beta(\phi(1)) < \beta(\phi(2)) < \cdots < \beta(\phi(n)) < \cdots .$$

It follows that $(z_{\phi(n)})_{n\in\mathbb{N}}$ converges to 0 in X.

case 2: $\{\alpha(n) : n \in \mathbb{N}\}$ is infinite and $\{\beta(n) : n \in \mathbb{N}\}$ is finite. Then there exists a subsequence $(z_{\psi(n)})_{n\in\mathbb{N}}$ of $(z_n)_{n\in\mathbb{N}}$ such that

$$\alpha(\psi(1)) < \alpha(\psi(2)) < \cdots < \alpha(\psi(n)) < \cdots$$

and

$$\beta(\psi(1)) = \beta(\psi(2)) = \cdots = \beta(\psi(n)) = \cdots .$$

Clearly $(z_{\psi(n)})_{n\in\mathbb{N}}$ converges to 0 in X.

case 3: $\{\alpha(n) : n \in \mathbb{N}\}$ is finite and $\{\beta(n) : n \in \mathbb{N}\}$ is infinite. Then there exists a subsequence $(z_{\varphi(n)})_{n\in\mathbb{N}}$ of $(z_n)_{n\in\mathbb{N}}$ such that

$$\alpha(\varphi(1)) = \alpha(\varphi(2)) = \cdots = \alpha(\varphi(n)) = \cdots$$

and

$$\beta(\varphi(1)) < \beta(\varphi(2)) < \cdots < \beta(\varphi(n)) < \cdots .$$

We have easily that $(z_{\varphi(n)})_{n\in\mathbb{N}}$ converges to $\frac{1}{k}$ in X, where $\alpha(\varphi(1)) = k \in \mathbb{N}$.

Thus we have $(z_n)_{n\in\mathbb{N}}$ has a convergent subsequence. Therefore, X does not satisfy (♣).

Example 1.5. The space of ordinals $X = [0, \omega_1]$, where ω_1 is the first uncountable ordinal, is a Hausdorff compact space and hence with unique sequential limits and (♣), but not Fréchet-Urysohn (see [6, Example 1.1(5)]).

Remark 1.6. (1) From Example 1.2 above, we have immediately that a Fréchet-Urysohn space with unique sequential limits and (♣) is neither countably compact nor sequentially compact in general.

(2) By Examples 1.2-1.5 and [5, 6.2. Example], we see that satisfying (♣) and having the Fréchet-Urysohn property are independent even if Hausdorff and moreover, a Fréchet-Urysohn space with unique sequential limits and (♣) need

not be Hausdorff even if compact and a topological space with unique sequential limits and (♣) need not be Fréchet-Urysohn even if compact Hausdorff.

Let X be a topological space with a topological property P. We recall that X is *P-closed* if and only if for each embedding $f : X \to Y$ from X into a topological space Y with the property P, $f(X)$ is closed in Y; that is, X is closed in every topological space with the property P containing X as a subspace, and X is *minimal* P if and only if X has no strictly coarser topology with the property P.

Some investigations of the properties of P-closed spaces and minimal P spaces have been undertaken for various properties P, among them the property of being Hausdorff, the property of being regular, the property of being Hausdorff extremally disconnected, and the property of being Hausdorff first countable etc. (see for examples [2]-[4], [7], [8], [10] and [12]).

In this paper, we introduce a new property (♣) of a topological space which is weaker than sequential compactness and give some examples related to the property (♣). We study a one-point sequentially compact extension and a sequentially compact modification of a Fréchet-Urysohn space with unique sequential limits and (♣). We then show that if P is the property of being Fréchet-Urysohn with unique sequential limits and (♣) and X is a topological space with the property P, then the following statements are equivalent:

(1) X is sequentially compact.
(2) X is countably compact.
(3) X is P-closed.
(4) X is minimal P.

All spaces under consideration are assumed to have unique sequential limits. Standard notations and terminology, not explained below, is the same as in [1] and [12].

2. Results

Let us recall that a sequentially compact extension of a topological space X is a sequentially compact space which contains X as a dense subset ([10]). First we study a one-point sequentially compact extension of a Fréchet-Urysohn space with (♣).

Theorem 2.1. *Let X be a non-sequentially compact and Fréchet-Urysohn space with (♣). Let $X_\infty = X \cup \{\infty\}$ with $\infty \notin X$ and $L_\infty = \{((x_n)_{n\in\mathbb{N}}, x) : (x_n)_{n\in\mathbb{N}}$ is a sequence of points of X which converges to x in $X\} \cup \{((x_n)_{n\in\mathbb{N}}, \infty) : (x_n)_{n\in\mathbb{N}}$ is a sequence of points of X which does not have any convergent subsequence in $X\} \cup \{((\infty), \infty)\}$, where (∞) is the constant sequence $\infty, \infty, \infty, \ldots$. Define a function $c_\infty : P(X_\infty) \to P(X_\infty)$ from the power set $P(X_\infty)$ of X_∞ into $P(X_\infty)$ itself by for each subset A of X_∞, $c_\infty(A) = \{x \in X_\infty : ((x_n)_{n\in\mathbb{N}}, x) \in L_\infty$ for some sequence $(x_n)_{n\in\mathbb{N}}$ of points of $A\}$. Then c_∞ is a topological closure operator on X_∞. Moreover, the space X_∞ endowed with the* *closure operator* c_∞ *is a sequentially compact and Fréchet-Urysohn space with unique sequential limits and* (♣) *and* X *is a dense subset of* X_∞.

Proof. It is clear that for each subset A of X_∞, either $c_\infty(A) = \overline{(A - \{\infty\})}$ or $c_\infty(A) = \overline{(A - \{\infty\})} \cup \{\infty\}$. Hence, by the definition of c_∞ and the Fréchet-Urysohn property of X, we have easily that c_∞ is a topological closure operator on X_∞ except for idempotence.

Clearly, $c_\infty(A) \subset c_\infty(c_\infty(A))$ for each subset A of X_∞. Conversely, let $x \in c_\infty(c_\infty(A))$. Either $x \neq \infty$ or $x = \infty$. We divide the proof into two cases.

case 1: $x \neq \infty$. By the definition of c_∞, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of points of $c_\infty(A)$ such that $(x_n)_{n\in\mathbb{N}}$ converges to x in X. Hence $x \in \overline{(A - \{\infty\})}$ and so $x \in c_\infty(A)$.

case 2: $x = \infty$. Suppose that $\infty \notin c_\infty(A)$. Then it is clear that $A \subset X$ and $c_\infty(A) = \overline{A}$. Since $\infty \in c_\infty(c_\infty(A)) = c_\infty(\overline{A})$, by the definition of c_∞, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of points of $\overline{A}$ which does not have any convergent subsequence in X. Since X is a Fréchet-Urysohn space, for each $n \in \mathbb{N}$, there exists a sequence $(x_{nm})_{m\in\mathbb{N}}$ of points of A such that $(x_{nm})_{m\in\mathbb{N}}$ converges to x_n in X. Since X satisfies (♣) , there exists a sequence $(z_n)_{n\in\mathbb{N}}$ of points of $\{x_{nm} : n, m \in \mathbb{N}\}$ such that $(z_n)_{n\in\mathbb{N}}$ does not have any convergent subsequence in X. It follows that $(z_n)_{n\in\mathbb{N}}$ is a sequence of points of A and $((z_n)_{n\in\mathbb{N}}, \infty) \in L_\infty$. Hence we have $\infty \in c_\infty(A)$, which is a contradiction. Thus we have $c_\infty(A) = c_\infty(c_\infty(A))$ for each subset A of X_∞ and therefore, c_∞ is a topological closure operator on X_∞.

It is easy to prove that the space X_∞ endowed with the closure operator c_∞ is a sequentially compact and Fréchet-Urysohn space with unique sequential limits and (♣) and X is dense subset of X_∞. □

Hereafter we call the space X_∞ endowed with the closure operator c_∞ the one-point sequentially compact extension of a Fréchet-Urysohn space X with (♣). It is possible that we use the term of a one-point sequentially compactfication for a one-point sequentially compact extension.

Next we study a sequentially compact modification of a Fréchet-Urysohn space with (♣).

Theorem 2.2. *Let* X *be a non-sequentially compact and Fréchet-Urysohn space with* (♣). *Let* $z \in X$ *fixed,* $X_* = X$ *and* $L_* = \{((x_n)_{n\in\mathbb{N}}, x) : (x_n)_{n\in\mathbb{N}}$ *is a sequence of points of* X *which converges to* x *in* $X\} \cup \{((x_n)_{n\in\mathbb{N}}, z) : (x_n)_{n\in\mathbb{N}}$ *is a sequence of points of* X *which does not have any convergent subsequence in* $X\}$. *Define a function* $c_* : P(X_*) \to P(X_*)$ *from the power set* $P(X_*)$ *of* X_* *into* $P(X_*)$ *itself by for each subset* A *of* X_*, $c_*(A) = \{x \in X_* : ((x_n)_{n\in\mathbb{N}}, x) \in L_*$ *for some sequence* $(x_n)_{n\in\mathbb{N}}$ *of points of* $A\}$. *Then* c_* *is a topological closure operator on* X_*. *Moreover, the space* X_* *endowed with the closure operator* c_* *is a sequentially compact and Fréchet-Urysohn space with unique sequential limits and* (♣).

Proof. Note that for each subset A of X_*, either $c_*(A) = \overline{A}$ or $c_*(A) = \overline{A} \cup \{z\}$. Hence it is also clear that c_* is a topological closure operator on X_* except for idempotence.

Let A be a subset of X_*. Obviously, $c_*(A) \subset c_*(c_*(A))$ and hence it suffices to show $c_*(c_*(A)) \subset c_*(A)$. Let $x \in c_*(c_*(A))$. Either $x \neq z$ or $x = z$. If $x \neq z$, then clearly $x \in \overline{A}$ and hence $x \in c_*(A)$. If $x = z$, then by the definition of c_* there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of points of $c_*(A)$ such that either $(x_n)_{n\in\mathbb{N}}$ converges to z in X or $(x_n)_{n\in\mathbb{N}}$ does not have any convergent subsequence in X. If there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of points of $c_*(A)$ such that $(x_n)_{n\in\mathbb{N}}$ converges to z in X, then clearly $z \in c_*(A)$. If there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of points of $c_*(A)$ such that $(x_n)_{n\in\mathbb{N}}$ does not have any convergent subsequence in X, then $\{n \in \mathbb{N} : x_n = z\}$ is finite. Without loss of generality, assume $x_n \neq z$ for all $n \in \mathbb{N}$. Then $x_n \in \overline{A}$ for each $n \in \mathbb{N}$. Since X is a Fréchet-Urysohn space, for each $n \in \mathbb{N}$, there exists a sequence $(x_{nm})_{m\in\mathbb{N}}$ of points of A such that $(x_{nm})_{m\in\mathbb{N}}$ converges to x_n in X. By (♣) of X, there exists a sequence $(z_n)_{n\in\mathbb{N}}$ of points of $\{x_{nm} : n, m \in \mathbb{N}\}$ such that $(z_n)_{n\in\mathbb{N}}$ does not have any convergent subsequence in X. It follows that $(z_n)_{n\in\mathbb{N}}$ is a sequence of points of A and $((z_n)_{n\in\mathbb{N}}, z) \in L_*$ and hence we have $z \in c_*(A)$. Thus we have $c_*(A) = c_*(c_*(A))$ for each subset A of X_* and therefore, c_* is a topological closure operator on X_*.

It is also easy to show that the space X_* endowed with the closure operator c_* is a sequentially compact and Fréchet-Urysohn space with unique sequential limits and (♣). □

Hereafter we call the space X_* endowed with the closure operator c_* the sequentially compact modification based at a point z of a Fréchet-Urysohn space X with (♣).

It is obvious that the topology for X_* is strictly coarser than the topology for X and for each sequence $(x_n)_{n\in\mathbb{N}}$ of points of X, if $(x_n)_{n\in\mathbb{N}}$ converges to $x \in X$ in X, then it converges to x in X_∞ and in X_*.

Finally we prove our main theorem using Theorems 2.1 and 2.2.

Theorem 2.3. *Let P be the property of being Fréchet-Urysohn with* (♣) *and X a topological space with the property P. Then the following statements are equivalent*:

(1) X *is sequentially compact.*

(2) X *is countably compact.*

(3) X *is P-closed.*

(4) X *is minimal P.*

Proof. (1)⇒(2): It is obvious (see [1] and [11]).

(2)⇒(1): Suppose on the contrary that X is not a sequentially compact space. Then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of points of X which does not have any convergent subsequence in X. By the Fréchet-Urysohn property of X, $\{x_n : n \in \mathbb{N}\}$ is closed in X. Note that every topological space with unique sequential limits is a T_1-space. Now let $U_n = X - \{x_i : n \leq i \in \mathbb{N}\}$ for each $n \in \mathbb{N}$. It follows that $\{U_n : n \in \mathbb{N}\}$ is a countable open cover of X which does not have any finite subcover, which is a contradiction.

(1)⇒(3): Let $f : X \to Y$ be an embedding from a sequentially compact space X with P into a topological space Y with P and let $(y_n)_{n\in\mathbb{N}}$ be a sequence of points of $f(X)$ which converges to a point y in Y. It is sufficient to show that $y \in f(X)$. Since f is an embedding, for each $n \in \mathbb{N}$, there exists uniquely $x_n \in X$ with $f(x_n) = y_n$. By sequential compactness of X, there exist a subsequence $(x_{\phi(n)})_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ and $x \in X$ such that $(x_{\phi(n)})_{n\in\mathbb{N}}$ converges to x in X. Since f is continuous and X and Y are Fréchet-Urysohn spaces, $(f(x_{\phi(n)}))_{n\in\mathbb{N}}$ converges to $f(x)$ in Y. Hence, since Y has unique sequential limits, $f(x) = y$, and thus we have $y \in f(X)$.

(3)⇒(1): Suppose that X is not a sequentially compact space. Then we have that the inclusion map $i : X \to X_\infty$ defined by for each $x \in X$, $i(x) = x$ is an embedding, where X_∞ is the one-point sequentially compact extension of X mentioned in Theorem 2.1. Clearly X and $i(X)$ are homeomorphic and hence X_∞ contains X as a subspace. By Theorem 2.1, X is a proper dense subset of X_∞. Thus $i(X)$ is not closed in X_∞, which is a contradiction.

(1)⇒(4): Suppose that X is not minimal P. Then there exist a continuous bijective function $f : X \to Y$ from X onto a topological space Y with P and a closed subset F of X such that $f(F)$ is not closed in Y. Since $f(F)$ is not closed in Y and Y is a Fréchet-Urysohn space, there exist a sequence $(y_n)_{n\in\mathbb{N}}$ of points of $f(F)$ and $y \in Y$ such that $(y_n)_{n\in\mathbb{N}}$ converges to y in Y and $y \notin f(F)$. By bijectiveness of f, for each $n \in \mathbb{N}$, there exist uniquely $x_n \in F$ and $x \in X$ such that $f(x_n) = y_n$ and $f(x) = y$. Since X is a sequentially compact space and F is closed in X, F is sequentially compact and hence there exist a subsequence $(x_{\psi(n)})_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ and $p \in F$ such that $(x_{\psi(n)})_{n\in\mathbb{N}}$ converges to p in F. So, $(x_{\psi(n)})_{n\in\mathbb{N}}$ converges to p in X. Since X and Y are Fréchet-Urysohn spaces, by continuity of f, $(f(x_{\psi(n)}))_{n\in\mathbb{N}}$ converges to $f(p)$ in Y. It follows that $f(p) = y$ since Y has unique sequential limits and hence $x = p \in F$. Thus we have $y \in f(F)$, which is a contradiction.

(4)⇒(1): Suppose that X is not a sequentially compact space. Then we have that the identity map $i_X : X \to X_*$ defined by for each $x \in X$, $i_X(x) = x$ is a continuous bijective function, where X_* is the sequentially compact modification based at a point z of X mentioned in Theorem 2.2. Since X is not sequentially compact, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of points of X which does not have any convergent subsequence and $x_n \neq z$ for all $n \in \mathbb{N}$. By the Fréchet-Urysohn property of X, $\{x_n : n \in \mathbb{N}\}$ is closed in X. Note that $c_*(\{x_n : n \in \mathbb{N}\}) = \{x_n : n \in \mathbb{N}\} \cup \{z\}$. It follows that $\{x_n : n \in \mathbb{N}\}$ is not closed in X_*. Thus i_X is not closed, which is a contradiction. □

According to Theorem 2.3, we have the following corollaries.

Corollary 2.4. *Let X be a topological space with P. Then the following statements are equivalent*:

(1) *X is sequentially compact.*
(2) *X is countably compact.*
(3) *Every closed subset of X is P-closed.*
(4) *Every closed subset of X is minimal P.*

Proof. Note that every closed subspace of a sequentially compact space with P is a sequentially compact space with P. Thus it follows from Theorem 2.3. □

Corollary 2.5. *Let Q be the property of being Hausdorff first countable with (♣) and X a minimal Q (Q-closed) space. Then X is sequentially compact if and only if it is minimal P (resp. P-closed).*

Proof. It is obvious. □

Remark 2.6. (1) Externally, "(1) ⇔ (3)" and "(2) ⇔ (3)"in Corollary 2.4 are very similar to the following well-known theorem: A Hausdorff space X is compact if and only if every closed subset of X is H-closed (see [12, 17L]).

(2) Note that a sequentially compact space with P need not be compact in general. For example, the space of ordinals $Y = [0, \omega_1)$ (called the open ordinal space [9, p. 68]), where ω_1 is the first uncountable ordinal, is a Hausdorff, sequentially compact and first countable space, and hence Y is a sequentially compact space with P, but not compact.

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