SEMI-PRECONVEX SETS ON PRECONVEXITY SPACES

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Abstract. In this paper, we introduce the concept of the semi-preconvex set on preconvexity spaces. We study some properties for the semi-preconvex set. Also we introduce the concepts of the $sc$-convex function and $s^*c$-convex function. Finally, we characterize $sc$-convex functions, $s^*$-convex functions and semi-preconvex sets by using the co-convexity hull and the convexity hull.

1. Introduction

In [1], Guay introduced the concept of preconvexity spaces defined by a binary relation on the power set $P(X)$ of a set $X$ and investigated some properties. He showed that a preconvexity on a set yields a convexity space in the same manner as a proximity [4] yields a topological space. The author introduced the concepts of the co-convexity hull and co-convex sets on preconvexity spaces in [3]. In particular, we showed that the complement of a co-convex set is a convex set and the union of co-convex sets is a co-convex set. And we characterized $c$-convex functions and $c$-concave functions by using the co-convexity hull and the convexity hull.

In this paper, we introduce the semi-preconvex set defined by the co-convexity hull on a preconvexity space and study some basic properties. And we introduce the concepts of $sc$-convex functions and $s^*c$-convex functions which are defined by the semi-preconvex sets. In particular, the $sc$-convex function is a generalized $c$-convex function.

Finally, some properties of $sc$-convex functions, $s^*c$-convex functions and semi-preconvex sets are discussed.

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2. Preliminaries

Definition 2.1 ([1]). Let $X$ be a nonempty set. A binary relation $\sigma$ on $P(X)$ is called a preconvexity on $X$ if the relation satisfies the following properties; we write $x \sigma A$ for $\{x\} \sigma A$:

1. If $A \subset B$, then $A \sigma B$.
2. If $A \sigma B$ and $B = \emptyset$, then $A = \emptyset$.
3. If $A \sigma B$ and $b \sigma C$ for all $b \in B$, then $A \sigma C$.
4. If $A \sigma B$ and $x \in A$, then $x \sigma B$.

The pair $(X, \sigma)$ is called a preconvexity space. Let $(X, \sigma)$ be a preconvexity space and $A \subset X$. $G(A) = \{x \in X : x \sigma A\}$ is called the convexity hull of a subset $A$. $A$ is called convex [1] if $G(A) = A$.

$I_\sigma(A) = \{x \in A : x \notin (X - A)\}$ (simply, $I(A)$) is called the co-convexity hull [3] of a subset $A$. And $A$ is called a co-convex set if $I(A) = A$ [3]. Let $\mathcal{I}(X) = \{A \subset X : I(A) = A\}$ and $\mathcal{G}(X) = \{A \subset X : G(A) = A\}$.

Theorem 2.2 ([3]). Let $(X, \sigma)$ be a preconvexity space and $A \subset X$. Then

1. $I(A) = X - G(X - A)$.
2. $G(A) = X - I(X - A)$.

Theorem 2.3 ([1], [3]). For a preconvexity space $(X, \sigma)$,

1. $G(\emptyset) = \emptyset$, $I(X) = X$.
2. $A \subset G(A)$, $I(A) \subset A$ for all $A \subset X$.
3. If $A \subset B$, then $G(A) \subset G(B)$, $I(A) \subset I(B)$.
4. $G(G(A)) = G(A)$, $I(I(A)) = I(A)$ for $A \subset X$.

Theorem 2.4 ([1], [3]). Let $\sigma$ be a preconvexity on $X$ and $A, B \subset X$. Then

1. $A \sigma B$ if and only if $A \subset G(B)$ if and only if $I(X - B) \subset X - A$.
2. $A \sigma B$ if and only if $G(A) \sigma G(B)$ if and only if $I(X - B) \sigma I(X - A)$.

We recall that the notions of $c$-convex function and $c$-concave function: Let $(X, \sigma)$ and $(Y, \mu)$ be two preconvexity spaces. A function $f : X \to Y$ is said to be $c$-convex [2] if for $C, D \subset Y$ whenever $C \mu D$, $f^{-1}(C) \sigma f^{-1}(D)$. A function $f : X \to Y$ is said to be $c$-convex [1] if $A \sigma B$ implies $f(A) \mu f(B)$. And $f$ is $c$-convex if and only if for each $U \in \mathcal{I}(Y)$, $f^{-1}(U) \in \mathcal{I}(X)$ [3].

3. Semi-preconvex sets

Definition 3.1. Let $(X, \sigma)$ be a preconvexity space and $A \subset X$. $A$ is called a semi-preconvex set if $A \sigma I(A)$. And $A$ is called a cosemi-preconvex set if the complement of $A$ is a semi-preconvex set.

Let $S_\sigma(X)$ (resp., $SC_\sigma(X)$) denote the set of all semi-preconvex sets (resp., cosemi-preconvex sets) in a preconvexity space $(X, \sigma)$.

From Theorem 2.2 and Theorem 2.4, we get the following theorem.

Theorem 3.2. Let $(X, \sigma)$ be a preconvexity space and $A \subset X$. Then
(1) A is a semi-preconvex set if and only if $A \subset G(I(A))$.
(2) A is a cosemi-preconvex set if and only if $I(G(A)) \subset A$.

**Theorem 3.3.** Every co-convex set is a semi-preconvex set in a preconvexity space $(X, \sigma)$.

*Proof.* Let $A$ be a co-convex set; then by the concept of co-convex sets, $A = I(A)$. By Definition 2.1, $A \subset I(A)$. □

**Theorem 3.4.** Every convex set is a cosemi-preconvex set in a preconvexity space $(X, \sigma)$.

*Proof.* Let $A$ be a convex set; then $G(A) = A$. Thus $IG(A) \subset G(A) = A$. □

**Theorem 3.5.** In a preconvexity space $(X, \sigma)$, $X$ and $\emptyset$ are both semi-preconvex sets and cosemi-preconvex sets.

*Proof.* Since $X$ and $\emptyset$ are both co-convex sets and convex sets, we get the result. □

**Theorem 3.6.** In a preconvexity space $(X, \sigma)$, the arbitrary union of semi-preconvex sets is a semi-preconvex set.

*Proof.* Let $A = \{A_\alpha : A_\alpha \text{ is a semi-preconvex set} \} \subset S_\sigma(X)$. We show that $\cup A = I(\cup A)$. For Definition 2.1(3), let $x \in \cup A$; then there exists a semi-preconvex set $A_\alpha$ containing $x$. Since $A_\alpha \subset \cup A$, from Definition 2.1(4), it follows $xI(A_\alpha)$. Since $A_\alpha \subset \cup A$, $I(A_\alpha) \subset I(\cup A)$ and the transitive property gives $xI(\cup A)$. Finally, we get $\cup A = I(\cup A)$ by Definition 2.1(3). □

**Theorem 3.7.** In a preconvexity space $(X, \sigma)$, the arbitrary intersection of cosemi-preconvex sets is a cosemi-preconvex set.

*Proof.* See Theorem 3.6. □

**Definition 3.8.** Let $(X, \sigma)$ be a preconvexity space and $A \subset X$.

(1) $SG(A) = \cap \{F : A \subset F, F^c \in S_\sigma(X)\}$.  
(2) $SI(A) = \cup \{U : U \subset A, U \in S_\sigma(X)\}$.

From Theorem 3.3, Theorem 3.6, Theorem 3.7, and Definition 3.8, we get the following theorem:

**Theorem 3.9.** Let $(X, \sigma)$ be a preconvexity space and $A, B \subset X$.

(1) $I(A) \subset SI(A) \subset A$.
(2) $A \subset SG(A) \subset G(A)$.
(3) $A$ is semi-preconvex if and only if $A = SI(A)$.
(4) $A$ is cosemi-preconvex if and only if $A = SC(X)$.

**Theorem 3.10.** Let $(X, \sigma)$ be a preconvexity space and $A, B \subset X$.

(1) $SI(X) = X$.
(2) $SI(A) \subset A$. 
(3) If $A \subset B$, then $SI(A) \subset SI(B)$.
(4) $SI(SI(A)) = SI(A)$.

**Proof.** (1), (2) and (3) are obvious.
(4) Since $SI(A) \subset A$, $SI(SI(A)) \subset SI(A)$ by (3).

For the converse, let $x \in SI(A)$; then since $x \in SI(A) \subset SI(A)$ and $SI(A)$ is a semi-preconvex set, by Definition 3.8(2), we get $x \in SI(SI(A))$. \qed

From Theorem 3.5, Theorem 3.7, Definition 3.8, and Theorem 3.9, we have the following theorem:

**Theorem 3.11.** Let $(X, \sigma)$ be a preconvexity space and $A, B \subseteq X$.

1. $SG(\emptyset) = \emptyset$.
2. $A \subseteq SG(A)$.
3. If $A \subseteq B$, then $SG(A) \subseteq SG(B)$.
4. $SG(SG(A)) = SG(A)$.

4. **sc-convex functions and $s^\ast c$-convex functions**

**Definition 4.1.** Let $(X, \sigma)$ and $(Y, \mu)$ be two preconvexity spaces. A function $f : X \to Y$ is said to be sc-convex if for each $A \in \mathcal{I}(Y)$, $f^{-1}(A) \in S_\sigma(X)$.

Every $c$-convex function is sc-convex but the converse is not always true as the following example:

**Example 4.2.** Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}\}$. Define $A \sigma B$ to mean $A \subseteq cl(B)$, the closure of $B$ in $X$. Then $\sigma$ is a preconvexity on $X$. In the preconvexity space $(X, \sigma)$, $\mathcal{G}(X) = \{\emptyset, X, \{b, c\}\}$, $\mathcal{I}(X) = \emptyset$ and $S_\sigma(X) = \emptyset$. Consider a function $f : (X, \sigma) \to (X, \sigma)$ defined as the following: $f(a) = a, f(b) = a, f(c) = c$. Then $f$ is sc-convex but it is not $c$-convex because for the co-convex set $\{a\}$, $f^{-1}(\{a\}) = \{a, b\}$ is semi-preconvex but not co-convex.

**Theorem 4.3.** Let $f : X \to Y$ be a function on two preconvexity spaces $(X, \sigma)$ and $(Y, \mu)$. Then $f$ is sc-convex if and only if for each $A \subset Y$, $f^{-1}(I_\mu(A)) \sigma I_\sigma(f^{-1}(A))$.

**Proof.** Let $f$ be sc-convex and $A \subset Y$; then since $I_\mu(A) \subset A$, by Theorem 2.3(3), we get $I_\mu(f^{-1}(I_\mu(A))) \subset I_\sigma(f^{-1}(A))$. Since $I_\mu(A) \in \mathcal{I}(Y)$ and $f$ is sc-convex, $f^{-1}(I_\mu(A)) \sigma I_\sigma(f^{-1}(I_\mu(A)))$. The transitive property gives $I_\mu(A) \sigma I_\sigma(f^{-1}(A))$.

For the converse, let $A \in \mathcal{I}(Y)$; then since $A = I_\mu(A)$,

\[ f^{-1}(A) = f^{-1}(I_\mu(A)) \sigma I_\sigma(f^{-1}(A)). \]

Thus $f^{-1}(A) \in S_\sigma(X)$. \qed

**Theorem 4.4.** Let $f : X \to Y$ be a function on two preconvexity spaces $(X, \sigma)$ and $(Y, \mu)$. Then the following things are equivalent:
Suppose a function \( f \) and \( g \) are \( \mathcal{S} \)-convex. Let \( f \) be \( \mathcal{S} \)-convex. From Theorem 3.2, it is obvious.

Let \( \mathcal{S} \)-convex. Let \( \mathcal{S} \)-convex. As the following example:

Example 4.7. In Example 4.2, consider a function \( f : (X, \sigma) \rightarrow (X, \sigma) \) defined as the following: \( f(a) = c, f(b) = b, f(c) = c \). Then \( f \) is \( \mathcal{S} \)-convex but it is not \( \mathcal{S} \)-convex because for a semi-preconvex set \( \{a, b\} \), \( f^{-1}\{a, b\} = \{b\} \) is not semi-preconvex.

Theorem 4.8. Let \( (X, \sigma) \) and \( (Y, \mu) \) be two preconvexity spaces. A function \( f : X \rightarrow Y \) is \( \mathcal{S} \)-convex if and only if for \( A \subseteq Y \) whenever \( A \mu I_\mu(A), f^{-1}(A) \sigma I_\sigma(f^{-1}(A)) \).

Proof. From Theorem 3.2, it is obvious.

Theorem 4.9. Let a function \( f : X \rightarrow Y \) be \( c \)-concave on two preconvexity spaces \( (X, \sigma) \) and \( (Y, \mu) \). Then if \( f \) is \( \mathcal{S} \)-convex, then it is \( \mathcal{S} \)-convex.

Proof. Suppose \( f \) is \( c \)-concave and \( \mathcal{S} \)-convex. Let \( A \in \mathcal{S}_\mu(Y) \); then \( A \mu I_\mu(A) \). By hypothesis and Theorem 4.3, \( f^{-1}(A) \sigma f^{-1}(I_\mu(A)) \sigma I_\sigma(f^{-1}(A)) \). Thus from Theorem 4.8, \( f \) is \( \mathcal{S} \)-convex.

From Theorem 4.4 and Corollary 4.5, we get the following results:
Theorem 4.10. Let $f : X \rightarrow Y$ be a function on two preconvexity spaces $(X, \sigma)$ and $(Y, \mu)$. Then the following things are equivalent:

1. $f$ is $s^*c$-convex
2. $f(SC(A)) \subset SC(f(A))$ for all $A \subset X$.
3. $SC(f^{-1}(B)) \subset f^{-1}(SC(B))$ for all $B \subset Y$.
4. $f^{-1}(SI(B)) \subset SI(f^{-1}(B))$ for all $B \subset Y$.
5. For each $U \in SC_\mu(Y)$, $f^{-1}(U) \in SC_\sigma(X)$.

We get the following implications:

$$c - \text{convex} \Rightarrow sc - \text{convex} \Leftarrow s^*c - \text{convex}$$

References


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