ON FIXED POINT THEOREMS IN
INTUITIONISTIC FUZZY METRIC SPACES

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Abstract. In this paper, we give some new fixed point theorems for contractive type mappings in intuitionistic fuzzy metric spaces. We improve and generalize the well-known fixed point theorems of Banach [4] and Edelstein [8] in intuitionistic fuzzy metric spaces. Our main results are intuitionistic fuzzy version of Fang’s results [10]. Further, we obtain some applications to validate our main results to product spaces.

1. Introduction

In 1965, the concept of fuzzy sets was introduced initially by Zadeh [28]. Since then, many authors have expansively developed the theory of fuzzy sets and applications. Especially, Deng [7], Erceg [9], Kaleva and Seikkala [14], Kramosil and Michalek [15] have introduced the concepts of fuzzy metric spaces in different ways. Mishra et al. [18] and Singh and Tomar [25] obtained some fixed point theorems and these fixed point theorems applied to product spaces.

Alaca et al. [2] using the idea of intuitionistic fuzzy sets [3, 6], they defined the notion of intuitionistic fuzzy metric space (shortly I-FM space) as Park [20] with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric space due to Kramosil and Michalek [15]. Further, they introduced the notion of Cauchy sequences in an I-FM spaces and proved the well-known fixed point theorems of Banach [4] and Edelstein [8] extended to I-FM spaces with the help of Grabiec [11]. Turkoglu et al. [27] introduced the concept of compatible maps and compatible maps of types (α) and (β) in I-FM spaces and gave some relations between the concepts of compatible maps and compatible maps of types (α) and (β). Turkoglu et al. [26] gave generalization of Jungck’s common fixed point theorem [13] to I-FM spaces. They first formulate the definition of weakly commuting and R-weakly commuting mappings in I-FM spaces and proved the intuitionistic fuzzy version of Pant’s theorem.

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Many authors studied the concept of I-FM space and its applications [1, 2, 12, 20, 21, 22, 23].

In the present paper, we give some new fixed point theorems for contractive type mappings in I-FM spaces. We improve and generalize the well-known fixed point theorems of Banach [4] and Edelstein [8] in I-FM spaces. Our main results are intuitionistic fuzzy version of Fang’s results [10]. Finally, we obtain some applications to validate our main results on the product of an I-FM space.

2. Intuitionistic fuzzy metric spaces

Definition 1 ([24]). A binary operation $\ast : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is continuous $t$-norm if $\ast$ is satisfying the following conditions:

(i) $\ast$ is commutative and associative;
(ii) $\ast$ is continuous;
(iii) $a \ast 1 = a$ for all $a \in [0, 1]$;
(iv) $a \ast b \leq c \ast d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Definition 2 ([24]). A binary operation $\triangleright : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is continuous $t$-conorm if $\triangleright$ is satisfying the following conditions:

(i) $\triangleright$ is commutative and associative;
(ii) $\triangleright$ is continuous;
(iii) $a \triangleright 0 = a$ for all $a \in [0, 1]$;
(iv) $a \triangleright b \leq c \triangleright d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

The concepts of triangular norms (t-norms) and triangular conorms (t-conorms) are known as the axiomatic skeletons that we use for characterizing fuzzy intersections and unions, respectively. These concepts were originally introduced by Menger [17] in his study of statistical metric spaces.

The following definition and the fundamental properties of I-FM spaces due to Kramosil and Michalek [15] was given by Alaca et al. [2].

Definition 3 ([2]). A 5-tuple $(X, M, N, \ast, \triangleright)$ is said to be an I-FM space if $X$ is an arbitrary set, $\ast$ is a continuous $t$-norm, $\triangleright$ is a continuous $t$-conorm and $M, N$ are fuzzy sets on $X^2 \times [0, \infty)$ satisfying the following conditions:

(i) $M(x, y, t) + N(x, y, t) \leq 1$ for all $x, y \in X$ and $t > 0$;
(ii) $M(x, y, 0) = 0$ for all $x, y \in X$;
(iii) $M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$;
(iv) $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t > 0$;
(v) $M(x, y, t) \ast M(y, z, s) \leq M(x, z, t + s)$ for all $x, y, z \in X$ and $s, t > 0$;
(vi) for all $x, y \in X$, $M(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]$ is left continuous;
(vii) $\lim_{t \rightarrow \infty} M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$;
(viii) $N(x, y, 0) = 1$ for all $x, y \in X$;
(ix) $N(x, y, t) = 0$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$;
(x) $N(x, y, t) = N(y, x, t)$ for all $x, y \in X$ and $t > 0$;
(xi) $N(x, y, t) \triangleright N(y, z, s) \geq N(x, z, t + s)$ for all $x, y, z \in X$ and $s, t > 0$.
(xii) for all $x, y \in X$, $N(x, y, t) : [0, \infty) \to [0, 1]$ is right continuous;
(xiii) $\lim_{t \to \infty} N(x, y, t) = 0$ for all $x, y$ in $X$.

Then $(M, N)$ is called an intuitionistic fuzzy metric on $X$. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between $x$ and $y$ with respect to $t$, respectively.

**Remark 1.** Every fuzzy metric space $(X, M, \ast)$ is an I-FM space of the form

$$(X, M, 1 - M, \ast, \diamond)$$

such that t-norm $\ast$ and t-conorm $\diamond$ are associated ([16]), i.e., $x \diamond y = 1 - ((1 - x) \ast (1 - y))$ for all $x, y \in X$.

**Remark 2.** In I-FM space $X$, $M(x, y, \cdot)$ is non-decreasing and $N(x, y, \cdot)$ is non-increasing for all $x, y \in X$.

**Definition 4** ([2]). Let $(X, M, N, \ast, \diamond)$ be an I-FM space. Then

(i) a sequence $\{x_n\}$ in $X$ is said to be Cauchy sequence if, for all $t > 0$ and $p > 0$,

$$\lim_{n \to \infty} M(x_{n+p}, x_n, t) = 1, \quad \lim_{n \to \infty} N(x_{n+p}, x_n, t) = 0.$$  

(ii) a sequence $\{x_n\}$ in $X$ is said to be convergent to a point $x \in X$ if, for all $t > 0$,

$$\lim_{n \to \infty} M(x_n, x, t) = 1, \quad \lim_{n \to \infty} N(x_n, x, t) = 0.$$  

Since $\ast$ and $\diamond$ are continuous, the limit is uniquely determined from (v) and (xi), respectively.

**Definition 5** ([2]). An I-FM space $(X, M, N, \ast, \diamond)$ is said to be complete if and only if every Cauchy sequence in $X$ is convergent.

**Definition 6** ([2]). An I-FM space $(X, M, N, \ast, \diamond)$ is said to be compact if every sequence in $X$ contains a convergent subsequence.

**Lemma 1** ([2]). (i) If $\lim_{n \to \infty} x_n = x$ and $\lim_{n \to \infty} y_n = y$, then,

$(2.1)$ $M(x, y, t) \leq \lim_{n \to \infty} M(x_n, y_n, t)$ and $N(x, y, t) \geq \lim_{n \to \infty} N(x_n, y_n, t)$

for all $t > 0$.

(ii) If $\lim_{n \to \infty} x_n = x$ and $\lim_{n \to \infty} y_n = y$, then,

$(2.2)$ $M(x, y, t) \geq \lim_{n \to \infty} \sup M(x_n, y_n, t)$ and $N(x, y, t) \leq \lim_{n \to \infty} \inf N(x_n, y_n, t)$

for all $t > 0$.

Particularly, if $M(x, y, \cdot)$ is continuous at point $t$, then

$$\lim_{n \to \infty} M(x_n, y_n, t) = M(x, y, t) \quad \text{and} \quad \lim_{n \to \infty} N(x_n, y_n, t) = N(x, y, t).$$

The following two theorems extend the well-known fixed point theorems of Banach [4] and Edelstein [8] to I-FM spaces in the sense of Kramosil and Michalek [15] was given by Alaca et al. [2].
Theorem 1 ([2]). Let \((X, M, N, \ast, \Diamond)\) be a complete I-FM space. Let \(T : X \rightarrow X\) be a mapping satisfying

\[
M(Tx, Ty, kt) \geq M(x, y, t) \quad \text{and} \quad N(Tx, Ty, kt) \leq N(x, y, t)
\]

for all \(x, y \in X, 0 < k < 1\). Then \(T\) has a unique fixed point.

Theorem 2 ([2]). Let \((X, M, N, \ast, \Diamond)\) be a compact space. Let \(T : X \rightarrow X\) be a mapping satisfying

\[
M(Tx, Ty, \cdot) > M(x, y, \cdot) \quad \text{and} \quad N(Tx, Ty, \cdot) < N(x, y, \cdot)
\]

for all \(x \neq y\).

\[
\begin{cases}
M(Tx, Ty, \cdot) \geq M(x, y, \cdot) \quad \text{and} \quad M(Tx, Ty, \cdot) \neq M(x, y, \cdot) \\
N(Tx, Ty, \cdot) \leq N(x, y, \cdot) \quad \text{and} \quad N(Tx, Ty, \cdot) \neq N(x, y, \cdot)
\end{cases}
\]

Then \(T\) has a unique fixed point.

3. Main results

Lemma 2. Let \(\ast\) be a continuous t-norm and \(\Diamond\) be a continuous t-conorm. Then for each \(\lambda \in (0, 1)\), there is a sequence \(\{\lambda_n\}\) in \((0, 1)\) such that

\[
(1 - \lambda_n) \ast (1 - \lambda_n) > 1 - \lambda_{n-1} \quad \text{and} \quad \lambda_n \Diamond \lambda_n < \lambda_{n-1}, \quad n = 1, 2, \ldots
\]

where \(\lambda_0 = \lambda\) (obviously, the sequence \(\{\lambda_n\}\) satisfying condition (3.1) is decreasing).

Proof. Since \(\ast\) is continuous at point Definition 3 [(vii) and (xiii)] and \(a \ast b \leq 1 \ast 1 = 1\) and \((1 - a) \Diamond (1 - a) \geq 0 \Diamond 0 = 0\) for all \(a, b \in [0, 1]\), we get

\[
\sup_{0 < \mu < 1} [(1 - \mu) \ast (1 - \mu)] = 1, \quad \inf_{0 < \mu < 1} [\mu \ast \mu] = 0.
\]

Hence, for each \(\lambda \in (0, 1)\), there exists \(\lambda_1 \in (0, 1)\) such that

\[
(1 - \lambda_1) \ast (1 - \lambda_1) > 1 - \lambda \quad \text{and} \quad \lambda_1 \Diamond \lambda_1 < \lambda.
\]

Similarly, from (3.2) there exists \(\lambda_2 \in (0, 1)\) such that

\[
(1 - \lambda_2) \ast (1 - \lambda_2) > 1 - \lambda_1 \quad \text{and} \quad \lambda_2 \Diamond \lambda_2 < \lambda_1.
\]

Continuing this procedure we can obtain a sequence \(\{\lambda_n\} \subset (0, 1)\) satisfying condition (3.1). This completes the proof. \(\square\)

Lemma 3 ([5]). Let the function \(\phi(t) : [0, \infty) \rightarrow [0, \infty)\) satisfying the following condition:

\(\phi_1 \) \(\phi(t)\) is strictly increasing, \(\phi(0) = 0\) and \(\lim_{t \to \infty} \phi^n(t) = 0\) for all \(t \geq 0\),

where \(\phi^n(t)\) denotes the \(n\)-th iterative function of \(\phi(t)\). Then \(\phi(t) > t, \phi^n(t) > \phi^{n-1}(t), \forall t > 0, n = 1, 2, \ldots\).
Lemma 4. Let \((X, M, N, *, \emptyset)\) be an I-FM space. Let \(T : X \to X\) be a mapping satisfying
\[
(3.3) \quad M(Tx, Ty, t_1) > M(x, y, t_1) \quad \text{and} \quad N(Tx, Ty, t_1) < N(x, y, t_1),
\]
where \(t_1\) is a fixed positive number. Then there exists a continuity point \(t_0\) of \(M(x, y, \cdot)\) such that
\[
(3.4) \quad M(Tx, Ty, t_0) > M(x, y, t_0) \quad \text{and} \quad N(Tx, Ty, t_0) < N(x, y, t_0).
\]

Proof. Since \(M(Tx, Ty, \cdot) - M(x, y, \cdot)\) and \(N(Tx, Ty, \cdot) - N(x, y, \cdot)\) are left-
continuous and right-continuous, respectively, at point \(t_1\), by (3.3) there exists
\(0 < t_2 < t_1\) such that
\[
M(Tx, Ty, t) > M(x, y, t) \quad \text{and} \quad N(Tx, Ty, t) < N(x, y, t)
\]
for all \(t \in [t_2, t_1]\). Note that the set of discontinuous points of \(M(x, y, \cdot)\) and \(N(x, y, \cdot)\) are countable at most. Thus, there exists \(t_0 \in [t_2, t_1]\) such that
\(M(x, y, \cdot)\) and \(N(x, y, \cdot)\) are continuous at \(t_0\). Thus (3.4) holds. This completes
the proof. \(\Box\)

Theorem 3. Let \((X, M, N, *, \emptyset)\) be a complete I-FM space. Let \(T : X \to X\) be a mapping satisfying the following conditions:
(i) there exists \(x_0 \in X\) such that
\[
(3.5) \quad \lim_{t \to \infty} M(x_0, T^i x_0, t) = 1 \quad \text{and} \quad \lim_{t \to \infty} N(x_0, T^i x_0, t) = 0, \quad i = 1, 2, \ldots;
\]
(ii) there exists a mapping \(m : X \to \mathbb{N}\) such that for any \(x, y \in X\),
\[
M(T^{m(x)} x, T^{m(y)} y, t) \geq M(x, y, \phi(t)) \quad \text{and}
\]
\[
N(T^{m(x)} x, T^{m(y)} y, t) \leq N(x, y, \phi(t)),
\]
where the function \(\phi(t)\) satisfies condition (\(\phi_1\)) and
\(\phi_2\) \(\lim_{t \to \infty} [\phi(t) - t] = \infty\).

Then \(T\) has a unique fixed point \(x_*\), and the quasi-iterative sequence \(\{x_n : T^{m(x_{n-1})} x_{n-1}\}\) converges to \(x_*\).

Proof. First we prove that
\[
(3.7) \quad \sup_{s > 0} \inf_{x \in O_T(x_0)} M(x_0, x, s) = 1 \quad \text{and} \quad \inf_{s > 0} \sup_{x \in O_T(x_0)} N(x_0, x, s) = 0,
\]
where \(O_T(x_0) = \{x_0, T x_0, T^2 x_0, \ldots\}\) is called the orbit of \(x_0\) for \(T\). For any \(n \in \mathbb{N}\) with \(n > m(x_0)\), we can denote
\[
n = km(x_0) + s, \quad \text{where} \quad 0 \leq s < m(x_0).
\]
Note that \(\phi(t) > t\) for all \(t > 0\) and \(\lim_{t \to \infty} [\phi(t) - t] = \infty\). By (3.5), we have
\[
(3.8) \quad \lim_{t \to \infty} M(x_0, T^i x_0, \phi(t)) = 1 \quad \text{and} \quad \lim_{t \to \infty} N(x_0, T^i x_0, \phi(t)) = 0.
\]
Therefore and thus, from (3.6), we get

\[ M(0, T^m x_0, \phi(t) - t) = 1 \text{ and } \lim_{t \to \infty} N(x_0, T^m x_0, \phi(t) - t) = 0. \]

Moreover, from Lemma 2, for any \( \lambda \in (0, 1) \), there exists a sequence \( \{\lambda_n\} \) in (0, 1) such that

\[(1 - \lambda_n) * (1 - \lambda_n) > 1 - \lambda_{n-1} \text{ and } \lambda_n \cup \lambda_n < \lambda_{n-1}, \quad (\lambda_0 = \lambda) \text{ for } n = 1, 2, \ldots.
\]

Thus, it follows from (3.8) and (3.9) that for given \( \lambda_k \) there exists \( t_0 > 0 \) such that

\[
\min_{1 \leq i \leq m(x_0)} M(x_0, T^i x_0, \phi(t)) > 1 - \lambda_k \text{ and } \max_{1 \leq i \leq m(x_0)} N(x_0, T^i x_0, \phi(t)) < \lambda_k,
\]

and

\[
M(x_0, T^m(x_0), \phi(t) - t) > 1 - \lambda_k \text{ and } N(x_0, T^m(x_0), \phi(t) - t) < \lambda_k, \quad \forall t > t_0.
\]

Thus, from (3.6), we get

\[
M(x_0, T^m x_0, \phi(t)) = M(x_0, T^{km(x_0)} + x_0, \phi(t)) \\
\geq M(x_0, T^m x_0, \phi(t) - t) * M(T^m x_0, T^{km(x_0)} + x_0, t) \\
\geq M(x_0, T^m x_0, \phi(t) - t) * M(x_0, T^{(k-1)m(x_0)} + x_0, \phi(t)) \geq \cdots \\
\geq M(x_0, T^m x_0, \phi(t) - t) * \cdots * M(x_0, T^m x_0, \phi(t) - t) \\
\geq (1 - \lambda_k) * \cdots * (1 - \lambda_k) > (1 - \lambda_{k-1}) * \cdots * (1 - \lambda_{k-1}) \\
> \cdots > (1 - \lambda_1) * (1 - \lambda_1) > 1 - \lambda, \quad \forall t > t_0,
\]

and

\[
N(x_0, T^m x_0, \phi(t)) = N(x_0, T^{km(x_0)} + x_0, \phi(t)) \\
\leq N(x_0, T^m x_0, \phi(t) - t) \cup N(T^m x_0, T^{km(x_0)} + x_0, t) \\
\leq N(x_0, T^m x_0, \phi(t) - t) \cup N(x_0, T^{(k-1)m(x_0)} + x_0, \phi(t)) \leq \cdots \\
\leq N(x_0, T^m x_0, \phi(t) - t) \cup \cdots \cup N(x_0, T^m x_0, \phi(t) - t) \\
\cup N(x_0, T^m x_0, \phi(t)) \\
< \lambda_k \cup \cdots \cup \lambda_k < \lambda_{k-1} \cup \cdots \cup \lambda_{k-1} \\
< \cdots < \lambda_1 \cup \lambda_1 < \lambda, \quad \forall t > t_0.
\]

Therefore

\[
\inf_{x \in O_T(x_0)} M(x_0, x, \phi(t)) \geq 1 - \lambda \text{ and } \sup_{x \in O_T(x_0)} N(x_0, x, \phi(t)) \leq \lambda, \quad \forall t > t_0.
\]
Hence
\[
\sup_{s>0} \inf_{x \in O_T(x_0)} M(x_0, x, s) \geq 1 - \lambda \quad \text{and} \quad \inf_{s>0} \sup_{x \in O_T(x_0)} N(x_0, x, s) \leq \lambda.
\]

By the arbitrariness of \(\lambda\), we have
\[
\sup_{s>0} \inf_{x \in O_T(x_0)} M(x_0, x, s) = 1 \quad \text{and} \quad \inf_{s>0} \sup_{x \in O_T(x_0)} N(x_0, x, s) = 0.
\]

Next, we prove that the quasi-iterative sequence \(\{x_n = T^{m(x_{n-1})}x_{n-1}\}_{n=1}^{\infty}\) is a Cauchy sequence. For convenience, put \(m_i = m(x_i), i = 0, 1, \ldots\). Then by (3.5),
\[
M(x_n, x_{n+p}, t) = M(T^{m_{n+1}}x_{n-1}, T^{m_{n+p-1}+m_{n+p-2}+\cdots+m_{n-1}}x_{n-1}, t)
\]
\[
\geq M(x_{n-1}, T^{m_{n+p-1}+m_{n+p-2}+\cdots+m_{n-1}}x_{n-1}, \phi(t))
\]
\[
\geq M(x_{n-2}, T^{m_{n+p-1}+m_{n+p-2}+\cdots+m_{n-1}}x_{n-2}, \phi^2(t))
\]
\[
\geq \cdots \geq M(x_0, T^{m_{n-1}+m_{n-2}+\cdots+m_0}x_0, \phi^n(t))
\]
\[
\geq \inf_{x \in O_T(x_0)} M(x_0, x, \phi^n(t))
\]
\[
\geq \sup_{0<s>\phi^n(t)} \inf_{x \in O_T(x_0)} M(x_0, x, s), \forall t > 0,
\]
and
\[
N(x_n, x_{n+p}, t) = N(T^{m_{n+1}}x_{n-1}, T^{m_{n+p-1}+m_{n+p-2}+\cdots+m_{n-1}}x_{n-1}, t)
\]
\[
\leq N(x_{n-1}, T^{m_{n+p-1}+m_{n+p-2}+\cdots+m_{n-1}}x_{n-1}, \phi(t))
\]
\[
\leq N(x_{n-2}, T^{m_{n+p-1}+m_{n+p-2}+\cdots+m_{n-1}}x_{n-2}, \phi^2(t))
\]
\[
\leq \cdots \leq N(x_0, T^{m_{n-1}+m_{n-2}+\cdots+m_0}x_0, \phi^n(t))
\]
\[
\leq \sup_{x \in O_T(x_0)} N(x_0, x, \phi^n(t))
\]
\[
\leq \inf_{0<s>\phi^n(t)} \sup_{x \in O_T(x_0)} N(x_0, x, s), \forall t > 0.
\]

Then by condition \((\phi_1)\) and (3.7) we have
\[
\lim_{n \to \infty} M(x_n, x_n, t) = 1, \quad \lim_{n \to \infty} N(x_n, x_n, t) = 0, \quad \forall t > 0.
\]

This means that \(\{x_n\}\) is a Cauchy sequence in \(X\). By the completeness of \(X\), there exists \(\lim_{n \to \infty} x_n = x_* \in X\).

Now we prove that \(x_*\) is the unique fixed point of \(T^{m_*}\), where \(m_* = n(x_*)\).

By Definition 3 [(v) and (xii)] and (3.6), we have
\[
M(x_*, T^{m_*}x_*, t) \geq M(x_*, T^{m_*}x_*, \frac{t}{2}) \circ M(T^{m_*}x_*, T^{m_*}x_*, \frac{t}{2})
\]
and
\[
N(x_*, T^{m_*}x_*, t) \leq N(x_*, T^{m_*}x_*, \frac{t}{2}) \circ N(T^{m_*}x_*, T^{m_*}x_*, \frac{t}{2}).
\]
Then
\[ M(x_*, T^{m_i}x_*, t) \geq M \left( x_*, T^{m_i}x_n, \frac{t}{2} \right) \ast M \left( x_n, x_*, \phi \left( \frac{t}{2} \right) \right), \]
\[ N(x_*, T^{m_i}x_*, t) \leq N \left( x_*, T^{m_i}x_n, \frac{t}{2} \right) \Diamond N \left( x_n, x_*, \phi \left( \frac{t}{2} \right) \right). \]
(3.10)

It is easy to prove that
\[ \lim_{n \to \infty} M(x_*, T^{m_i}x_n, u) = 1 \quad \text{and} \quad \lim_{n \to \infty} N(x_*, T^{m_i}x_n, u) = 0, \ \forall u > 0. \]

In fact,
\[
M(x_*, T^{m_i}x_n, u) \geq M \left( x_*, x_n, \frac{1}{2} u \right) \ast M \left( x_n, T^{m_i}x_n, \frac{1}{2} u \right)
= M \left( x_*, x_n, \frac{1}{2} u \right) \ast M \left( T^{m_{i-1}}x_{n-1}, T^{m_{i-1}+m_i}x_{n-1}, \frac{1}{2} u \right)
\geq M \left( x_*, x_n, \frac{1}{2} u \right) \ast M \left( x_{n-1}, T^{m_i}x_{n-1}, \phi \left( \frac{1}{2} u \right) \right) \geq \cdots
\geq M \left( x_*, x_n, \frac{1}{2} u \right) \ast M \left( x_{n-1}, T^{m_i}x_{n-1}, \phi^n \left( \frac{1}{2} u \right) \right) \to 1
\]

and
\[
N(x_*, T^{m_i}x_n, u) \leq N \left( x_*, x_n, \frac{1}{2} u \right) \Diamond N \left( x_n, T^{m_i}x_n, \frac{1}{2} u \right)
= N \left( x_*, x_n, \frac{1}{2} u \right) \Diamond N \left( T^{m_{i-1}}x_{n-1}, T^{m_{i-1}+m_i}x_{n-1}, \frac{1}{2} u \right)
\leq N \left( x_*, x_n, \frac{1}{2} u \right) \Diamond N \left( x_{n-1}, T^{m_i}x_{n-1}, \phi \left( \frac{1}{2} u \right) \right) \leq \cdots
\leq N \left( x_*, x_n, \frac{1}{2} u \right) \Diamond N \left( x_{n-1}, T^{m_i}x_{n-1}, \phi^n \left( \frac{1}{2} u \right) \right) \to 0
\]

for \( n \to \infty \). Then, letting \( n \to \infty \) on the right side of (3.10), and noting the continuity of \( \ast \) and \( \Diamond \) we have
\[ M(x_*, T^{m_i}x_*, t) = 1 \quad \text{and} \quad N(x_*, T^{m_i}x_*, t) = 0, \ \forall t > 0. \]

This implies that \( T^{m_i}x_* = x_* \), i.e., \( x_* \) is a fixed point of \( T^{m(x_*)} \). To show uniqueness, assume that \( T^{m(x_*)}y = y \) for \( y \in X \). Then
\[ M(x_*, y, t) = M(T^{m(x_*)}x_*, T^{m(x_*)}y, t) \geq M(x_*, y, \phi(t)) \]
and
\[ N(x_*, y, t) = N(T^{m(x_*)}x_*, T^{m(x_*)}y, t) \leq N(x_*, y, \phi(t)). \]

On the other hand, as \( M(x_*, y, t) \) is non-decreasing and \( N(x_*, y, t) \) is non-increasing, we have
\[ M(x_*, y, t) \leq M(x_*, y, \phi(t)) \quad \text{and} \quad N(x_*, y, t) \geq N(x_*, y, \phi(t)). \]
Hence
\[ M(x_*, y, t) = M(x_*, y, \phi(t)) = M(x_*, y, \phi^n(t)), \forall t > 0, \]
and
\[ N(x_*, y, t) = N(x_*, y, \phi(t)) = N(x_*, y, \phi^n(t)), \forall t > 0. \]
By the condition \((\phi_1)\),
\[ M(x_*, y, t) = 1 \text{ and } N(x_*, y, t) = 0, \forall t > 0. \]
Then by Definition 3 \((iii) \text{ and } (ix)\) we have \(x_* = y\).

Finally, we prove that \(x_*\) is unique fixed point of \(T\), too. In fact, since \(T^{m(x_*)}x_* = x_*\), it follows that \(Tx_* = T(T^{m(x_*)}x_*) = T^{m_1}(Tx_*)\). Hence, \(Tx_* = x_*\).

Uniqueness is obvious. This completes the proof. □

**Corollary 1.** Let \((X, M, N, *, \Diamond)\) be a complete I-FM space. Let \(T : X \to X\) be a mapping satisfying the following conditions:

(i) there exists \(x_0 \in X\) such that
\[ \lim_{t \to \infty} M(x_0, T^i x_0, t) = 1 \text{ and } \lim_{t \to \infty} N(x_0, T^i x_0, t) = 0, \quad i = 1, 2, \ldots; \]

(ii) there exists a mapping \(m : X \to N\) such that for any \(x, y \in X\),
\[ M(T^{m(x)}x, T^{m(y)}y, t) \geq M(x, y, \frac{t}{k}) \text{ and } N(T^{m(x)}x, T^{m(y)}y, t) \leq N(x, y, \frac{t}{k}), \]
where \(0 < k < 1\).

Then the conclusion of Theorem 3 remains true.

**Proof.** Taking \(\phi(t) = \frac{t}{k}\). Obviously, \(\phi(t)\) satisfies the conditions \((\phi_1)\) and \((\phi_2)\). Therefore the conclusion follows from Theorem 3 directly. □

**Corollary 2.** Let \((X, M, N, *, \Diamond)\) be a complete I-FM space. Let \(T : X \to X\) be a mapping. If there exists a mapping \(m : X \to N\) such that for any \(x, y \in X\),
\[ M(T^{m(x)}x, T^{m(y)}y, t) \geq M(x, y, \phi(t)) \text{ and } N(T^{m(x)}x, T^{m(y)}y, t) \leq N(x, y, \phi(t)), \]
where the function \(\phi(t)\) satisfies conditions \((\phi_1)\) and \((\phi_2)\). Then \(T\) has a unique fixed point \(x_*\), and the iterative sequence \(\{T^n x\}\) converges to \(x_*\) for every \(x \in X\).

**Proof.** From Theorem 3, we need only to show that the iterative sequence \(\{T^n x\}\) converges to \(x_*\). For any \(n \in \mathbb{N}\) with \(n > m(x_*)\),
\[ n = km(x_*) + s, \quad 0 \leq s < x_* \]
Since
\[ M(x_*, T^n x, t) = M(T^{m(x_*)} x_*, T^{km(x_*)+s} x, t) \]
\[ \geq M(x_*, T^{(k-1)m(x_*)+s} x, \phi(t)) \]
\[ \geq \cdots \geq M(x_*, T^s x, \phi^k(t)) \to 1 \]
and
\[ N(x_*, T^n x, t) = N(T^{m(x_*)} x_*, T^{km(x_*)+s} x, t) \]
\[ \leq N(x_*, T^{(k-1)m(x_*)+s} x, \phi(t)) \]
\[ \leq \cdots \leq N(x_*, T^s x, \phi^k(t)) \to 0 \]
for \( n \to \infty \). It follows that
\[ \lim_{n \to \infty} M(x_*, T^n x, t) = 1 \quad \text{and} \quad \lim_{n \to \infty} N(x_*, T^n x, t) = 0, \forall t > 0. \]
Then we get \( \lim_{n \to \infty} T^n x = x_* \). This completes the proof. \( \Box \)

Remark 3. Taking \( \phi(t) = \frac{1}{t} (0 < k < 1) \) and \( m(x) \equiv 1 \) in Corollary 2, we at once obtain Theorem 1. Hence Theorem 1 is a special case of Corollary 2.

Theorem 4. Let \( (X, M, N, *, \Diamond) \) be a complete I-FM space with \( t \ast t \geq t \) and \( (1-t) \Diamond (1-t) \leq (1-t) \) for all \( t \in [0, 1] \), and \( T : X \to X \) be a continuous mapping satisfying
\[ M(Tx, Ty, \cdot) > M(x, Tx, \cdot) \ast M(y, Ty, \cdot) \ast M(x, y, \cdot), \]
\[ N(Tx, Ty, \cdot) < N(x, Tx, \cdot) \diamond N(y, Ty, \cdot) \Diamond N(x, y, \cdot) \]
for all \( x \neq y \). If there exists \( x_0 \in X \) such that \( \{T^n x_0\}_{n=0}^\infty \) has an accumulation point \( x_* \in X \), and
\[ M(T^{-1} x_0, T^n x_0, t) \leq M(T^{n-1} x_0, T^{n+1} x_0, t), \]
\[ N(T^{-1} x_0, T^n x_0, t) \geq N(T^{n-1} x_0, T^{n+1} x_0, t), \forall t > 0, \ n = 1, 2, \ldots, \]
then \( x_* \) is the unique fixed point of \( T \), and \( \lim_{n \to \infty} T^n x_0 = x_* \).

Proof. Assume \( T^n x_0 \neq T^{n+1} x_0 \) for each \( n \in \mathbb{N} \). (If not, there is \( n_0 \in \mathbb{N} \) such that \( T^{n_0} x_0 \neq T^{n_0+1} x_0 \). This means that \( x_* = T^{n_0} x_0 \) is a fixed point of \( T \), and \( \lim_{n \to \infty} T^n x_0 = x_* \). Since \( \{T^n x_0\}_{n=0}^\infty \) has an accumulation point \( x_* \in X \), there exists a subsequence \( \{T^n x_0\} \), \( \lim_{n \to \infty} T^n x_0 = x_* \). \( \{M(T^n x_0, T^{n+1} x_0, t)\} \) is non-decreasing and bounded and \( \{N(T^n x_0, T^{n+1} x_0, t)\} \) is non-increasing and bounded. Thus, we have
\[ \{M(T^n x_0, T^{n+1} x_0, t)\} \quad \text{and} \quad \{N(T^n x_0, T^{n+1} x_0, t)\}, \]
\[ \{M(T^{n+1} x_0, T^{n+2} x_0, t)\} \quad \text{and} \quad \{N(T^{n+1} x_0, T^{n+2} x_0, t)\}. \]
are convergent to a common limit, i.e.,
\[
\lim_i M(T^{n_i}x_0, T^{n_i+1}x_0, t) = \lim_i M(T^{n_{i+1}}x_0, T^{n_{i+2}}x_0, t), \\
\lim_i N(T^{n_i}x_0, T^{n_i+1}x_0, t) = \lim_i N(T^{n_{i+1}}x_0, T^{n_{i+2}}x_0, t), \forall t > 0.
\]

By the continuity of \(T\), we have
\[
\lim_i T^{n_i}x_0 = \lim_i T(T^{n_i}x_0) = T x_0.
\]

Suppose \(T x_0 \neq x_0\). Putting \(y = T x\) in (3.11), we have
\[
M(x, Tx, \cdot) < M(Tx, T^2x, \cdot) \text{ and } N(x, Tx, \cdot) > N(Tx, T^2x, \cdot)
\]
for every \(x \neq Tx\).

So by Lemma 4, there exists a continuous point \(t_0\) of \(M(x, Tx, \cdot)\) and \(N(x, Tx, \cdot)\) such that \(M(Tx, T^2x, \cdot) > M(x, Tx, t_0)\) and \(N(Tx, T^2x, \cdot) < N(x, Tx, t_0)\). On the other hand, from Lemma 1,
\[
M(x, Tx, t_0) = \lim_i M(T^{n_i}x_0, T(T^{n_i}x_0), t_0)
= \lim_i M(T^{n_i+1}x_0, T^{n_i+2}x_0, t_0)
\geq M(Tx, T^2x, t_0)
\]
and
\[
N(x, Tx, t_0) = \lim_i N(T^{n_i}x_0, T(T^{n_i}x_0), t_0)
= \lim_i N(T^{n_i+1}x_0, T^{n_i+2}x_0, t_0)
\leq N(Tx, T^2x, t_0)
\]
a contradiction. Therefore \(T x_0 = x_0\), i.e., \(x_0\) is a fixed point of \(T\). Uniqueness follows at once from (3.11).

Finally, we prove that \(\lim_{n \to \infty} T^n x_0 = x_0\). Since \(\lim_i T^{n_i}x_0 = x_0\) and \(\lim_i T^{n_i+1}x_0 = T x_0 = x_0\), by Lemma 1,
\[
\lim \inf_i M(T^{n_i}x_0, T^{n_i+1}x_0, t) \geq M(x, x, t) = 1
\]
and
\[
\lim \sup_i N(T^{n_i}x_0, T^{n_i+1}x_0, t) \leq N(x, x, t) = 0, \forall t > 0.
\]
So \(\lim_i M(T^{n_i}x_0, T^{n_i+1}x_0, t) = 1\) and \(\lim_i N(T^{n_i}x_0, T^{n_i+1}x_0, t) = 0, \forall t > 0\). For any \(n \in \mathbb{N}\) with \(n > n_i\), there exists \(n_i\) with \(n_{i+1} \geq n > n_i\). From (3.11),
\[
M(T^n x_0, x, t) \geq M(T^{n-1}x_0, T^n x_0, t) \ast M(T^{n-1}x_0, x, t)
\geq M(T^{n-1}x_0, T^n x_0, t) \ast M(T^{n-2}x_0, T^{n-1}x_0, t)
\ast M(T^{n-2}x_0, x, t)
= M(T^{n-2}x_0, T^{n-1}x_0, t) \ast M(T^{n-2}x_0, x, t)
\geq \cdots \geq M(T^n x_0, T^{n+1}x_0, t) \ast M(T^n x_0, x, t)
\]
and
\[
N(T^n x_0, x_*, t) \leq N(T^{n-1} x_0, T^n x_0, t) \triangleq N(T^n x_0, x_*, t)
\]
\[
\leq N(T^n x_0, T^{n+1} x_0, t) \triangleq N(T^n x_0, x_*, t)
\]
\[
\leq \cdots \leq N(T^n x_0, T^{n+1} x_0, t) \triangleq N(T^n x_0, x_*, t).
\]
Letting \( n \to \infty \) \((n_i \to \infty)\), we have
\[
\lim_n M(T^n x_0, x_*, t) \geq 1 \text{ and } \lim_n N(T^n x_0, x_*, t) \leq 0, \forall t > 0.
\]
Hence we get \( \lim_n T^n x_0 = x_* \). This completes the proof. \( \Box \)

Remark 4. Theorem 2 (i.e., Theorem 1 of [2]) is the immediate consequence of Theorem 4.

4. Applications to product spaces

In this chapter, we apply Theorem 3, Corollary 1 and Corollary 2 to obtain fixed point type theorems on the product of an I-FM space.

Theorem 5. Let \( X \) be a complete I-FM space and \( T : X \times X \to X \) such that be a mapping satisfying the following conditions:

(i) there exists \((x_0, y_0) \in X \times X\) such that
\[
\lim_{t \to \infty} M((x_0, y_0), T^i(x_0, y_0), t) = 1 \text{ and }
\]
\[
\lim_{t \to \infty} N((x_0, y_0), T^i(x_0, y_0), t) = 0, \ i = 1, 2, \ldots ;
\]

(ii) there exists a mapping \( m : X \times X \to \mathbb{N} \) such that for any \((x, y), (u, v)\) in \( X \times X\),
\[
M(T^m(x,y) (x, y), T^m(x,y) (u, v), t) \geq M((x, y), (u, v), \phi(t)),
\]
\[
N(T^m(x,y) (x, y), T^m(x,y) (u, v), t) \leq N((x, y), (u, v), \phi(t)),
\]
where the function \( \phi(t) \) satisfies condition (\( \phi_1 \)) and
\[
(\phi_2) \lim_{t \to \infty} [\phi(t) - \ell] = \infty.
\]
Then there exists exactly one point \( q \in X \) such that \( T^m(q,y) (q, y) = q \) for all \( y \in X \) for each \( m(q, y) \in \mathbb{N} \).

Proof. For a fixed \( x \in X \) and \( y = v \), the inequality (ii) corresponds to the condition (ii) of Theorem 3. Therefore for each \( x \in X \), there exists one and only one \( x(y) \) in \( X \) such that \( T^{m(x(y), y)} (x(y), y) = x(y) \) and \( T^{m(x(y), y)} (x(v), v) = x(v) \), \( m(x(y), y) \in \mathbb{N} \).
Now, for every \( y, v \in X \), from (ii) we get
\[
\begin{align*}
M(x(y), x(v), t) &= M(T^{m(x(y), y)}(x(y), y), T^{m(x(y), y)}(x(v), v), t) \\
&\geq M(x(y), x(v), \phi(t)), \\
N(x(y), x(v), t) &= N(T^{m(x(y), y)}(x(y), y), T^{m(x(y), y)}(x(v), v), t) \\
&\leq N(x(y), x(v), \phi(t)).
\end{align*}
\]
On the other hand, as \( M(x(y), x(v), t) \) is non-decreasing and \( N(x(y), x(v), t) \) is non-increasing, we have
\[
\begin{align*}
M(x(y), x(v), t) &\leq M(x(y), x(v), \phi(t)) \quad \text{and} \\
N(x(y), x(v), t) &\geq N(x(y), x(v), \phi(t)).
\end{align*}
\]
Hence
\[
\begin{align*}
M(x(y), x(v), t) &= M(x(y), x(v), \phi(t)) = M(x(y), x(v), \phi^n(t)), \quad \forall t > 0, \\
N(x(y), x(v), t) &= N(x(y), x(v), \phi(t)) = N(x(y), x(v), \phi^n(t)), \quad \forall t > 0.
\end{align*}
\]
By the condition \((\phi_1)\),
\[
M(x(y), x(v), t) = 1 \text{ and } N(x(y), x(v), t) = 0, \quad \forall t > 0.
\]
Then by Definition 3 \(\text{[(iii) and (ix)]}\) we have \(x(y) = x(v)\). So, \(u(\cdot)\) is some constant \(q \in X\) and conclusions of the theorem are obtained. \(\square\)

If \(\phi(t) = \frac{t}{k}\) in Theorem 5, we obtain an application on the product of an I-FM space of Corollary 1.

**Corollary 3.** Let \(X\) be a complete I-FM space and \(T : X \times X \to X\) such that be a mapping satisfying the following conditions:

(i) there exists \((x_0, y_0) \in X \times X\) such that
\[
\lim_{t \to \infty} M((x_0, y_0), T^i(x_0, y_0), t) = 1 \quad \text{and}
\lim_{t \to \infty} N((x_0, y_0), T^i(x_0, y_0), t) = 0, \quad i = 1, 2, \ldots;
\]

(ii) there exists a mapping \(m : X \times X \to \mathbb{N}\) such that for any \((x, y), (u, v)\) in \(X \times X\),
\[
\begin{align*}
M(T^{m(x, y)}(x, y), T^{m(x, y)}(u, v), t) &\geq M((x, y), (u, v), \frac{t}{k}), \\
N(T^{m(x, y)}(x, y), T^{m(x, y)}(u, v), t) &\leq N((x, y), (u, v), \frac{t}{k}),
\end{align*}
\]
where \(0 < k < 1\). Then the conclusion of Theorem 5 remains true.

**Proof.** Taking \(\phi(t) = \frac{t}{k}\). Obviously, \(\phi(t)\) satisfies the conditions \((\phi_1)\) and \((\phi_2)\). Therefore the conclusion follows from Theorem 5 directly. \(\square\)
Corollary 4. Let $X$ be a complete I-FM space and $T : X \times X \to X$ be a mapping. If there exists a mapping $m : X \times X \to \mathbb{N}$ such that for any $(x, y)$ in $X$,

\begin{align*}
M(T^m(x,y)(x, y), T^m(x,y)(u, v), t) & \geq M((x, y), (u, v), \phi(t)), \\
N(T^m(x,y)(x, y), T^m(x,y)(u, v), t) & \leq N((x, y), (u, v), \phi(t)),
\end{align*}

where the function $\phi(t)$ satisfies conditions $(\phi_1)$ and $(\phi_2)$. Then there exists exactly one point $q \in X$ such that $T^m(x,y)(q, y) = q$ for all $y \in X$ for each $m(x, y) \in \mathbb{N}$.

Proof. It is clear from proof of Theorem 5. \hfill \Box

Remark 5. Taking $\phi(t) = \frac{t}{k} \ (0 < k < 1)$ and $m(x, y) \equiv 1$ in Corollary 4, we obtain an application on product space of Theorem 1.

Conclusion. Essentially, from (2.4) it is easy to see that $T$ is continuous and (3.11) hold for any $x_0 \in X$. In addition, from the compactness of $X$, $\{T^nx_0\}$ has an accumulation point. Hence Theorem 2 follows immediately from Theorem 4. Thus we improve and generalize the well-known fixed point theorems of Banach [4] and Edelstein [8] were given by Alaca et al. [2] in intuitionistic fuzzy metric spaces. Our main results are intuitionistic fuzzy version of Fang’s results [10]. These fixed point theorems are applied to obtain solutions of fixed point type equations on product spaces.

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References


