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Abstract. We will discuss some very old and some new open problems concerning infinite dimensional algebras. All these problems have been inspired by combinatorial group theory.

I. The Burnside and Kurosh problems

In 1902 W. Burnside formulated his famous problems for torsion groups:

(1) let G be a finitely generated torsion group, that is, for an arbitrary element $g \in G$ there exists $n = n(g) \geqslant 1$, such that $g^n = 1$. Does it imply that G is finite?

(2) Let a group G be finitely generated and torsion of bounded degree, that is, there exists $n \geqslant 1$ such that for an arbitrary element $g \in G$ $g^n = 1$. Does it imply that G is finite?

W. Burnside [7] and I. Schur [43] proved (1) for linear groups. The positive answer for (2) is known for $n = 2, 3$ (W. Burnside, [6]), $n = 4$ (I. N. Sanov, [42]) and $n = 6$ (M. Hall, [17]).

In 1964 E. S. Golod and I. R. Shafarevich ([12], [13]) constructed a family of infinite finitely generated p–groups (for an arbitrary element g there exists $n = n(g) \geqslant 1$ such that $g^{p^n} = 1$) for an arbitrary prime p. This was a negative answer to the question (1). Other finitely generated torsion groups were constructed by S. V. Alyoshin [1], R. I. Grigorchuk [14], N. Gupta –S. Sidki [16], V. I. Sushchansky [48]. In 1968 P. S. Novikov and S. I. Adian constructed infinite finitely generated groups of bounded odd degree $n \geqslant 4381$. In 1994 S. Ivanov [19] extended this to $n = 2^k, k \geqslant 32$, so now we can say that the question (2) has negative solution for all sufficiently large n.

Remark though that all the counterexamples above are not finitely presented. The following important problem still remains open.

Problem 1. Do there exist infinite finitely presented torsion groups?

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A. Yu. Ol'shansky and M. V. Sapir [36] constructed a finitely presented group, which in torsion–by–cyclic.

In 1940 A. G. Kurosh [24] (and independently J. Levitzky, see [2]) formulated an analog of the Burnside problems for algebras:

(1) Let K be a field and let A be a finitely generated K-algebra, which is algebraic, that is, for an arbitrary element $a \in A$ there exists a polynomial $f_a(t) \in K[t]$ such that $f_a(a) = 0$. Does it imply that the algebra A is finite dimensional?
(2) What about algebras, which are algebraic of bounded degree, that is, the degrees of all polynomials $f_a(t)$, $a \in A$, are uniformly bounded from above?

Of particular interest is the case of nil algebras when all polynomials $f_a(t)$ are powers of t.

The negative answer to the question (1) is provided by the same Golod-Shararevich examples [12], [13], however, the answer to the question (2) is YES (N. Jacobson [20], I. Kaplansky [21], J. Levitzky [26], A. I. Shirshov [46]). Just as in group theory the following problem is still open.

Problem 2. Is it true that every finitely presented algebraic (nil) algebra is finite dimensional?

Let $A = A_1 + A_2 + \cdots$ be a graded algebra, $A_iA_j \subseteq A_{i+j}$, $\dim A_i < \infty$ for all $i, j \geqslant 1$. We say that the algebra A is *quadratic* (see [32], [39]) if it is generated by A_1 and presented (in these generators) by relators which all look as $\sum\limits_{i,j} \alpha_{ij}x_ix_j = 0$, $\alpha_{ij} \in K$.

By a Veronese subalgebra $A^{(n)}$ of A we mean $A^{(n)} = A_n + A_{2n} + A_{3n} + \cdots$. J. Bachelin [4] noticed that if a graded algebra A is presented by a finite system of relators, all of degrees $< d$, then for an arbitrary $n \geqslant d$ the Veronese subalgebra $A^{(n)}$ is quadratic. This reduces the Kurosh problem for finitely presented graded algebras to the quadratic case.

Problem 3. Is it true that a quadratic nil algebra is nilpotent?

I would expect this problem and, more generally, The Kurosh Problem for finitely presented algebras to have positive solution. This opens a possibility of an interesting dichotomy for groups. Recall that a group is residually finite if the intersection of all its subgroups of finite index is trivial. The Burnside Problem for groups of bounded degree has negative solution in the class of all groups but positive solution in the class of residually finite groups ([51], [55], [56]), because residually p–groups behave somewhat like algebras (see [57]). It is conceivable that infinite finitely presented torsion groups exist in the class of all groups, but, perhaps, not in the class of residually finite groups.

II. The Golod–Shafarevich construction

Let's recall the notion of a pro-p group. A group G is said to be residually-p if the intersection of all subgroups of G of p–power index is trivial. Taking all these subgroups for the basis of neighborhoods of 1 we define a topology on G. If this topology is complete then G is called a pro-p group. Another way to say it: a pro–p group is an inverse limit of finite p–groups. If the group G is residually–p but the topology is not compete then we can embed G into the completion $G_{\widehat{p}}$, which is called the pro–p completion of G.

For an arbitrary (not necessarily residually–p) group G we can pass to the quotient $G/\cap\{H \lhd G \mid |G:H| = p^k,\ n \geqslant 1\}$ and then to its pro–p completion, which is denoted as $G_{\widehat{p}}$ and called the pro–p completion of G.

The free group $F_m\langle X\rangle$ on the set of free generators $X = \{x_1, \ldots, x_m\}$ is residually–p for an arbitrary prime p. Its pro–p completion $F = F_m\langle X\rangle_{\widehat{p}}$ is free in the category of pro–p groups: an arbitrary mapping of X into an arbirtary pro–p group G uniquely extends to a continuous homomorphism $F \to G$.

Consider the field $K = \mathbb{Z}/p\mathbb{Z}$, the group algebra KF and its fundamental ideal $\omega = \{\sum\limits_i \alpha_i f_i,\ \alpha_i \in K,\ f_i \in F,\ \sum \alpha_i = 0\}$. Then $\bigcap\limits_{i \geqslant 1} \omega^i = (0)$. The descending chain of invariant subgroups

$$F = F_1 > F_2 > \cdots,\ F_i = F \cap (1 + \omega^i),$$

is called the Zassenhaus filtration, $\bigcap\limits_{i \geqslant 1} F_i = (1)$.

Let $1 \neq f \in F,\ f \in F_i \smallsetminus F_{i+1}$. Then we say that $\deg(f) = i$.

For a subset $R \subset F$ let $N(R)$ be the smallest closed normal subgroup of F, which contains R. The group $G = F/N(R) = \langle X \mid R = 1\rangle$ is said to be presented by the set of generators X and the set of relators R in the category of pro–p groups. If $G = \langle X \mid R = 1\rangle$ is a presentation of a discrete group then the same presentation in the category of pro–p groups defines the pro–p completion $G_{\widehat{p}}$.

Suppose that the set R does not contain elements of degree 1, but contains r_i elements of degree $i,\ i \geqslant 2$. The formal power series $H_R(t) = \sum\limits_{i=2}^{\infty} r_i t^i$ is called the Hilbert series of R. We will consider also the Hilbert series $H_{KG}(t) = 1 + \sum\limits_{i=1}^{\infty} \dim\left(\dfrac{\omega(KG)^i}{\omega(KG)^{i+1}}\right) t^i$. Golod and Shafarevich [13] (see also Roquette [41] and E. B. Vinberg [50]) proved that

$$\frac{H_{KG}(t)(1 - mt + H_R(t))}{1 - t} \geqslant \frac{1}{1 - t}$$

formally. This inequality has the following interesting corollary. Suppose that there exists a number $0 < t_0 < 1$ such that the series $H_R(t)$ converges at t_0 and $1 - mt_0 + H_R(t_0) < 0$. Then the group G is infinite.

Definition 1. We say that a pro–p group G is a Golod-Shafarevich (GS) group if it has a presentation with the above property.

Every GS–group is infinite. For example, if $|R| < m^2/4$ then such a t_0 exists and therefore G is a GS–group. In [58] it is proved that every GS–group contains a nonabelian free pro–p as a subgroup.

A discrete group is GS if its pro–p completion is GS.

Examples of GS–groups.

(1) As we have mentioned above, E.S. Golod constructed a finitely generated p–group, which is GS and therefore infinite.

(2) A. Lubotzky [28] showed that the fundamental group of a hyperbolic 3–manifold has a finite index subgroup which is GS, for all but finitely many p.

(3) Let S be a finite set of primes, $p \notin S$. Denote the maximal p–extension of $\mathbb{Q}$ unramified outside S (allowing ramification at infinity) as $\mathbb{Q}_S$, $G_S = \mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q})$. I. Shafarevich [45] proved that G_S has a presentation $\langle x_1, \ldots, x_m \mid {x_i}^{p^{k_i}} = [x_i, a_i],\ 1 \leqslant i \leqslant m\rangle, m = |S|$ and therefore is GS for $m > 4$.

In the same way we can define GS–algebras [13]. The most natural context for it is the category of topological profinite algebras, but we will restrict ourselves only to graded algebras.

Let $K\langle X\rangle$ be the free associative algebra on the set of free generators $X = \{x_1, \ldots,\ x_m\}$ over the field K. Assigning the degree 1 to generators from X we define a gradation on $K\langle X\rangle = K \cdot 1 + \sum\limits_{i=1}^{\infty} K\langle X\rangle_i$.

Let R be a graded subspace of $\sum\limits_{i=2}^{\infty} K\langle X\rangle_i,\ R_i = R \cap K\langle X\rangle_i$. The formal series $H_R(t) = \sum\limits_{i=2}^{\infty} (\dim R_i)t^i$ is called the Hilbert series of R. The algebra $A = \langle X \mid R = (0)\rangle$ presented by the generators X and the relators R is graded, $A = \sum\limits_{i=0}^{\infty} A_i,\ H_A(t) = \sum\limits_{i=0}^{\infty} (\dim A_i)t^i$. E. S. Golod and I. R. Shafarevich proved that $H_A(t)(1 - mt + H_R(t)) \geqslant 1$. As in the case of groups this inequality implies that if there exists a number t_0 such that $H_R(t)$ converges at t_0 and $1 - mt_0 + H_R(t_0) < 0$ then A is infinite dimensional. If A has such a presentation then we say that A is a GS–algebra.

J. Wilson [54] proved that every discrete GS–group has an infinite torsion homomorphic image and every GS–algebra has an infinite–dimensional nil homomorphic image.

III. Growth and GK–dimension

Let G be a group which is generated by a finite set S, $1 \in S = S^{-1}$. Consider the ascending chain of finite subsets $S^n = \{s_1 \dots s_n \mid s_i \in S\}, S = S^1 \subset S^2 \subset \cdots$, $G = \cup_{n=1}^{\infty} S^n$. We say that the group G has a polynomial growth if there exists a polynomial $f(t)$ such that $|S| \leqslant f(n)$ for all n. This property does not depend on a generating set and has origin in geometry (see [9], [34], [44]). Answering a question of J. Milnor, M. Gromov [15] proved that a group has polynomial growth if and only if it has a nilpotent subgroup of finite index. In particular it implies that a torsion group having a polynomial growth is finite. I am not aware of any proof of this fact that does not rely on Gromov's theorem.

Now let A be a K–algebra, which is generated by a finite dimensional subspace S. Let S^n be the subspace of A which is spanned by all products in S of length $\leqslant n$ (with all possible brackets). As above $S = S^1 < S^2 < \cdots$, $\dim_K S^n < \infty$, $\underset{n\geqslant 1}{\cup} S^n = A$.

The algebra A has a finite Gelfand–Kirillov dimension (or a polynomial growth) if there exists a polynomial $f(t)$ such that $\dim_K S^n \leqslant f(n)$ for all $n \geqslant 1$. In this case the Gelfand-Kirillov dimension is defined as $GK \dim(A) = \limsup_{n\to\infty} \dfrac{\ln \dim_K S^n}{\ln n}$ (see [11], [23]).

Problem 4. (L. Small). Does there exist an infinite dimensional nil algebra of finite GK–dimension?

Recently T. Lenagan and A. Swoktunowicz [27] constructed infinite dimensional nil algebras of finite GK–dimensions over countable fields.

Problem 5. Is it true that an arbitrary GS–algebra has an infinite dimensional homomorphic image of finite GK–dimension?

If the answer is YES then it provides examples of infinite dimensional nil algebras of finite GK–dimension over arbitrary fields.

IV. Self–similar Lie algebras

As we have mentioned above, after the counterexamples of Golod and Shafarevich new finitely generated infinite torsion groups were constructed by (i) S. V. Alyoshin, R. I. Grigorchuk, N. Gupta –S. Sidki, V. Sushchansky, not to mention (ii) infinite torsion groups of bounded degree of P. S. Novikov and S. I. Adian and Tarski Monsters of A. Yu. Ol'shansky. The groups (i) are residually finite whereas the groups (ii) are not. The Grigorchuk groups are of particular interest since they are of intermediate growth: the sequence $|S^1| < |S^2| < \cdots$ grows faster than any polynomial but slower than any exponential function. Is there an analog for algebras?

The Grigorchuk group is a group of automorphisms of a regular rooted tree. It is natural therefore to look for "Grigorchuk algebras" among algebras of differential operators in infinitely many variables (which correspond to infinitely many vertices of a tree). The first such construction was suggested by V. Petrogradsky for fields of characteristic 2 [37]. I. Shestakov and E. Zelmanov [49] generalized it and extended to algebras of arbitrary positive characteristics.

Let K be a field of characteristic $p > 0$; $T = K[t_0, t_1, \dots \mid {t_i}^p = 0,\ i \geqslant 0]$ is the polynomial algebra in countably many truncated variables. Denote $\partial_i = d/dt_i$. An arbitrary derivation of the algebra T can be represented as an infinite sum $\sum\limits_{i=0}^{\infty} a_i\partial_i,\ a_i \in T$.

Let D_0 consist of derivations $\sum\limits_{i=0}^{\infty} a_i\partial_i,\ a_i \in T$, with the following properties:

(1) every coefficient a_i depends only on $t_0, \dots, t_{i-2}$,
(2) there exists a constant $c > 0$ such that each a_i involves only monomials of degree $\geqslant i(p-1) - c$

It is not difficult to check that D_0 is a Lie algebra, which is closed with respect to p–powers.

Let T^s denote the ideal of T spanned by all monomials of degree $\geqslant s$. Let D be the algebra of all derivations of T. Clearly, $\bigcap\limits_{s\geqslant 1} T^s D = (0)$, the subspaces $T^s D \cap D$ define a topology on D.

Let $D_{\text{fin}} = \{\sum\limits_{i=0}^{r} a_i\partial_i\}$ be the subspace of D consisting of finite sums and let $\overline{D}_{\text{fin}} = \bigcap\limits_{s\geqslant 1}(D_{\text{fin}} + (T^s D \cap D))$ be the closure of D_{fin}. Consider the Lie algebra

$$L = D_0 \cap \overline{D}_{\text{fin}}.$$

Theorem 1 ([37], [49]). (a) *An arbitrary element of L is nilpotent;*
(b) *Consider the elements $v_n = \partial_n + \sum\limits_{i=n+1}^{\infty}(t_0 \cdots t_{i-2})^{p-1}\partial_i$ of L. The Lie algebra Lie $\langle v_1, v_2\rangle$ generated by v_1, v_2 is not nilpotent and has Gelfand–Kirillov dimension lying between* 1 *and* 2*.*

Let A be the associative subalgebra of $\mathrm{End}_k T$ generated by v_1, v_2.

Problem 6. Is A a nil algebra?

If yes, then A would provide another counterexample to The Kurosh Problem. Remark, that A is of subexponential growth (see [47]).

An important still unresolved part of The Kurosh Problem is The Kurosh Problem for division algebras.

Problem 7. Do there exist infinite dimensional finitely generated algebraic division algebras?

Even a weaker version of this question is still open.

Problem 8. Do there exist infinite dimensional finitely generated division algebras?

There is a reasonable expectation that the Lie algebras described above may be of some help.

Let U be the universal associative enveloping algebra of Lie $\langle v_1, v_2\rangle$. Let Z be the center of U. From the case (a) of the theorem above it follows that for an arbitrary element $a \in \mathrm{Lie}\langle v_1, v_2\rangle$ some power a^{p^k} falls into Z. Consider the ring of fractions $D = (Z \setminus \{0\})^{-1}U$.

Problem 9. Is D an algebraic division algebra?

V. Property tau and dimension expanders

Let $X = (V, E)$ be a finite connected graph. Let $\varepsilon > 0$. For a subset of vertices $W \subset V$ let ∂W denote its boundary $\partial W = \{v \in V \mid \mathrm{dist}(v, W) = 1\}$. We say that X is an ε–expander if for every $W \subset V$ with $|W| \leqslant \frac{1}{2}|V|$ we have $|W \cup \partial W| \geqslant (1+\varepsilon)|W|$ (see [38], [53]).

Of particular interest are families of k–regular ε–expander graphs, where k and ε are fixed but $|V| \to \infty$. The first explicit examples of such families are due to G. A. Margulis [33]. He noticed that if G is a group with the Kazhdan property(T) (see [22]), which is generated by a finite set S and $\rho_i : G \to G_i$ is a family of epimorphisms onto finite groups G_i, $S_i = \rho_i(S), G_i = \langle S_i\rangle$, then the family of Cayley graphs $\mathrm{Cay}(G_i, S_i)$ is an expander family. In fact, to produce a family of expanders the group G does not need to satisfy the property (T). This led to the following definition (see [30]): *a group G generated by a finite set S has property (τ) if there exists $\varepsilon > 0$ such that for every homomorphism onto a finite group G', $S \to S'$, the Cayley graph $Cay(G', S')$ is an ε–expander.*

Motivated by some considerations from theoretical computer science B. Barak, R. Impagliazzo, A. Shpilka and A. Wigderson [5] suggested a notion of a dimension expander.

Definition 2. Let K be a field, $m \geqslant 1$, $\varepsilon > 0$, V a vector space of dimension n over K and $T_1, \ldots, T_m$ are K–linear transformations from V to V. We say that the pair $(V, \{T_i\}_{i=1}^m)$ in an ε–dimension expander if for every subspace W of dimension $\leqslant n/2$ we have $\dim(W + \sum\limits_{i=1}^{m} T_i W) \geqslant (1+\varepsilon)\dim W$.

In [53] A. Wigderson posed a problem of finding an explicit construction of a family of dimension expanders with K, m, ε fixed but $\dim_K V \to \infty$.

It is tempting to approach the problem through infinite dimensional “algebras with property τ”, whose finite dimensional irreducible representations would produce a desired family of dimension expanders. We suggest the following definition.

Definition 3 ([31])**.** A residually finite associative K–algebra A generated by a finite dimensional space S satisfies the property τ if there exists $\varepsilon > 0$ such that for every infinite dimensional residually finite

A–module ${}_AV$ without nonzero finite dimensional submodules for every finite dimensional subspace W of V we have $\dim(W + SW) \geqslant (1+\varepsilon)\dim W$.

Theorem 2 ([31])**.** *Let p be a prime number, $K_p = \mathbb{Z}/p\mathbb{Z}$, $G = SL^1(n, K_p[t])$, $n \geqslant 3$, is the congruence subgroup of the special linear group over the ring of polynomials $K_p[t]$. Let K be a field of zero characteristic. Then the group algebra KG has property τ.*

This theorem provides examples of dimension expanders over fields of zero characteristics. The case of finite fields is still open.

Problem 10. Let p and q be distinct primes. Does the group algebra

$$K_q[SL^1(n, K_p[t])], n \geqslant 3$$

satisfy property τ?

We even don't know an answer to the following question.

Problem 11. Let K be a field of zero characteristic. Does the group algebra $K[SL(3,\mathbb{Z})]$ satisfy τ?

VI. Golod–Shafarevich construction and property τ

We will start with the following long standing question on hyperbolic 3–manifolds.

The Virtual Haken Conjecture. Every irreducible compact 3–dimensional hyperbolic manifold has a finite sheeted cover which is Haken, or, equivalently, every cocompact lattice Γ in $SL(2, \mathbb{C})$ has a finite index subgroup which is either a free product with amalgam, or an HNN–construction.

M. Lackenby recently showed [25] that an important part of the conjecture above would follow if one proved that the cocompact lattice Γ does not have property τ.

Conjecture 1. (A. Lubotzky–P. Sarnak, [29]) *A lattice in $SL(2,\mathbb{C})$ does not have property τ.*

We have mentioned above that for all but finitely many prime p such that a lattice has a subgroup of finite index which is GS (section II, example 2). The Lubotzky–Sarnak conjecture would follow, therefore, if all GS pro–p groups did not have τ. However recently M. Ershov [10] proved that positive parts of certain Kac–Moody groups over finite fields are GS–groups and are known to have property T. ([8], [40]).

What distinguishes GS–groups of hyperbolic 3–manifolds among all GS–groups? For example, GS–groups of hyperbolic 3–manifolds are hereditary GS, that is, every subgroup of finite index of such a group is again GS. It follows because a subgroup of finite index of a 3–manifold group is again a 3–manifold group [28].

T. Voden [52] addressed this question for graded algebras.

Theorem 3 ([52])**.** (a) *Let A be an m generated graded algebra, presented by $< \frac{1}{4}(\frac{m}{2}-1)^2$ homogeneous relators of degree $\geqslant 2$. Then infinitely many Veronese subalgebras $A^{(n)}$ are GS;*

(b) *if in addition A is quadratic and the number of relators is $< \frac{4}{25}m^2$ then infinitely many Veronese subalgebras are GS;*

(c) *for generic quadratic algebras* (see [3]) *the bound $\frac{4}{25}m^2$ is sharp. If the number of relators is $< \frac{4}{25}m^2$ then all but finitely many Veronese subalgebras are GS; if the number of relators is $\geqslant \frac{4}{25}m^2$ then all but finitely many Veronese subalgebras are not GS.*

Conjecture 2. *A GS–algebra does not have τ.*

Remark that this result would follow if the conjecture about infinite homomorphic images of finite GK–dimension from the section III were true.

As in the section I we can reduce the question to quadratic algebras and even to generic quadratic algebras.

Conjecture 3. *Let $A = A_1 + A_2 + \cdots$ be a graded algebra generated by A_1, $\dim A_1 = m$ and presented by $< m^2/4$ generic quadratic relators. Then all but finitely many Veronese subalgebras can be epimorphically mapped onto the polynomial algebra $K[t]$.*

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