ADDITIVITY OF LIE MAPS ON OPERATOR ALGEBRAS

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Abstract. Let \( \mathcal{A} \) be a standard operator algebra which does not contain the identity operator, acting on a Hilbert space of dimension greater than one. If \( \Phi \) is a bijective Lie map from \( \mathcal{A} \) onto an arbitrary algebra, that is

\[
\Phi(AB - BA) = \Phi(A)\Phi(B) - \Phi(B)\Phi(A)
\]

for all \( A, B \in \mathcal{A} \), then \( \Phi \) is additive. Also, if \( \mathcal{A} \) contains the identity operator, then there exists a bijective Lie map of \( \mathcal{A} \) which is not additive.

1. Introduction

Throughout, for a Hilbert space \( \mathcal{H} \), we write \( B(\mathcal{H}) \) for the algebra of all bounded linear operators on \( \mathcal{H} \). Usually, a standard operator algebra on \( \mathcal{H} \) will mean a subalgebra of \( B(\mathcal{H}) \) containing all finite rank operators. Let \( \mathcal{A} \) and \( \mathcal{B} \) be two algebras or rings. A map \( \Phi : \mathcal{A} \to \mathcal{B} \) is called a Lie map if it is multiplicative with respect to the Lie product \( AB - BA \), that is

\[
\Phi(AB - BA) = \Phi(A)\Phi(B) - \Phi(B)\Phi(A)
\]

for all \( A, B \in \mathcal{A} \).

Characterizing the interrelation between the multiplicative and the additive structures of a ring is an interesting topic. This question was first studied by Martindale who obtained the surprising result that every bijective multiplicative map from a prime ring containing a nontrivial idempotent onto an arbitrary ring is necessarily additive [10]. For operator algebras, the same problem was treated in [1, 7, 15]. In the papers [2, 3, 8, 9, 11, 12, 13], the additivity of maps on operator algebras which are multiplicative with respect to other products, such as the Jordan product \( AB + BA \) or the Jordan triple product \( ABA \), were investigated. Also, the papers [4, 5, 6, 14] studied the similar questions for elementary maps and Jordan elementary maps on rings or operator algebras.

In this note, we shall study the additivity of Lie maps on operator algebras. More precisely, it will be proved that every bijective Lie map on a standard operator algebra \( \mathcal{A} \) which does not contain the identity operator, acting on a Hilbert space \( \mathcal{H} \) of dimension greater than one, is automatically additive. In...
particular, if \( \dim \mathcal{H} = \infty \) and \( \mathcal{A} \) is either the ideal of all finite rank operators or the ideal of all compact operators in \( B(\mathcal{H}) \), then every bijective Lie map on \( \mathcal{A} \) is additive. Furthermore, we show that if \( \mathcal{A} \) contains the identity operator, then there must exist a bijective Lie map on \( \mathcal{A} \) which is not additive.

It should be mentioned that Lu in [8] proved that a bijective Jordan map on a standard operator algebra which is allowed to contain the identity operator, is additive. Although the basic ideas used in our proof are similar to those in [8], some concrete techniques are new.

2. Result and Proof

Our main result reads as follows.

**Theorem 1.** Let \( \mathcal{H} \) be a real or complex Hilbert space with \( \dim \mathcal{H} > 1 \), \( \mathcal{A} \subseteq B(\mathcal{H}) \) be a standard operator algebra which does not contain the identity operator \( I \) and \( \mathcal{B} \) be an arbitrary algebra. If \( \Phi : \mathcal{A} \to \mathcal{B} \) is a bijective Lie map, then \( \Phi \) is necessarily additive.

We shall organize the proof of Theorem 1 in a series of lemmas, in which the notation of the theorem will be kept. Since \( \dim \mathcal{H} > 1 \), we can take a non-trivial orthogonal projection \( P_1 \) which has finite rank. Then \( P_1 \in \mathcal{A} \). Put \( P_2 = I - P_1 \). Note that \( P_2 \) is not in \( \mathcal{A} \). Let \( \mathcal{A}_{ij} = P_i \mathcal{A} P_j \), \( i, j = 1, 2 \). Then

\[
\mathcal{A} = \mathcal{A}_{11} \oplus \mathcal{A}_{12} \oplus \mathcal{A}_{21} \oplus \mathcal{A}_{22}
\]

which is the Peirce decomposition of \( \mathcal{A} \). This idea is essentially from Martin-dale [10].

**Lemma 1.** \( \Phi(0) = 0 \).

**Proof.** It is obvious. \( \square \)

**Lemma 2.** If \( A, B, S \in \mathcal{A} \) such that \( \Phi(S) = \Phi(A) + \Phi(B) \), then for all \( T \in \mathcal{A} \), we have

1. \( \Phi(ST - TS) = \Phi(AT - TA) + \Phi(BT - TB) \),
2. \( \Phi(TS - ST) = \Phi(TA - AT) + \Phi(TB - BT) \).

**Proof.** Let \( T \in \mathcal{A} \). Then

\[
\Phi(ST - TS) = \Phi(S)\Phi(T) - \Phi(T)\Phi(S)
= (\Phi(A) + \Phi(B))\Phi(T) - \Phi(T)(\Phi(A) + \Phi(B))
= \Phi(A)\Phi(T) - \Phi(T)\Phi(A) + \Phi(B)\Phi(T) - \Phi(T)\Phi(B)
= \Phi(AT - TA) + \Phi(BT - TB).
\]

So (1) holds. Similarly, we can prove (2). \( \square \)

In the following, the notation \( A_{ij} \) will denote an arbitrary element in \( \mathcal{A}_{ij} \).

**Lemma 3.** Let \( S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathcal{A} \).
(1) If \( T_{ij}S_{jk} = 0 \) for all \( T_{ij}, 1 \leq i, j, k \leq 2 \), then \( S_{jk} = 0 \). If \( S_{kl}T_{ij} = 0 \) for all \( T_{ij}, 1 \leq i, j, k \leq 2 \), then \( S_{kl} = 0 \);
(2) If \( ST_{ij} - T_{ij}S \in \mathcal{A}_{ij} \) for all \( T_{ij}, 1 \leq i \neq j \leq 2 \), then \( S_{ji} = 0 \);
(3) If \( ST_{ij} - T_{ij}S \in \mathcal{A}_{ij} \) for all \( T_{ij}, 1 \leq i \neq j \leq 2 \), then \( S_{ji} = 0 \) and \( S_{jj} = \lambda P_j \) for some scalar \( \lambda \);
(4) If \( ST_{ij} - T_{ij}S \in \mathcal{A}_{ij} \) for all \( T_{ij}, 1 \leq i \neq j \leq 2 \), then \( S_{ij} = 0 \) and \( S_{jj} = \lambda P_j \) for some scalar \( \lambda \);
(5) If \( ST_{ij} - T_{ij}S \in \mathcal{A}_{ij} \) for all \( T_{ij}, j = 1, 2 \), then \( S_{ji} = S_{ij} = 0 \) for \( 1 \leq i \neq j \leq 2 \).

**Proof.** (1) It is [8, Lemma 2(ii)].

(2) By the hypothesis, we have obviously \( S_{ji}T_{ij} = P_j(ST_{ij} - T_{ij}S) = 0 \) for all \( T_{ij} \) with \( i \neq j \). Hence \( S_{ji} = 0 \) by (1).

(3) Similar to (2), we can easily obtain that \( S_{ji} = 0 \). Also, for every \( T_{jj} \), we have \( P_j(ST_{jj} - T_{jj}S)P_j = 0 \). Hence \( T_{jj}S_{jj} = S_{jj}T_{jj} \), which implies that \( S_{jj} \) commutes all operators in \( B(P_j, \mathcal{H}) \). It is well known that \( S_{jj} = \lambda P_j \) for some scalar \( \lambda \).

Similarly, we can prove (4) and (5).

**Lemma 4.** For \( 1 \leq i \neq j \leq 2 \), we have

1. \( \Phi(A_{ii} + A_{ij}) = \Phi(A_{ii}) + \Phi(A_{ij}) \);
2. \( \Phi(A_{ii} + A_{ij}) = \Phi(A_{ij}) + \Phi(A_{ii}) \).

**Proof.** (1) We only give the proof of (1), and for (2) the proof goes similarly. Since \( \Phi \) is surjective, there is \( S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathcal{A} \) such that

\[
\Phi(S) = \Phi(A_{ii}) + \Phi(A_{ij}).
\]

For any \( T_{jj} \), by Lemma 2 and noticing that \( i \neq j \), we have

\[
\Phi(ST_{jj} - T_{jj}S) = \Phi(A_{ii}T_{jj} - T_{jj}A_{ii}) + \Phi(A_{ij}T_{jj} - T_{jj}A_{ij})
\]

\[
= \Phi(0) + \Phi(A_{ij}T_{jj}) = \Phi(A_{ij}T_{jj}).
\]

It follows from the injectivity of \( \Phi \) that

(1) \( ST_{jj} - T_{jj}S = A_{ij}T_{jj} \in \mathcal{A}_{ij} \).

So \( S_{ji} = 0 \) and \( S_{jj} = \lambda P_j \) for some scalar \( \lambda \) by Lemma 3(3). Also, from (1) we get that \( S_{jj}T_{jj} = A_{ij}T_{jj} \) for all \( T_{jj} \), and hence \( S_{jj} = A_{ij} \) by Lemma 3(1).

For every \( T_{ij} \), applying Lemma 2 we can similarly get that \( \Phi(ST_{ij} - T_{ij}S) = \Phi(A_{ii}T_{ij}) \), which implies \( ST_{ij} - T_{ij}S = A_{ii}T_{ij} \). Therefore

\[
A_{ii}T_{ij} = P_i(ST_{ij} - T_{ij}S)P_j = S_{ii}T_{ij} - T_{ij}S_{jj} = S_{ii}T_{ij} - \lambda T_{ij}
\]

and so \( S_{ii} = A_{ii} + \lambda P_i \). Thus

\[
S = S_{ii} + S_{ij} + S_{ji} + S_{jj}
\]

\[
= (A_{ii} + \lambda P_i) + A_{ij} + 0 + \lambda P_j
\]

\[
= A_{ii} + A_{ij} + \lambda.
\]

Since \( I \notin \mathcal{A} \), we have \( \lambda = 0 \). This proves \( S = A_{ii} + A_{ij} \), as required. □
Lemma 5. \( \Phi(T_{ii}A_{ij} + B_{ij}S_{jj}) = \Phi(T_{ii}A_{ij}) + \Phi(B_{ij}S_{jj}) \) for \( 1 \leq i \neq j \leq 2 \).

Proof. Making use of Lemma 4, we have

\[
\Phi(T_{ii}A_{ij} + B_{ij}S_{jj}) = \Phi(T_{ii} + B_{ij}) (A_{ij} + S_{jj}) - (A_{ij} + S_{jj}) (T_{ii} + B_{ij})
\]

Thus

\[
\Phi(T_{ii}A_{ij} + B_{ij}S_{jj}) = (\Phi(T_{ii}) + \Phi(B_{ij})) (A_{ij} + S_{jj}) - (A_{ij} + S_{jj}) (T_{ii} + B_{ij})
\]

By Lemma 3(1) again, we have

\[
\Phi(T_{ii}A_{ij} + B_{ij}S_{jj}) = (A_{ij} + S_{jj}) (T_{ii} + B_{ij})
\]

For any \( T_{ii}, T_{jj} \), by Lemmas 2, 5, we have

\[
\Phi(ST_{jj} - T_{jj}S) = \Phi(A_{ij}T_{jj}) + \Phi(B_{ij}T_{jj}),
\]

and

\[
\Phi(ST_{jj} + T_{jj}ST_{ii}) = \Phi(A_{ij}T_{jj} + B_{ij}T_{jj})
\]

Thus \( T_{ii}ST_{jj} + T_{jj}ST_{ii} = T_{ii}(A_{ij} + B_{ij})T_{jj} \). Multiplying this equality by \( P_j \) from the left, we get \( T_{ii}ST_{jj} = T_{ii}(A_{ij} + B_{ij})T_{jj} \). It follows from Lemma 3(1) that \( S_{ij} = A_{ij} + B_{ij} \).

For every \( T_{ij} \), applying Lemma 2 to (2) and (3) respectively, we get

\[
ST_{ij} - T_{ij}S = 0,
\]

\[
(ST_{jj} - T_{jj}S)T_{ij} - T_{ij}(ST_{jj} - T_{jj}S) = 0.
\]

It follows easily that \( S_{ii} = 0 \) and \( T_{ij}(ST_{jj} - T_{jj}S) = 0 \). Hence \( S_{jj}T_{ij} = T_{jj}S_{ij} \)

Thus \( S_{ii}T_{ij} = \lambda T_{ij} \) and \( S_{ii}T_{ij} = T_{ij}S_{jj} = P_i(ST_{ij} - T_{ij}S)P_j = 0 \).

By Lemma 3(1) again, we have \( S_{ii} = \lambda P_i \). Therefore,

\[
S = \lambda P_i + (A_{ij} + B_{ij}) + 0 + \lambda P_j = A_{ij} + B_{ij} + M.
\]
Recalling that $I \notin \mathcal{A}$, we have $\lambda = 0$ and hence $S = A_{ij} + B_{ij}$. This completes the proof. □

**Lemma 7.** $\Phi(A_{ii} + B_{ii}) = \Phi(A_{ii}) + \Phi(B_{ii})$ for $i = 1, 2$.

*Proof.* Choose $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathcal{A}$ such that

\[ (4) \quad \Phi(S) = \Phi(A_{ii}) + \Phi(B_{ii}). \]

Take $j \neq i$. For any $T_{jj}$, applying Lemma 2 to (4) we obtain $ST_{jj} - T_{jj}S = 0$. Thus, by Lemma 3(3)-(4) we have

\[ S_{ij} = S_{ji} = 0 \quad \text{and} \quad S_{jj} = \lambda P_{j} \]

for some scalar $\lambda$. Further, for any $T_{ij}$, applying Lemmas 2, 6, it follows from (4) that

\[ \Phi(ST_{ij} - T_{ij}S) = \Phi(A_{ii}T_{ij}) + \Phi(B_{ii}T_{ij}) = \Phi(A_{ii}T_{ij} + B_{ii}T_{ij}), \]

Thus $S_{ii}T_{ij} - MT_{ij} = S_{ii}T_{ij} - T_{ij}S_{jj} = P_{j}(ST_{ij} - T_{jj}S)P_{j} = (A_{ii} + B_{ii})T_{ij}$, from which we get $S_{ii} = A_{ii} + B_{ii} + \lambda P_{i}$ by Lemma 3(1). So

\[ S = (A_{ii} + B_{ii} + \lambda P_{i}) + 0 + 0 + \lambda P_{j} = A_{ii} + B_{ii} + \lambda I. \]

Since $I \notin \mathcal{A}$, we have $\lambda = 0$ and $S = A_{ii} + B_{ii}$, as desired. □

**Lemma 8.** $\Phi(A_{11} + A_{22}) = \Phi(A_{11}) + \Phi(A_{22})$.

*Proof.* Choose $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathcal{A}$ such that

\[ (5) \quad \Phi(S) = \Phi(A_{11}) + \Phi(A_{22}). \]

For any $T_{11}$, by Lemma 2 we get $ST_{11} - T_{11}S = A_{11}T_{11} - T_{11}A_{11}$, which implies $T_{11}S_{12} = S_{21}T_{11} = 0$. So $S_{12} = S_{21} = 0$. Also, we have

\[ T_{11}(S_{11} - A_{11}) = (S_{11} - A_{11})T_{11} \]

and hence there exists a scalar $\lambda$ such that $S_{11} = A_{11} + \lambda P_{1}$.

For any $T_{12}$, applying Lemmas 2, 6, we obtain from (5) that

\[ \Phi(ST_{12} - T_{12}S) = \Phi(A_{11}T_{12} - T_{12}A_{11}) + \Phi(A_{22}T_{12} - T_{12}A_{22}) = \Phi(A_{11}T_{12}) + \Phi(-T_{12}A_{22}) = \Phi(A_{11}T_{12} - T_{12}A_{22}). \]

It follows that $ST_{12} - T_{12}S = A_{11}T_{12} - T_{12}A_{22}$. Hence

\[ S_{11}T_{12} - T_{12}S_{22} = A_{11}T_{12} - T_{12}A_{22}, \]

in which putting $A_{11} + \lambda P_{1}$ for $S_{11}$, we have $S_{22} = A_{22} + \lambda P_{2}$ by Lemma 3(1). So

\[ S = (A_{11} + \lambda P_{1}) + 0 + 0 + (A_{22} + \lambda P_{2}) = A_{11} + A_{22} + \lambda I. \]

Then $S = A_{11} + A_{22}$ since $I \notin \mathcal{A}$, completing the proof. □

**Lemma 9.** $\Phi(A_{12} + A_{21}) = \Phi(A_{12}) + \Phi(A_{21})$. 
Proof. Choose $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathcal{A}$ such that
\begin{equation}
\Phi(S) = \Phi(A_{12}) + \Phi(A_{21}).
\end{equation}
For any $T_{12}$, by Lemma 2 one has
\begin{equation}
ST_{12} - T_{12}S = A_{21}T_{12} - T_{12}A_{21}.
\end{equation}
Multiplying this equality by $P_1$ from the right, we get $T_{12}S_{21} = T_{12}A_{21}$, which implies $S_{21} = A_{21}$. With the same discussion for $T_{21}$, we can get $S_{12} = A_{12}$.

For any $T_{12}$, by Lemma 2 we get from (6)
\begin{equation}
\Phi(ST_{11} - T_{11}S) = \Phi(-T_{11}A_{12}) + \Phi(A_{21}T_{11}).
\end{equation}
Moreover, for any $T_{21}$, applying Lemma 2 to the above equality, we have
\begin{equation}
T_{11}ST_{21} + T_{21}(ST_{11} - T_{11}S) = T_{11}A_{12}T_{21} - T_{21}T_{11}A_{12}.
\end{equation}
Multiplying this equality by $P_1$ from the right and noting that $S_{12} = A_{12}$, we get $T_{21}(ST_{11} - T_{11}S)P_1 = 0$. Then $S_{11}T_{11} = T_{11}S_{11}$, and so $S_{11} = \lambda P_1$ for some scalar $\lambda$. Also, observing that $S_{11}T_{12} - T_{12}S_{22} = 0$ from (7), we have $S_{22} = \lambda P_2$. So $S = A_{12} + A_{21} + \lambda I$. Hence $S = A_{12} + A_{21}$ since $I \notin \mathcal{A}$. \hfill \square

**Lemma 10.** \Phi(A_{11} + A_{12} + A_{21}) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}).

**Proof.** Let $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathcal{A}$ such that
\begin{equation}
\Phi(S) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}).
\end{equation}
Then by Lemmas 4, 9, we have that
\begin{equation}
\Phi(S) = \Phi(S_{11} + S_{12} + S_{21} + S_{22}),
\end{equation}
\begin{equation}
\Phi(S) = \Phi(S_{11} + S_{12} + S_{21} + S_{22}),
\end{equation}
\begin{equation}
\Phi(S) = \Phi(S_{11} + S_{12} + S_{21} + S_{22}).
\end{equation}
For any $T_{21}$, by Lemma 2 we get
\begin{equation}
ST_{21} - T_{21}S = A_{12}T_{21} - T_{21}A_{12},
\end{equation}
from (8). Multiplying this equality by $P_1$ from the left, we get $S_{12}T_{21} = A_{12}T_{21}$ and so $S_{12} = A_{12}$. Similarly, one has $S_{21} = A_{21}$ from (9). Further, for any $T_{22}$, applying Lemma 2 to (10), we obtain
\begin{equation}
ST_{22} - T_{22}S = A_{12}T_{22} - T_{22}A_{12}.
\end{equation}
Multiplying this equality by $P_2$ from both sides, we see that $S_{22}T_{22} - T_{22}S_{22} = 0$. It follows that there exists a scalar $\lambda$ such that $S_{22} = \lambda P_2$. Also, multiplying (11) by $P_2$ from the left and by $P_1$ from the right respectively, we get
\begin{equation}
S_{22}T_{21} - T_{21}S_{11} = -T_{21}A_{11}.
\end{equation}
So $S_{11} = A_{11} + \lambda P_1$. Thus $S = A_{11} + A_{12} + A_{21} + \lambda I$ and consequently, $S = A_{11} + A_{12} + A_{21}$. The proof is complete. \hfill \square

**Lemma 11.** \Phi(A_{11} + A_{12} + A_{21} + A_{22}) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}) + \Phi(A_{22}).
Suppose that $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathcal{A}$ are such that
\[ \Phi(S) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}) + \Phi(A_{22}). \]

Then by the Lemma 10, we can write
\[ \Phi(S) = \Phi(A_{11} + A_{12} + A_{21} + A_{22}). \]

For any $T_{11}$, by Lemma 2 we see that
\[ ST_{11} - T_{11}S = (A_{11} + A_{21})T_{11} - T_{11}(A_{11} + A_{12}). \]

By multiplying this equality by $P_2$ from the left and the right respectively, it is easily seen that $S_{12} = A_{21}$ and $S_{12} = A_{12}$. Also, multiplying this equality by $P_1$ from both sides, we can get $(S_{11} - A_{11})T_{11} = T_{11}(S_{11} - A_{11})$. It follows that there exists a scalar $\lambda$ such that $S_{11} = A_{11} + \lambda P_1$.

For any $T_{12}$, applying Lemma 2 to (12), we get
\[ \Phi(ST_{12} - T_{12}S) = \Phi(A_{11}T_{12} + A_{21}T_{12} - T_{12}A_{21}) + \Phi(-T_{12}A_{22}). \]

Again, for any $T_{11}$, by Lemmas 2, 4 and 6, we obtain from the above equality that
\[
\Phi(-T_{12}ST_{11} - T_{11}ST_{12} + T_{11}T_{12}S) \\
= \Phi(-T_{12}A_{12}T_{11} - T_{11}A_{11}T_{12} + T_{11}T_{12}A_{21}) + \Phi(T_{11}T_{12}A_{22}) \\
= \Phi(T_{11}T_{12}A_{21} - T_{12}A_{21}T_{11}) + \Phi(-T_{11}A_{11}T_{12}) + \Phi(T_{11}T_{12}A_{22} - T_{11}A_{11}T_{12}) \\
= \Phi(T_{11}T_{12}A_{21} - T_{12}A_{21}T_{11} + T_{11}T_{12}A_{22} - T_{11}A_{11}T_{12}).
\]

It follows that
\[ T_{11}T_{12}S - T_{12}ST_{11} - T_{11}ST_{12} \\
= T_{11}T_{12}A_{21} - T_{12}A_{21}T_{11} + T_{11}T_{12}A_{22} - T_{11}A_{11}T_{12}, \]
in which multiplying by $P_2$ from the right and making use of $S_{11} = A_{11} + \lambda P_1$, we get $S_{22} = A_{22} + \lambda P_2$. Hence $S = A_{11} + A_{12} + A_{21} + A_{22} + \lambda I$. So $S = A_{11} + A_{12} + A_{21} + A_{22}$ because of $I \notin \mathcal{A}$, completing the proof.

**Proof of Theorem 1.** Let $A, B \in \mathcal{A}$. By writing $A = A_{11} + A_{12} + A_{21} + A_{22}$ and $B = B_{11} + B_{12} + B_{21} + B_{22}$, then it is easily seen that $\Phi(A + B) = \Phi(A) + \Phi(B)$ making use of Lemmas 6, 7 and 11. We are done.

Theorem 1 has the following obvious corollary.

**Corollary 1.** Let $\mathcal{H}$ be a real or complex Hilbert space with $\dim \mathcal{H} = \infty$, and $\mathcal{A} \subseteq B(\mathcal{H})$ be either the ideal of all finite rank operators or the ideal of all compact operators. Then every bijective Lie map from $\mathcal{A}$ onto an arbitrary algebra is additive.
We shall conclude by considering the case that $\mathcal{A}$ contains the identity operator $I$ in Theorem 1. In this case, define the map $\Phi : \mathcal{A} \to \mathcal{A}$ by
\[
\Phi(A) = \begin{cases} 
2A, & \text{if } A \in \mathbb{F}I, \\
A, & \text{otherwise,}
\end{cases}
\]
where $\mathbb{F}$ denotes the real or complex field. Then
(1) $\Phi$ is bijective;
(2) $\Phi$ is not additive;
(3) $\Phi$ is a Lie map.
In fact, (1) and (2) are obvious. Let us prove that (3). Suppose $A, B \in \mathcal{A}$.
We distinguish two cases.
Case 1. At least one of $A, B$ is in $\mathbb{F}I$. Then clearly, $AB - BA = 0$ and
\[
\Phi(AB - BA) = 0 = \Phi(A)\Phi(B) - \Phi(B)\Phi(A).
\]
Case 2. Both $A$ and $B$ are not in $\mathbb{F}I$. If $AB - BA \notin \mathbb{F}I$, then
\[
\Phi(A)\Phi(B) - \Phi(B)\Phi(A) = AB - BA = \Phi(AB - BA).
\]
Suppose now that $AB - BA = \lambda I$ for some $\lambda \in \mathbb{F}$. We then have $\sigma(AB) = \lambda + \sigma(BA)$, where $\sigma(\cdot)$ denotes the spectrum of an operator. It is well known that $\sigma(AB) \cup \{0\} = \sigma(BA) \cup \{0\}$. This leads to $\lambda = 0$ and hence
\[
\Phi(AB - BA) = 0 = AB - BA = \Phi(A)\Phi(B) - \Phi(B)\Phi(A).
\]
So $\Phi$ is a Lie map.

References


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