X-LIFTING MODULES OVER RIGHT PERFECT RINGS

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Abstract. Keskin and Harmanci defined the family \( B(M, X) = \{ A \leq M \mid \exists Y \leq X, \exists f \in \text{Hom}_R(M, X/Y), \text{Ker} f/A \ll M/A \} \). And Orhan and Keskin generalized projective modules via the class \( B(M, X) \).

In this note we introduce \( X \)-local summands and \( X \)-hollow modules via the class \( B(M, X) \). Let \( R \) be a right perfect ring and let \( M \) be an \( X \)-lifting module. We prove that if every co-closed submodule of any projective module \( P \) contains \( \text{Rad}(P) \), then \( M \) has an indecomposable decomposition. This result is a generalization of Kuratomi and Chang’s result [9, Theorem 3.4]. Let \( X \) be an \( R \)-module. We also prove that for an \( X \)-hollow module \( H \) such that every non-zero direct summand \( K \) of \( H \) with \( K \in B(H, X) \), if \( H \oplus H \) has the internal exchange property, then \( H \) has a local endomorphism ring.

1. Introduction

Extending modules and lifting modules have been studied extensively in recent years by many ring theorists (see, for example, [3], [5]-[14]).

Let \( M \) and \( X \) be \( R \)-modules. In [8], D. Keskin and A. Harmanci defined the family \( B(M, X) = \{ A \leq M \mid \exists Y \leq X, \exists f \in \text{Hom}_R(M, X/Y), \text{Ker} f/A \ll M/A \} \). They considered the following conditions:

\( B(M, X)-(D_1) \): For any \( A \in B(M, X) \), there exists a direct summand \( A^* \leq M \) such that \( A/A^* \ll M/A^* \).

\( B(M, X)-(D_2) \): For any \( A \in B(M, X) \), if \( B \leq B \leq M \), \( M/A \simeq B \) implies \( A \leq M \).

\( B(M, X)-(D_3) \): For any \( A \in B(M, X) \) and \( B \leq B \leq M \), if \( A \leq A \) and \( M = A + B \) then \( A \cap B \leq A \).

They defined that \( M \) is said to be \( X \)-discrete if \( B(M, X)-(D_1) \) and \( B(M, X)-(D_2) \) hold, and is said to be \( X \)-quasi-discrete if \( B(M, X)-(D_1) \) and \( B(M, X)-(D_3) \) hold. Furthermore, \( M \) is said to be \( X \)-lifting if \( B(M, X)-(D_1) \) holds. We have just seen that the following implications hold:

\( "X\text{-discrete} \implies X\text{-quasi-discrete} \implies X\text{-lifting}" \).
Throughout this paper, all rings $R$ considered are associative rings with identity and all $R$-modules are unital.

Let $M$ be a right $R$-module and $N$ a submodule of $M$. The notation $N \leq_{\oplus} M$ means that $N$ is a direct summand of $M$.

A submodule $K$ of $M$ is called a small submodule (or superfluous submodule) of $M$, abbreviated $K \ll M$, in the case when, for every submodule $L \leq M$, $K + L = M$ implies $L = M$.

2. Preliminaries

Let $A$ and $P$ be submodules of $M$ with $P \in \mathbf{B}(M, X)$. $P$ is called an X-supplement of $A$ if it is minimal with the property $A + P = M$ equivalently, if $M = A + P$ and $A \cap P \ll P$.

The module $M$ is called X-amply supplemented if for any submodules $A$, $B$ of $M$ with $A \in \mathbf{B}(M, X)$ and $M = A + B$ there exists an X-supplement $P$ of $A$ such that $P \leq B$.

Let $N_1 \leq N_2 \leq M$. $N_1$ is a co-essential submodule of $N_2$ in $M$, abbreviated $N_1 \leq_{c} N_2$ in $M$, if the kernel of the canonical map $M/N_1 \longrightarrow M/N_2 \longrightarrow 0$ is small in $M/N_1$, or equivalently, if $M = N_2 + X$ with $N_1 \leq X$ implies $M = X$.

A submodule $N$ of $M$ is said to be co-closed in $M$ (or a co-closed submodule of $M$), if $N$ has no proper co-essential submodule in $M$, i.e., $N' \leq_{c} N$ in $M$ implies $N = N'$. It is easy to see that any direct summand of a module $M$ is co-closed in $M$. Note that every X-supplement submodule of $M$ is co-closed in $M$.

For $N' \leq N \leq M$, $N'$ is called a co-closure of $N$ in $M$ if $N'$ is a co-closed submodule of $M$ with $N' \leq_{c} N$ in $M$. Any submodule of a module has a closure, however, co-closure does not exist in general.

**Lemma 2.1** ([9, Lemma 1.4] and [5, 3.2, 3.7]). Let $A \leq B \leq M$. Then the following hold:

1. $A \leq_{c} B$ in $M$ if and only if $M = A + K$ for any submodule $K$ of $M$ with $M = B + K$.
2. If $A \ll M$ and $B$ is co-closed in $M$, $A \ll B$.

**Lemma 2.2** ([16, Lemma 41.14]). Any projective module satisfies the following condition:

(D) If $M_1$ and $M_2$ are direct summands of $M$ such that $M_1 \cap M_2 \ll M$ and $M = M_1 + M_2$, then $M = M_1 \oplus M_2$.

**Lemma 2.3** ([13, Theorem 3.5]). If $M$ is a lifting module with the condition (D), then $M$ can be expressed as a direct sum of hollow modules.

**Lemma 2.4** ([1, Lemma 17.17]). Suppose that $M$ has a projective cover. If $P$ is projective with an epimorphism $\varphi : P \longrightarrow M$, then $P$ has a decomposition $P = P_1 \oplus P_2$ such that $P_1 \leq \ker \varphi$ and $\varphi|_{P_2} : P_2 \longrightarrow M$ is a projective cover of $M$. 

Theorem 2.5 (\cite[Theorem 1.1.24]{3}). For an $R$-module $M$, the following hold:

1. If $M$ is a quasi-injective module, then $M$ is a fully invariant submodule of $E(M)$.
2. If $M$ is a quasi-injective module, then any direct decomposition $E(M) = E_1 \oplus \cdots \oplus E_n$ induces $M = (M \cap E_1) \oplus \cdots \oplus (M \cap E_n)$.
3. If $M$ is a quasi-projective module with a projective cover $\varphi : P \longrightarrow M$, $\mathrm{Ker} \varphi$ is a fully invariant submodule of $P$; whence any endomorphism of $P$ induces an endomorphism of $M$.
4. If $M$ is a quasi-projective module with a projective cover $\varphi : P \longrightarrow M$, then any direct decomposition $P = P_1 \oplus \cdots \oplus P_n$ induces $M = \varphi(P_1) \oplus \cdots \oplus \varphi(P_n)$.

A ring $R$ is called right perfect if every right $R$-module has a projective cover.

Proposition 2.6. The following statements are equivalent:

(i) Every cyclic right $R$-module has a projective cover;
(ii) $R_R$ is a lifting module.

Proof. (i) $\implies$ (ii) Let $A$ be a submodule of $R_R$ and let $\varphi : R \longrightarrow R/A$ be the canonical epimorphism. Since $R/A$ has a projective cover, by Lemma 2.4, there exists a decomposition $R_R = eR \oplus (1 - e)R$ such that $(\varphi \mid_{eR}) : eR \longrightarrow R/A \longrightarrow 0$ is a projective cover and $(1 - e)R \leq A$. This implies $\mathrm{Ker} (\varphi \mid_{eR}) = A \cap eR \ll eR$. i.e., $R = eR \oplus (1 - e)R$ such that $A \cap eR \ll eR$. Thus $R_R$ is lifting.

(ii) $\implies$ (i) Suppose that $R_R$ is lifting. We claim that $R/A$ has a projective cover. Since $R_R$ is lifting, for any $A \leq R$, there exists $A^* \leq \leq eA$ such that $R = A^* \oplus A^{**}$. Then $\pi_{|A^*} : A^{**} \longrightarrow R/A \longrightarrow 0$ is a projective cover of $R/A$, where $\pi : R \longrightarrow R/A \longrightarrow 0$ is the canonical epimorphism. \(\square\)

As corollaries of Proposition 2.6, we obtain the following two results.

Corollary 2.7. Let $P$ be a projective module. Then the following statements are equivalent:

(i) Every factor module of $P$ has a projective cover;
(ii) $P$ is lifting.

Corollary 2.8. The following statements are equivalent:

(i) Every simple right $R$-module has a projective cover;
(ii) $R_R$ satisfies the lifting property for simple factor modules.

Lemma 2.9 (\cite[Lemma 3.1]{9} and \cite[3.2]{5}). Let $f : M \longrightarrow N$ be an epimorphism. Suppose $K \leq \leq K'$ in $M$. Then $f(K) \leq \leq f(K')$ in $N$.

Lemma 2.10 (\cite[Lemma 2.2]{8}). Let $M$, $N$ and $X$ be $R$-modules. Then the following hold:

1. If $A \in \mathcal{B}(M, X)$ and $B \leq A$ with $A/B \ll M/B$, then $B \in \mathcal{B}(M, X)$.
(2) Let \( h : M \longrightarrow N \) be an epimorphism and \( A \in \mathcal{B}(M, X) \) with \( \text{Ker } h \leq A \). Then \( h(A) \in \mathcal{B}(N, X) \). Conversely, if \( h(A) \in \mathcal{B}(N, X) \) and \( \text{Ker } h \leq A \), then \( A \in \mathcal{B}(M, X) \).

(3) Let \( B \leq A \leq M \). Then \( A \in \mathcal{B}(M, X) \) if and only if \( A/B \in \mathcal{B}(M/B, X) \).

(4) Let \( h : N \longrightarrow M \) be an epimorphism and \( A \in \mathcal{B}(M, X) \). Then \( h^{-1}(A) \in \mathcal{B}(N, X) \).

3. Main results

**Theorem 3.1.** Let \( R \) be a ring. The following conditions are equivalent:

1. \( R \) is right perfect;
2. Every projective right \( R \)-module is lifting;
3. Every quasi-projective right \( R \)-module is lifting;
4. Every countably generated free right \( R \)-module is lifting.

**Proof.** (1) \( \iff \) (2) This follows from Corollary 2.7.

(2) \( \implies \) (3) Let \( Q_R \) be a quasi-projective module and let \( A \) be a submodule of \( Q \). Consider the canonical epimorphism \( f : Q \longrightarrow Q/A \). We can take a projective module \( P_R \) such that \( Q \) is a homomorphic image of \( P \), i.e., we have an epimorphism \( g : P \longrightarrow Q \). Since \( P \) is a lifting module, by Lemma 2.4, there exists a decomposition \( P = P_1 \oplus P_2 \) such that \( P_1 \leq g^{-1}(A) \), \( fg \mid P_2 \); \( P_2 \longrightarrow Q/A \) is a projective cover. As \( Q \) is a quasi-projective module, the decomposition \( P = P_1 \oplus P_2 \) induces a direct decomposition \( Q = g(P_1) \oplus g(P_2) \) by Theorem 2.5. Then \( g(P_1) \leq A \) and \( g(P_2) \cap A \ll g(P_2) \) hold.

(3) \( \implies \) (2) Obvious.

(4) \( \implies \) (1) By (4), \( R/J(R) \) is semisimple. Since \( R^{(N)} \) is lifting, there exists a decomposition \( R^{(N)} = X \oplus Y \) such that \( X \leq \text{Rad}(R^{(N)}) \) and \( \text{Rad}(R^{(N)}) \cap Y \ll Y \). Because \( \text{Rad}(R^{(N)}) = \text{Rad}(X) \oplus \text{Rad}(Y) \) and \( X \leq \text{Rad}(R^{(N)}) \), we see \( \text{Rad}(X) = X \), which implies \( X = 0 \) and \( R^{(N)}J(R) = \text{Rad}(R^{(N)}) \ll R^{(N)} \). Hence, by [1, Lemma 28.3], \( J(R) \) is right \( T \)-nilpotent. Thus \( R \) is right perfect. \( \square \)

A family \( \{ X_\lambda \mid \lambda \in \Lambda \} \) of submodules of a module \( M \) with \( X_\lambda \in \mathcal{B}(M, X) \) is called an \( X \)-local summand of \( M \), if \( \Sigma_{\lambda \in \Lambda} X_\lambda \) is direct and \( \Sigma_{\lambda \in F} X_\lambda \leq M \) for every finite subset \( F \subseteq \Lambda \).

By analogy with the proof of [14, Lemma 2.4] or [11, Theorem 2.17], we have the following lemma.

**Lemma 3.2.** If every \( X \)-local summand of a module \( M \) is a direct summand, then \( M \) has an indecomposable decomposition.

By Lemma 2.1(1), we have the following lemma.

**Lemma 3.3.** Assume \( P_i \leq c Q_i \) in \( P \) for every \( i \in I \). Then \( \Sigma_{i \in I} P_i \leq c \Sigma_{i \in I} Q_i \) in \( P \).
Lemma 3.4. Let \( \{P_i\}_{i \in I} \) be a set of \( R \)-modules. Assume \( P_i \in \mathcal{B}(M, X) \) for every \( i \in I \). Then \( \sum_{i \in I} P_i \subseteq \mathcal{B}(M, X) \).

Proof. Since \( P_i \in \mathcal{B}(M, X) \), there exist a submodule \( Y \) of \( X \) and a homomorphism \( f_i : M \rightarrow X/Y \) such that \( \ker f_i \subseteq M/P_i \). Put \( f = \sum_{i \in I} f_i \). Then \( f : M \rightarrow X/Y \) such that \( \ker f/\sum_{i \in I} P_i \subseteq M/\sum_{i \in I} P_i \). Thus \( \sum_{i \in I} P_i \subseteq \mathcal{B}(M, X) \). \( \square \)

Lemma 3.5. Let \( X \) be a right \( R \)-module. Suppose that \( R \) is a right perfect ring. Then every projective right \( R \)-module is \( X \)-lifting.

Proof. Let \( P \) be a projective module. For any \( A \in \mathcal{B}(P, X) \), consider the canonical epimorphism \( \varphi : P \rightarrow P/A \). Since \( P/A \) has a projective cover, by Lemma 2.4, there exists a decomposition \( P = P_1 \oplus P_2 \) such that \( P_1 \leq \ker \varphi \) and \( \varphi |_{P_2} : P_2 \rightarrow P/A \) is a projective cover of \( P/A \). Hence \( P \) is \( X \)-lifting. \( \square \)

Proposition 3.6. Let \( R \) be a right perfect ring and let \( M \) be an \( X \)-lifting module. Then \( M \) is \( X \)-amply supplemented.

Proof. Let \( A, B \leq M \) such that \( B \in \mathcal{B}(M, X) \) and \( M = A + B \). Since \( M = A + B \) and \( B \in \mathcal{B}(M, X) \), there exist \( Y \subseteq X \\) and \( f : M \rightarrow X/Y \) such that \( \ker f/B \subseteq M/B \). Consider the isomorphism \( \alpha : M/B \rightarrow A/A \cap B \). Then \( \alpha(\ker f/B) = \ker f/A \cap B \). Hence \( \ker f/A \cap B \subseteq M/A \cap B \). Therefore \( A \cap B \in \mathcal{B}(M, X) \). As \( M \) is \( X \)-lifting, there exists a direct summand \( K \) of \( M \) such that \( K \leq A \cap B \) in \( M \). Then \( A \cap B = K \oplus (K \cap (A \cap B)) \), \( M = (A \cap B) + K \) and \( (A \cap B) \cap K^* \subseteq K^* \). Thus \( M = B + (A \cap K^*) \).

Let \( D \) be a co-closure of \( A \cap K^* \) in \( M \). Then \( M = B + D \) and \( B \cap D \leq B \cap (A \cap K^*) \subseteq K^* \). Hence \( B \cap D \subseteq K^* \). Since \( D \) is co-closed in \( M \), \( B \cap D \leq D \) and \( B \cap D \subseteq M, B \cap D \subseteq D \). Thus \( D \) is an \( X \)-supplement of \( B \) in \( M \) such that \( D \leq A \). \( \square \)

Lemma 3.7 ([8, Lemma 3.2]). Every epimorphic image of an \( X \)-amply supplemented \( R \)-module is \( X \)-amply supplemented.

Lemma 3.8. Let \( M \) be an \( X \)-amply supplemented module and let \( \ker f \ll M \rightarrow N \rightarrow 0 \). Suppose \( K \) is co-closed in \( M \) with \( \ker f \leq K \). Then \( f(K) \) is co-closed in \( N \).

Proof. By Lemma 3.7, \( N \) is \( X \)-amply supplemented. Let \( L \leq \ker f(K) \) in \( N \). We claim that \( L = \ker f(K) \). Since \( f \) is an epimorphism, there exists a submodule \( T \) of \( K \) in \( M \) with \( f(T) = L \). Since \( N \) is \( X \)-amply supplemented, there exists an \( X \)-supplement \( P \) of \( f(K) \) such that \( P \subseteq N \). i.e., \( N = f(K) + P \) and \( f(K) \cap P = \epsilon \). Since \( f \) is an epimorphism, there exists a submodule \( Q \) of \( M \) with \( f(Q) = P \). Then \( M = K + Q + \ker f \). As \( \ker f \ll M, M = K + Q \). This implies \( N = f(K) + f(Q) = f(K) + P = L + P = f(T) + f(Q) \). Then \( M = T + Q + \ker f = T + Q \). Thus \( T \leq K \) in \( M \) by Lemma 2.1(1). As \( K \) is co-closed in \( M \), \( T = K \). Hence \( L = f(T) = f(K) \). Therefore \( f(K) \) is co-closed in \( N \). \( \square \)
Proposition 3.9. Suppose that $M$ is an $X$-lifting module. Then every co-closed submodule $K$ of $M$ with $K \in \mathcal{B}(M, X)$ is a direct summand.

Proof. Since $M$ is $X$-lifting, there exists a direct summand $K^*$ such that $K^* \leq_c K$ in $M$. As $K$ is co-closed in $M$, $K = K^* \leq_\oplus M$. \hfill $\square$

Theorem 3.10. Let $R$ be a right perfect ring and let $M$ be an $X$-lifting module. Assume that every co-closed submodule of any projective module $P$ contains $\text{Rad}(P)$. Then every $X$-local summand of $M$ is a direct summand.

Proof. Let $M$ be an $X$-lifting module and let $\Sigma_{i \in I}X_i$ be an $X$-local summand of $M$ with $X_i \in \mathcal{B}(M, X)$. Since $R$ is right perfect, $M$ has a projective cover, say $\text{Ker} f \leq P \xrightarrow{i} M \twoheadrightarrow 0$. By Lemma 3.5, $P$ is projective $X$-lifting. Since $X_i \in \mathcal{B}(M, X)$, $f^{-1}(X_i) \in \mathcal{B}(P, X)$ by Lemma 2.10(4). So there exists a decomposition $P = P_i \oplus F_i$ ($i \in I$) such that $P_i \leq_c f^{-1}(X_i)$ in $P$. By Lemma 2.9, $f(P_i) \leq_c f(f^{-1}(X_i)) = X_i$ in $M$. As $X_i$ is co-closed in $M$, $f(P_i) = X_i$. First we prove that $\Sigma_{i \in I}P_i$ is direct. Let $F$ be a finite subset of $I - \{i\}$. Since $\Sigma \oplus_{i \in I}X_i$ is an $X$-local summand of $M$, we see

$$f(P_i + \Sigma_{j \in F}P_j) = X_i \oplus (\Sigma_{j \in F}X_j) \leq_\oplus M.$$ 

So there exists a direct summand $Y$ of $M$ such that $M = X_i \oplus (\Sigma_{j \in F}X_j) \oplus Y$. As $P$ is lifting, there exists a decomposition $P = Q \oplus Q^*$ such that $Q \leq_c f^{-1}(Y)$ in $P$. Then $f(Q) = Y$. Thus we see

$$P = P_i + \Sigma_{j \in F}P_j + Q + \text{Ker} f = P_i + \Sigma_{j \in F}P_j + Q.$$ 

Then $P_i \cap (\Sigma_{j \in F}P_j + Q) \leq \text{Ker} f \leq P$. Similarly, we see $Q \cap (P_i + \Sigma_{j \in F}P_j) \leq P$ and $P_j \cap (P_i + \Sigma_{j \in F - \{j\}}P_j + Q) \leq P$. By Lemma 2.2, we obtain $P = P_i \oplus (\Sigma_{j \in F}P_j) \oplus Q$. Hence $\Sigma_{i \in I}P_i$ is direct. By the same argument, we see $\Sigma \oplus_{i \in I}P_i$ is an $X$-local summand of $P$. By Lemma 2.3, $\Sigma \oplus_{i \in I}P_i \leq_\oplus P$. So $f(\Sigma \oplus_{i \in I}P_i)$ is co-closed in $M$ by Lemma 3.8. Since $M$ is $X$-lifting, we see

$$\Sigma \oplus_{i \in I}X_i = f(\Sigma \oplus_{i \in I}P_i) \leq_\oplus M.$$ 

Thus any $X$-local summand of $M$ is a direct summand. \hfill $\square$

By Lemma 3.2 and Theorem 3.10, we obtain the first main theorem.

Theorem 3.11. Suppose that every co-closed submodule of any projective module $P$ contains $\text{Rad}(P)$. Then every $X$-lifting module over right perfect rings has an indecomposable decomposition.

Let $X$ be an $R$-module. A non-zero $R$-module $H$ is $X$-hollow if for any proper submodule $K$ of $H$ with $K \in \mathcal{B}(H, X)$, $K \ll H$.

Proposition 3.12. Let $H$ and $X$ be $R$-modules. Assume that every non-zero direct summand $K$ of $H$ with $K \in \mathcal{B}(H, X)$. Then $H$ is $X$-hollow if and only if $H$ is indecomposable $X$-lifting.
Proof. \((\Rightarrow)\) Assume \(H\) is \(X\)-hollow. Let \(K \in \mathcal{B}(H, X)\) with \(K \preceq H\). Since \(H\) is \(X\)-hollow, \(K \ll H\). So there exists a decomposition \(H = 0 \oplus H\) such that \(0 \leq cK\) in \(H\). Thus \(H\) is \(X\)-lifting. Now, assume that \(H = H_1 \oplus H_2\), \(H_i \neq 0, i = 1, 2\). Since \(H\) is \(X\)-hollow, \(H_i \ll H, i = 1, 2\). Hence \(H_i = 0\). This is a contradiction. Therefore \(H\) is indecomposable. \((\Leftarrow)\) Suppose that \(H\) is indecomposable \(X\)-lifting. Let \(K \in \mathcal{B}(H, X)\) with \(K \preceq H\). By hypothesis, there exists a decomposition \(H = K^* \oplus K^{**}\) such that \(K^* \leq cK\) in \(H\). As \(H\) is indecomposable, we have either \(K^* = 0\) or \(K^{**} = 0\). If \(K^* = 0\), then \(K \ll H\). In the second case, \(H = K\). This is a contradiction. \(\Box\)

A module \(M\) is said to have the (finite) exchange property if, for any (finite) index set \(I\), whenever \(M \oplus N = \oplus_{i \in I} A_i\) for modules \(N\) and \(A_i\), then \(M \oplus N = M \oplus (\oplus_{i \in I} B_i)\) for some submodules \(B_i \leq A_i\). A module \(M\) has the (finite) internal exchange property if, for any (finite) direct sum decomposition \(M = \oplus_{i \in I} M_i\) and any direct summand \(X\) of \(M\), there exist submodules \(M_i \leq M_i\) such that \(M = X \oplus (\oplus_{i \in I} M_i)\).

By Proposition 3.12, we obtain the second main theorem.

**Theorem 3.13** ([15, Proposition 1]). Let \(X\) be an \(R\)-module and let \(H\) be an \(X\)-hollow module. Assume that every non-zero direct summand \(K\) of \(H\) with \(K \in \mathcal{B}(H, X)\).

If \(H \oplus H\) has the internal exchange property, then \(H\) has a local endomorphism ring.

**Corollary 3.14** (cf., [5, 12.2]). Let \(X\) be an \(R\)-module and let \(H\) be an \(X\)-hollow module. Assume that every non-zero direct summand \(K\) of \(H\) with \(K \in \mathcal{B}(H, X)\). Then the following conditions are equivalent:

1. \(H\) has a local endomorphism ring;
2. \(H\) has the finite exchange property;
3. \(H\) has the exchange property.

**References**


Information Technology Manpower Development Program
Kyungpook National University
Taegu 702-701, Korea
E-mail address: yamaguchi21@hanmail.net