ON GENERALIZED NONLINEAR QUASI-VARIATIONAL-LIKE INCLUSIONS DEALING WITH \((h, \eta)-\)PROXIMAL MAPPING

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ON GENERALIZED NONLINEAR QUASI-VARIATIONAL-LIKE INCLUSIONS DEALING WITH \((h, \eta)\)-PROXIMAL MAPPING

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Abstract. In this paper, a new class of \((h, \eta)\)-proximal mappings for proper functionals in Hilbert spaces is introduced. The existence and Lipschitz continuity of the \((h, \eta)\)-proximal mappings for proper functionals are proved. A class of generalized nonlinear quasi-variational-like inclusions in Hilbert spaces is introduced. A perturbed three-step iterative algorithm with errors for the generalized nonlinear quasi-variational-like inclusion is suggested. The existence and uniqueness theorems of solution for the generalized nonlinear quasi-variational-like inclusion are established. The convergence and stability results of iterative sequence generated by the perturbed three-step iterative algorithm with errors are discussed.

1. Introduction

Variational inequality theory has become a very effective and powerful tool in pure and applied mathematics and has been used in a large variety of problems arising in differential equations, mechanics, contact problems in elasticity and general equilibrium, see [1]-[10], [12]-[14], [16]-[27].

monotone and generalized pseudocontractive mappings and proved the stability results of iterative sequence generated by the perturbed three-step iterative algorithm.

Inspired and motivated by the recent research works [3]-[6], [13]-[14], [16]-[20], [23]-[25], in this paper, we introduce and study a new class of generalized nonlinear quasi-variational-like inclusions, which includes as special cases, the classes of variational inclusions and quasivariational inclusions studied in [1], [3]-[6], [13]-[14], [16], [23]. Applying $\eta$-subdifferential and $(h, \eta)$-proximal mappings, we establish the equivalence between the generalized nonlinear quasi-variational-like inclusion and the fixed point problem. By using the equivalence, a new perturbed three-step iterative algorithm with errors is suggested and analyzed. The convergence criteria of the algorithm is also discussed. Our results extend and improve the recent results due to Cho-Kim-Huang-Kang [3], Ding [4] and [5], Ding-Luo [6], Hassouni-Moudafi [13], Kazmi [14], Liu-Kang [16] and Verma [23].

2. Preliminaries

Throughout this paper, we assume that $H$ is a real Hilbert space endowed with the norm $\| \cdot \|$ and inner product $(\cdot, \cdot)$, respectively. Let $a, b, c, d, g : H \to H$ and $\eta, M, N : H \times H \to H$ be mappings and $\phi : H \times H \to \mathbb{R} \cup \{+\infty\}$ be a proper functional such that for each fixed $y \in H$, $\phi(\cdot, y) : H \to H$ is lower semicontinuous and $\eta$-subdifferentiable on $H$ and $g(H) \cap \text{dom} \phi(\cdot, y) \neq \emptyset$.

Given $f \in H$, we consider the following generalized nonlinear quasi-variational-like inclusion problem:

Find $x \in H$ such that $gx \in \text{dom} \phi(\cdot, x)$ and

$$\langle N(ax, bx) - M(cx, dx) - f, \eta(y, gx) \rangle \geq \phi(gx, x) - \phi(y, x), \quad \forall y \in H.$$  (2.1)

Special cases:

If $f = 0$, $N(ax, bx) = ax - bx$ and $M(cx, dx) = 0$ for all $x, y \in H$, then the problem (2.1) reduces to the following variational inclusion problem:

Find $x \in H$ such that $gx \in \text{dom} \phi(\cdot, x)$ and

$$\langle ax - bx, \eta(y, gx) \rangle \geq \phi(gx) - \phi(y), \quad \forall y \in H,$$
which was introduced and studied by Ding-Luo [6].

If $f = 0$, $\phi(x, y) = \phi(x)$, $N(ax, bx) = ax - bx$, $M(cx, dx) = 0$ and $\eta(y, x) = y - x$ for all $x, y \in H$, then the problem (2.1) reduces to the following inclusion problem:

Find $x \in H$ such that $gx \in \text{dom} \phi(\cdot, x)$ and

$$\langle ax - bx, y - gx \rangle \geq \phi(gx) - \phi(y), \quad \forall y \in H,$$
which was studied by Hassouni-Moudafi [13].

If $f = 0$, $\eta(x, y) = x - y$, $N(ax, bx) = ax - bx$, $M(cx, dx) = 0$ and $K(x) = mx + K$ for all $x, y \in H$, where $m : H \to H$ is a mapping and $K$ is a closed
convex subset of $H$, $\phi : H \times H \to H$ is defined by

$$\phi(x, y) = I_{K(y)}(x), \quad \forall x, y \in H,$$

and $I_{K(y)}(x)$ is the indicator function of $K(y)$, that is,

$$I_{K(y)}(x) = \begin{cases} 0, & \text{if } x \in K(y), \\ +\infty, & \text{otherwise,} \end{cases}$$

then the problem (2.1) reduces to the following strongly nonlinear quasi-variational inequality problem:

Find $x \in H$ such that $gx \in K(x)$ and

$$\langle ax - bx, y - gx \rangle \geq 0, \quad \forall y \in K(x),$$

which includes a number of variational inequalities, quasi-variational inequalities, complementarity and quasi-complementarity problems as special cases, studied previously by many authors.

Now we recall the following definitions and results.

**Definition 2.1.** Let $h : H \to H$, $\eta : H \times H \to H$ be mappings.

1. $h$ is said to be $\delta$-$\eta$-strongly monotone if there exists a constant $\delta > 0$ such that

$$\langle hx - hy, \eta(x, y) \rangle \geq \delta \|x - y\|^2, \quad \forall x, y \in H;$$

2. $h$ is said to be $\sigma$-Lipschitz continuous if there exists a constant $\sigma > 0$ such that

$$\|hx - hy\| \leq \sigma \|x - y\|, \quad \forall x, y \in H;$$

3. $\eta$ is said to be $\alpha$-Lipschitz continuous if there exists a constant $\alpha > 0$ such that

$$\|\eta(x, y)\| \leq \alpha \|x - y\|, \quad \forall x, y \in H.$$

**Definition 2.2.** Let $a, b, c : H \to H$ be mappings. A mapping $N : H \times H \to H$ is called

1. strongly monotone with respect to $a$ in the first argument if there exists a constant $r > 0$ such that

$$\langle N(ax, u) - N(ay, u), x - y \rangle \geq r \|x - y\|^2, \quad \forall x, y, u \in H;$$

2. relaxed coercive with respect to $a$ in the first argument if there exist constants $\gamma > 0$ and $r > 0$ such that

$$\langle N(ax, u) - N(ay, u), x - y \rangle \geq r \|x - y\|^2 - \gamma \|ax - ay\|^2, \quad \forall x, y, u \in H;$$

3. Lipschitz continuous in the first argument if there exists a constant $t > 0$ satisfying

$$\|N(x, u) - N(y, u)\| \leq t \|x - y\|, \quad \forall x, y, u \in H;$$
generalized pseudocontractive with respect to $a$ in the first argument if there exists a constant $s > 0$ such that
\[ \langle N(ax, u) - N(ay, u), x - y \rangle \leq s\|x - y\|^2, \quad \forall x, y, u \in H; \]

(5) mixed Lipschitz continuous with respect to $a$ and $b$ in the first and second arguments if there exists a constant $\beta > 0$ such that
\[ \|N(ax, bx) - N(ay, by)\| \leq \beta\|x - y\|, \quad \forall x, y \in H. \]

In a similar way, we can define the Lipschitz continuity of the mapping $N$ in the second argument.

**Definition 2.3** ([24]). A functional $f : H \times H \to \mathbb{R} \cup \{+\infty\}$ is said to be 0-diagonally quasi-concave in the first argument if for any finite set $\{x_1, \ldots, x_n\} \subset H$ and for any $y = \sum_{i=1}^{n} \lambda_i x_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$,
\[ \min_{1 \leq i \leq n} f(x_i, y) \leq 0. \]

**Definition 2.4.** Let $\eta : H \times H \to H$ be a mapping. A proper functional $\phi : H \to \mathbb{R} \cup \{+\infty\}$ is said to be $\eta$-subdifferentiable at a point $x \in H$ if there exists a point $f^* \in H$ such that
\[ \phi(y) \geq \phi(x) + \langle f^*, \eta(y, x) \rangle, \quad \forall y \in H, \]
where $f^*$ is called a $\eta$-subgradient of $\phi$ at $x$. The set of all $\eta$-subgradient of $\phi$ at $x$ is denoted by $\Delta \phi(x)$. The mapping $\Delta \phi : H \to 2^H$ denoted by
\[ \Delta \phi(x) = \{ f^* \in H : \phi(y) \geq \phi(x) + \langle f^*, \eta(y, x) \rangle, \forall y \in H \}, \quad \forall x \in H, \]
is said to be $\eta$-subdifferential of $\phi$.

**Definition 2.5.** Let $\phi : H \to \mathbb{R} \cup \{+\infty\}$ be a proper functional, $\eta : H \times H \to H$ and $h : H \to H$ be mappings, and $\rho > 0$ be a constant. Assume that for any given $x \in H$ there exists a unique point $u \in H$ such that
\[ \langle hu - x, \eta(y, u) \rangle \geq \rho \phi(u) - \rho \phi(y), \quad \forall y \in H. \]
Then the mapping $x \mapsto u$, denoted by $J_{\rho, h}^{\phi}(x)$, is said to be $(h, \eta)$-proximal mapping of $\phi$.

It follows from the definition of $J_{\rho, h}^{\phi}(x)$ that $x - hu \in \rho \Delta \phi(u)$, that is,
\[ J_{\rho, h}^{\phi}(x) = (h + \rho \Delta \phi)^{-1}(x). \]

**Definition 2.6** ([11]). Let $T : H \to H$ be a mapping and $x_0 \in H$. Assume that $x_{n+1} = f(T, x_n)$ define an iterative procedure which yields a sequence of points \( \{x_n\}_{n \geq 0} \subseteq H \). Suppose that $F(T) = \{ x \in H : x = Tx \} \neq \emptyset$ and $\{x_n\}_{n \geq 0}$ converges to some $u \in F(T)$. Let $\{z_n\}_{n \geq 0} \subset H$ and $\varepsilon_n = \|z_{n+1} - f(T, z_n)\|$ for $n \geq 0$. If $\lim_{n \to \infty} \varepsilon_n = 0$ implies that $\lim_{n \to \infty} z_n = u$, then the iterative procedure defined by $x_{n+1} = f(T, x_n)$ is said to be $T$-stable or stable with respect to $T$. 
Lemma 2.1 ([15]). Let \( \{a_n\}_{n \geq 0}, \{b_n\}_{n \geq 0}, \{c_n\}_{n \geq 0} \) and \( \{t_n\}_{n \geq 0} \) be four sequences of nonnegative numbers satisfying
\[
a_{n+1} \leq (1 - t_n)a_n + t_nb_n + c_n, \quad \forall n \geq 0,
\]
where \( \{t_n\}_{n \geq 0} \subset [0, 1], \sum_{n=0}^{\infty} t_n = +\infty, \lim_{n \to \infty} b_n = 0 \) and \( \sum_{n=0}^{\infty} c_n < +\infty. \)
Then \( \lim_{n \to \infty} a_n = 0. \)

Lemma 2.2 ([7]). Let \( D \) be a nonempty convex subset of a topological vector space and \( f : D \times D \to \mathbb{R} \cup \{+\infty\} \) be such that

(a) for each \( x \in D \), \( y \mapsto f(x, y) \) is lower semicontinuous on each compact subset of \( D \),
(b) for each finite set \( \{x_1, \ldots, x_m\} \subset D \) and for each \( y = \sum_{i=1}^{m} \lambda_i x_i \) with \( \lambda_i \geq 0 \) and \( \sum_{i=1}^{m} \lambda_i = 1 \), \( \min_{0 \leq i \leq m} f(x_i, y) \leq 0 \),
(c) there exist a nonempty compact convex subset \( D_0 \) of \( D \) and a nonempty compact subset \( K \) of \( D \) such that for each \( y \in D \setminus K \), there is an \( x \in \text{co}(D_0 \cup \{y\}) \) satisfying \( f(x, y) > 0 \).
Then there exists \( \hat{y} \in D \) such that \( f(x, \hat{y}) \leq 0 \).

Now we show the existence and Lipschitz continuity of the \((h, \eta)\)-proximal mapping for a proper functional \( \phi \) in Hilbert spaces.

Theorem 2.1. Let \( \eta : H \times H \to H \) be \( \tau \)-Lipschitz continuous such that \( \eta(x, y) = -\eta(y, x) \) for all \( x, y \in H \), \( \phi : H \to \mathbb{R} \cup \{+\infty\} \) be a lower semicontinuous \( \eta \)-subdifferentiable proper functional and \( h : H \to H \) be \( \gamma \)-\( \eta \)-strongly monotone and \( \lambda \)-Lipschitz continuous. Assume that for any given \( x \in H \), the functional \( k : H \times H \to \mathbb{R} \) defined by \( k(y, u) = \langle x - hu, \eta(y, u) \rangle \) is 0-DQC\( \nu \) in the first argument. Let \( \rho > 0 \) be a constant. Then for any given \( x \in H \), there exists a unique \( u \in H \) such that
\[
\langle hu - x, \eta(y, u) \rangle \geq \rho \phi(u) - \rho \phi(y), \quad \forall y \in H.
\]
That is, \( u = J_{\rho, h}^\Delta \phi(x) \).

Proof. For any given \( x \in H \), define a functional \( f : H \times H \to \mathbb{R} \cup \{+\infty\} \) by
\[
f(y, u) = \langle x - hu, \eta(y, u) \rangle + \rho \phi(u) - \rho \phi(y), \quad \forall y \in H.
\]
It follows from the continuity of \( \eta(y, u) \) and lower semicontinuity of \( \phi \) that for each \( y \in H \), \( u \mapsto f(y, u) \) satisfies the condition (b) of Lemma 2.2. Otherwise, there exists a finite set \( \{y_1, \ldots, y_m\} \subset H \) and some \( u_0 = \sum_{i=1}^{m} \lambda_i y_i \) with \( \lambda_i \geq 0 \) and \( \sum_{i=1}^{m} \lambda_i = 1 \) such that
\[
\langle x - hu_0, \eta(y_i, u_0) \rangle + \rho \phi(u_0) - \rho \phi(y_i) > 0, \quad 1 \leq i \leq m.
\]
Since \( \phi \) is \( \eta \)-subdifferentiable at \( u_0 \), there exists \( f_{u_0}^* \in H \) such that
\[
\phi(y) \geq \phi(u_0) + \langle f_{u_0}^*, \eta(y, u_0) \rangle, \quad \forall y \in H.
\]
By virtue of (2.3), (2.4), we infer that
\[
\langle x - hu_0, \eta(y_i, u_0) \rangle + \rho \phi(y_i) - \rho \phi(u_0) \\
\geq \rho \langle f_{u_0}^*, \eta(y_i, u_0) \rangle, \quad 1 \leq i \leq m,
\]
which implies that
\[
\langle x - \rho f_{u_0} - hu_0, \eta(y, u_0) \rangle > 0, \quad 1 \leq i \leq m. 
\]
Since \(k(y, u_0) = \langle x - \rho f_{u_0} - hu_0, \eta(y, u_0) \rangle\) is 0-DQCV in the first argument, it follows that
\[
\min_{1 \leq i \leq m} \langle x - \rho f_{u_0} - hu_0, \eta(y, u_0) \rangle \leq 0 < \min_{1 \leq i \leq m} \langle x - \rho f_{u_0} - hu_0, \eta(y, u_0) \rangle, 
\]
which is a contradiction. Hence \(f\) satisfies the condition (b) of Lemma 2.2. Let \(\bar{y} \in \text{dom}\phi\). Since \(\phi\) is \(\eta\)-subdifferentiable at \(\bar{y}\), there exists \(f_{\bar{y}}^* \in H\) such that
\[
\phi(u) - \phi(\bar{y}) \geq \langle f_{\bar{y}}^*, \eta(u, \bar{y}) \rangle, \quad \forall u \in H, 
\]
which implies that
\[
f(\bar{y}, u) = \langle x - hu, \eta(\bar{y}, u) \rangle + \rho \phi(u) - \rho \phi(\bar{y}) \\
\geq \langle h\bar{y} - hu, \eta(\bar{y}, u) \rangle + \langle x - h\bar{y}, \eta(\bar{y}, u) \rangle + \rho \langle f_{\bar{y}}^*, \eta(u, \bar{y}) \rangle \\
\geq \gamma \|\bar{y} - u\|^2 - \tau \|x - h\bar{y}\| \|\bar{y} - u\| - \tau \rho \|f_{\bar{y}}^*\| \|\bar{y} - u\| \\
= \|\bar{y} - u\| \|\gamma \|\bar{y} - u\| - \tau \|x - h\bar{y}\| - \tau \rho \|f_{\bar{y}}^*\|.
\]
Let \(r = (1/\gamma)\tau \|x - h\bar{y}\| + \rho \|f_{\bar{y}}^*\|, K = \{u \in H : \|\bar{y} - u\| \leq r\}\) and \(D_0 = \{\bar{y}\}\). Clearly \(D_0\) and \(K\) are both weakly compact convex subsets of \(H\) and for each \(u \in H\backslash K\), there exists \(\bar{y} \in \text{co}(D_0 \cup \{\bar{y}\})\) such that \(f(\bar{y}, u) > 0\). Hence all the conditions of Lemma 2.2 are satisfied. By Lemma 2.2, there exists \(\bar{u} \in H\) such that \(f(\bar{y}, \bar{u}) \leq 0, \forall y \in H\), that is,
\[
\langle h\bar{u} - x, \eta(y, \bar{u}) \rangle \geq \rho \phi(\bar{u}) - \rho \phi(y), \quad \forall y \in H.
\]
Now we show that \(\bar{u}\) is a unique solution of the problem (2.2). Suppose that \(u_1, u_2 \in H\) are arbitrary two solutions of the problem (2.2). It follows that
\[
\langle hu_1 - x, \eta(\bar{u}, u_1) \rangle \geq \rho \phi(u_1) - \rho \phi(\bar{u}), \quad \forall y \in H, 
\]
(2.6)
\[
\langle hu_2 - x, \eta(y, u_2) \rangle \geq \rho \phi(u_2) - \rho \phi(y), \quad \forall y \in H. 
\]
(2.7)
Taking \(y = u_2\) in (2.6) and \(y = u_1\) in (2.7), and adding these inequalities, we have
\[
\langle hu_1 - x, \eta(u_2, u_1) \rangle + \langle hu_2 - x, \eta(u_1, u_2) \rangle \geq 0.
\]
Since \(\eta(x, y) = -\eta(y, x), \forall x, y \in H\) and \(h\) is \(\gamma\)-\(\eta\)-strongly monotone, by (2.8) we deduce that
\[
\gamma \|u_1 - u_2\|^2 \leq \langle hu_1 - hu_2, \eta(u_1, u_2) \rangle \leq 0, 
\]
which implies that \(u_1 = u_2\). This completes the proof.

Theorem 2.2. Let \(\eta : H \times H \to H\) be \(\tau\)-Lipschitz continuous such that \(\eta(x, y) = -\eta(y, x)\) for all \(x, y \in H\), \(\phi : H \to \mathbb{R} \cup \{+\infty\}\) be a lower semicontinuous \(\eta\)-subdifferentiable proper functional and \(h : H \to H\) be \(\gamma\)-\(\eta\)-strongly monotone and \(\lambda\)-Lipschitz continuous. Let \(\rho > 0\) be a constant. Assume that for any given \(x \in H\), the functional \(k : H \times H \to \mathbb{R}\) defined by \(k(y, u) = \langle x - hu, \eta(y, u) \rangle\)
is 0-DQCV in the first argument. Then the \((h, \eta)\)-proximal mapping \(J_{p,h}^{\Delta \phi}\) of \(\phi\) is \(\tau/\gamma\)-Lipschitz continuous.

**Proof.** Let \(x_1, x_2 \in H, u_1 = J_{p,h}^{\Delta \phi}(x_1)\) and \(u_2 = J_{p,h}^{\Delta \phi}(x_2)\). It follows from Theorem 2.1 that

\[
\langle hu_1 - x_1, \eta(y, u_1) \rangle \geq \rho \phi(u_1) - \rho \phi(y), \quad \forall y \in H,
\]

(2.9)

\[
\langle hu_2 - x_2, \eta(y, u_2) \rangle \geq \rho \phi(u_2) - \rho \phi(y), \quad \forall y \in H.
\]

(2.10)

Taking \(y = u_2\) in (2.9) and \(y = u_1\) in (2.10), and adding these inequalities, we conclude that

\[
\langle hu_1 - x_1, \eta(u_2, u_1) \rangle + \langle hu_2 - x_2, \eta(u_1, u_2) \rangle \geq 0.
\]

Since \(h\) is \(\gamma/\eta\)-strongly monotone and \(\eta\) is \(\tau\)-Lipschitz continuous with \(\eta(x, y) = -\eta(y, x)\) for all \(x, y \in H\), we infer that

\[
\gamma \|u_1 - u_2\|^2 \leq \langle hu_1 - hu_2, \eta(u_1, u_2) \rangle \leq \langle x_1 - x_2, \eta(u_1, u_2) \rangle \leq \|x_1 - x_2\| \|\eta(u_1, u_2)\| \leq \tau \|x_1 - x_2\| \|u_1 - u_2\|,
\]

which implies that

\[
\|J_{p,h}^{\Delta \phi}(x_1) - J_{p,h}^{\Delta \phi}(x_2)\| = \|u_1 - u_2\| \leq \frac{\tau}{\gamma} \|x_1 - x_2\|,
\]

that is, \(J_{p,h}^{\Delta \phi}\) is \(\tau/\gamma\)-Lipschitz continuous. This completes the proof. □

**Remark 2.1.** If \(\eta(x, y) = x - y\) for all \(x, y \in H\), where \(h\) is the identity mapping in \(H\), then Theorem 2.2 reduces to Lemma 2.2 in [13].

### 3. A perturbed three-step iterative algorithm with errors

We first transfer the problem (2.1) into a fixed point problem.

**Lemma 3.1.** \(x^* \in H\) is a solution of the problem (2.1) if and only if \(x^*\) satisfies the following relation

\[
gx^* = J_{p,h}^{\Delta \phi}([hx^* - \rho(N(ax^*, bx^*) - M(cx^*, dx^*) - f)],
\]

(3.1)

where \(J_{p,h}^{\Delta \phi}((\cdot, x^*)) = (h + \rho \Delta \phi(\cdot, x^*))^{-1}\) is the \((h, \eta)\)-proximal mapping of \(\phi(\cdot, x^*)\) and \(\rho > 0\) is a constant.

**Proof.** The fact directly follows from Definitions 2.4 and 2.5. □

Based on Lemma 3.1, we suggest the following perturbed three-step iterative algorithm with errors.

**Algorithm 3.1.** Let \(N, M, \eta : H \times H \to H\), \(a, b, c, d, g, h : H \to H\) be mappings, and \(\{\phi_n\}_{n \geq 0} : H \times H \to \mathbb{R} \cup \{+\infty\}\) be a sequence of proper functionals such that for each given \(x \in H, n \geq 0\), the \((h, \eta)\)-proximal mapping of \(\phi_n(\cdot, x)\)
exists. Let $E x = h x - \rho N(ax, bx) + \rho M(cx, dx) + \rho f$ for all $x \in H$. For any given $u_0 \in H$, compute sequence $\{u_n\}_{n \geq 0}$ by the following iterative schemes:

$$
\begin{align*}
  u_{n+1} &= (1 - a_n - b_n)u_n \\
  &+ a_n [v_n - gv_n + J_{\rho,h}^\phi(v_n, Ev_n)] + b_n p_n, \\
  v_n &= (1 - a'_n - b'_n)u_n \\
  &+ a'_n [w_n - gw_n + J_{\rho,h}^\phi(w_n, Ew_n)] + b'_n q_n, \\
  w_n &= (1 - a''_n - b''_n)u_n \\
  &+ a''_n [u_n - gu_n + J_{\rho,h}^\phi(u_n, Eu_n)] + b''_n r_n, \quad n \geq 0,
\end{align*}
$$

(3.2)

where $\rho > 0$, $\{p_n\}_{n \geq 0}$, $\{q_n\}_{n \geq 0}$ and $\{r_n\}_{n \geq 0}$ are bounded sequences in $H$ introduced to take into account possible in exact computation and sequences $\{a_n\}_{n \geq 0}$, $\{b_n\}_{n \geq 0}$, $\{a'_n\}_{n \geq 0}$, $\{b'_n\}_{n \geq 0}$, $\{a''_n\}_{n \geq 0}$ and $\{b''_n\}_{n \geq 0}$ are in $[0, 1]$ satisfying

$$
\max\{a_n + b_n, a'_n + b'_n, a''_n + b''_n\} \leq 1, \quad n \geq 0,
$$

(3.3)

$$
\sum_{n=0}^{\infty} a_n = +\infty, \quad \lim_{n \to \infty} a_n b''_n = \lim_{n \to \infty} b'_n = 0
$$

(3.4)

and one of the following conditions:

$$
\sum_{n=0}^{\infty} b_n < +\infty;
$$

(3.5)

$$
\text{with } \lim_{n \to \infty} b_n = 0 \text{ and } b_n = a_n h_n, \quad \forall n \geq 0.
$$

(3.6)

Remark 3.1. In case $a''_n = b''_n = 0$ for all $n \geq 0$, then Algorithm 3.1 reduces to the Ishikawa type perturbed iterative algorithm in [3]. If $a'_n = b'_n = 0$, for all $n \geq 0$, then the Ishikawa type perturbed iterative algorithm reduces to the Mann type perturbed iterative algorithm in [23]. If $h$ is the identity mapping in $H$ and $M(x,y) = 0, N(x, y) = x - y = \eta(x, y)$ for all $x, y \in H$, and $b_n = b'_n = b''_n = a''_n = 0$ for all $n \geq 0$, Algorithm 3.1 reduces to the algorithm in [13].

4. Existence, convergence and stability

In this section, we give the existence and uniqueness theorems of solution for the generalized nonlinear quasi-variational-like inclusion, and show the convergence and stability results of iterative sequence generated by the perturbed three-step iterative algorithm with errors.
Theorem 4.1. Let $a, b, c, d, g : H \to H$ be mappings, $a, b, g$ be Lipschitz continuous with constants $\lambda_a, \lambda_b, \lambda_g$, respectively, and $g$ be $\sigma$-strongly monotone. Let $\eta : H \times H \to H$ be $\delta$-strongly monotone and $\tau$-Lipschitz continuous such that $\eta(x, y) = -\eta(y, x)$, $\forall x, y \in H$. Let $h : H \to H$ be $\gamma$-$\eta$-strongly monotone and $\lambda$-Lipschitz continuous. For each given $x \in H$, the functional $k : H \times H \to \mathbb{R}$ defined by $k(y, u) = (x - hu, \eta(y, u))$ is 0-DQCV in the first argument. Let $N : H \times H \to H$ be Lipschitz continuous in the first and second arguments with constants $\xi$ and $\zeta$, respectively, and be strongly monotone with respect to $a$ in the first argument with constant $\alpha$. Assume that $M : H \times H \to H$ is mixed Lipschitz continuous with respect to $c$ and $d$ in the first and second arguments with constant $\beta$. Let $\phi$ and $\phi_n : H \times H \to \mathbb{R} \cup \{+\infty\}$ be such that for each fixed $y \in H$ and $n \geq 0$, $\phi(-, y)$ and $\phi_n(-, y)$ are lower semicontinuous $\eta$-subdifferentiable proper functionals satisfying $g(H) \cap \text{dom} \phi(-, y) \neq \emptyset$, $g(H) \cap \text{dom} \phi_n(-, y) \neq \emptyset$ and

$$
\sup \{ \| J_{\rho, h}^{\Delta \phi_n(\cdot, x)}(z) - J_{\rho, h}^{\Delta \phi_n(\cdot, y)}(z) \|, \| J_{\rho, h}^{\Delta \phi_n(\cdot, x)}(z) - J_{\rho, h}^{\Delta \phi_n(\cdot, y)}(z) \| : n \geq 0 \}
\leq \mu \| x - y \|, \ \forall x, y, z \in H,
$$

(4.1)

$$
\lim_{n \to \infty} \| J_{\rho, h}^{\Delta \phi_n(\cdot, y)}(z) - J_{\rho, h}^{\Delta \phi_n(\cdot, y)}(z) \| = 0, \ \forall y, z \in H,
$$

where $\mu > 0$ is a constant. Let $\{x_n\}_{n \geq 0}$ be any sequence in $H$ and define $\{\varepsilon_n\}_{n \geq 0} \subset [0, \infty)$ by

$$
\varepsilon_n = \| x_{n+1} - [(1 - a_n - b_n)x_n
+ a_n(y_n - gy_n + J_{\rho, h}^{\Delta \phi_n(\cdot, y_n)}(Ex_n) + b_n p_n)] \|,
$$

$$
y_n = (1 - a_n' - b_n')x_n
+ a_n'[z_n - g z_n + J_{\rho, h}^{\Delta \phi_n(\cdot, z_n)}(Ex_n)] + b_n' q_n,
$$

$$
z_n = (1 - a_n'' - b_n'')x_n
+ a_n''[x_n - g x_n + J_{\rho, h}^{\Delta \phi_n(\cdot, x_n)}(Ex_n)] + b_n'' r_n, \quad n \geq 0.
$$

(4.2)

Let $K = (1 + \frac{1}{\tau}) \sqrt{1 - 2\sigma + \lambda_b^2} + \frac{1}{\tau} \lambda_b \sqrt{1 - 2\gamma + \lambda_b^2} + \mu$, $P = \lambda_b^2 \xi^2 + \tau^2 (\zeta \lambda_b + \beta)^2$, $Q = \alpha \tau^2$. If there exists a constant $\rho > 0$ satisfying one of the following conditions:

$$
P > 0, \quad |Q - (1 - K)(\zeta \lambda_b + \beta)\tau \delta| > (\tau^2 - \delta^2(1 - K)^2)P,
$$

(4.3)

$$
|\rho - \frac{Q - (1 - K)(\zeta \lambda_b + \beta)\tau \delta}{P}| < \sqrt{|Q - (1 - K)(\zeta \lambda_b + \beta)\tau \delta|^2 - P(\tau^2 - (1 - K)^2 \delta^2)},
$$

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\[ P < 0, \]
\[
\left| \frac{\rho - Q - (1 - K)(\zeta \lambda_b + \beta)\tau \delta}{P} \right| > \sqrt{\frac{[Q - (1 - K)(\zeta \lambda_b + \beta)\tau \delta]^2 - P(\tau^2 - (1 - K)^2\delta^2)}{P}},
\]
then the problem (2.1) has a unique solution \( x^* \in H \) and the iterative sequence \( \{x_n\}_{n \geq 0} \) defined by Algorithm 3.1 converges strongly to \( x^* \). Moreover, if there exists a constant \( A > 0 \) satisfying
\[
\text{(4.5)} \quad a_n \geq A, \quad n \geq 0,
\]
then \( \lim_{n \to \infty} x_n = x^* \) if and only if \( \lim_{n \to \infty} \varepsilon_n = 0 \).

Proof. First, we define a mapping \( F : H \to H \) by
\[
F(x) = x - gx + J_{\rho,h}^{\Delta \phi(x)}(Ex), \quad \forall x \in H.
\]
For any \( x, y \in H \), we conclude that
\[
\|F(x) - F(y)\|
\leq \|x - y - (gx - gy)\| + \|J_{\rho,h}^{\Delta \phi(x)}(Ex) - J_{\rho,h}^{\Delta \phi(x)}(Ey)\|
+ \|J_{\rho,h}^{\Delta \phi(x)}(Ey) - J_{\rho,h}^{\Delta \phi(y)}(Ey)\|
\leq \left(1 + \frac{\tau}{\delta}\right)\|x - y - (gx - gy)\|
+ \frac{\tau}{\delta}\|gx - gy - (hx - hy)\|
+ \frac{\tau}{\delta}\|x - y - \rho(N(ax, bx) - N(ay, by))\|
+ \frac{\rho}{\delta}\|M(cx, dx) - M(cy, dy)\| + \mu\|x - y\|
\leq \left(1 + \frac{\tau}{\delta}\right)\sqrt{1 - 2\sigma + \lambda_2^2}\|x - y\|
+ \frac{\tau}{\delta}\sqrt{1 - 2\rho\alpha + \rho^2\xi^2\lambda_2^2 + \rho\zeta\lambda_b}\|x - y\|
+ \frac{\tau\rho^2\beta}{\delta}\|x - y\| + \mu\|x - y\|
= \theta\|x - y\|,
\]
where
\[
\theta = \left(1 + \frac{\tau}{\delta}\right)\sqrt{1 - 2\sigma + \lambda_2^2} + \frac{\tau}{\delta}\sqrt{1 - 2\rho\alpha + \rho^2\xi^2\lambda_2^2 + \rho\zeta\lambda_b + \rho\beta}\|x - y\|
+ \sqrt{1 - 2\rho\alpha + \rho^2\xi^2\lambda_2^2 + \rho\zeta\lambda_b + \rho\beta} + \mu.
\]
It follows from one of (4.3), (4.4), and (4.6) that \( \theta < 1 \) and that \( F \) is a contraction mapping and that it has a unique fixed point \( x^* \in H \) such that \( x^* = F(x^*) \), that is,
\[
gx^* = J_{\rho,h}^{\Delta \phi(x^*)}[hx^* - \rho(N(ax^*, bx^*) - M(cx^*, dx^*) - f)].
Thus $x^*$ is a unique solution of the problem (2.1) in $H$.

Now we show that $\lim_{n \to \infty} x_n = x^*$. Notice that

$$x^* = (1 - a_n - b_n)x^* + a_n [x^* - gx^* + J_{\rho,h}^{\Delta \phi_n(x^*)}(Ex^*)] + b_n x^*$$

$$= (1 - a_n' - b_n')x^* + a_n' [x^* - gx^* + J_{\rho,h}^{\Delta \phi_n(x^*)}(Ex^*)] + b_n' x^*$$

$$= (1 - a_n'' - b_n'')x^* + a_n'' [x^* - gx^* + J_{\rho,h}^{\Delta \phi_n(x^*)}(Ex^*)] + b_n'' x^*, \quad n \geq 0.$$

Put

$$d_n = \| J_{\rho,h}^{\Delta \phi_n(x^*)}(Eu) - J_{\rho,h}^{\Delta \phi_n(y^*)}(Eu) \|, \quad \forall x, y, u \in H,$n \geq 0}.$$.

It follows from (4.1) that

$$\lim_{n \to \infty} d_n = 0.$$.

Using (3.2), (3.3), and (4.7), we get that

$$\| u_{n+1} - x^* \|$$

$$\leq (1 - a_n - b_n)\| u_n - x^* \| + a_n [\| v_n - x^* - (g v_n - g x^*) \|$$

$$+ \| J_{\rho,h}^{\Delta \phi_n(v_n)}(E v_n) - J_{\rho,h}^{\Delta \phi_n(x^*)}(E x^*) \|$$

$$+ \| J_{\rho,h}^{\Delta \phi_n(x^*)}(E x^*) - J_{\rho,h}^{\Delta \phi_n(x^*)}(E x^*) \|$$

$$+ \| J_{\rho,h}^{\Delta \phi_n(x^*)}(E x^*) - J_{\rho,h}^{\Delta \phi_n(x^*)}(E x^*) \|] + b_n \| p_n - x^* \|$$

$$\leq (1 - a_n - b_n)\| u_n - x^* \| + a_n \sqrt{1 - 2 \sigma^2 + \lambda} \| v_n - x^* \|$$

$$+ \frac{\sigma}{\delta} a_n \| E v_n - E x^* \| + a_n \| v_n - x^* \| + a_n d_n + b_n L$$

$$\leq (1 - a_n - b_n)\| u_n - x^* \| + a_n \sqrt{1 - 2 \sigma^2 + \lambda} \| v_n - x^* \|$$

$$+ \frac{\sigma}{\delta} a_n \| v_n - x^* - (g v_n - g x^*) \|$$

$$+ \frac{\sigma}{\delta} a_n \| g v_n - g x^* - (h g v_n - h g x^*) \|$$

$$+ \frac{\sigma}{\delta} a_n \| v_n - x^* - \rho (N(a v_n, b v_n) - N(ax^*, b x^*)) \|$$

$$+ \frac{\sigma}{\delta} a_n \| M(c v_n, d v_n) - M(c x^*, d x^*) \| + a_n \| v_n - x^* \|$$

$$+ a_n d_n + b_n L.$$
\[
(1 - a_n - b_n)\|u_n - x^*\| + a_n \left\{ \left( 1 + \frac{\tau}{\delta} \right) \sqrt{1 - 2\sigma + \lambda_g^2} \right. \\
+ \frac{\tau}{\delta} \left[ \lambda_g \sqrt{1 - 2\gamma + \lambda_L^2} + \frac{\tau}{\delta} \sqrt{1 - 2\rho\alpha + \rho^2\xi_2^2\lambda_L^2} \right] \\
+ \rho\lambda_\beta \zeta + \rho\beta + \mu \right\} \|v_n - x^*\| + a_n d_n + b_n L \\
= (1 - a_n - b_n)\|u_n - x^*\| + a_n d_n + b_n, \quad n \geq 0.
\]

Similarly, we deduce that
\[
\|v_n - x^*\| \leq (1 - a_n' - b_n')\|u_n - x^*\| + \theta a_n'\|w_n - x^*\| \\
+ a_n' d_n + b_n' L, \\
\|w_n - x^*\| \leq (1 - a_n'' - b_n'')\|u_n - x^*\| + \theta a_n''\|u_n - x^*\| \\
+ a_n'' d_n + b_n'' L, \quad n \geq 0.
\]

Substituting (4.10) into (4.9), we infer that
\[
\begin{align*}
\|u_{n+1} - x^*\| \\
&\leq [1 - a_n - b_n + a_n\theta(1 - a_n' - b_n') + a_n'\theta(1 - a_n'' - b_n'') \\
&+ a_n''\theta]\|u_n - x^*\| + a_n\theta a_n'(a_n''\theta d_n + \theta Lb_n'' + d_n) \\
&+ \theta Lb_n' + d_n + b_n L \\
&\leq (1 - (1 - \theta)a_n)\|u_n - x^*\| + a_n[a_n'(2d_n + Lb_n'') \\
&+ Lb_n' + d_n] + b_nL, \quad n \geq 0.
\end{align*}
\]

It follows from Lemma 2.3, (3.4), (3.8), (4.11) and one of (3.5) and (3.6) that \(\lim_{n \to \infty} x_n = x^*\).

Now we assume that (4.5) holds. Observe that each of (3.5) and (3.6) implies that
\[
\lim_{n \to \infty} b_n = 0.
\]

As in the proof of (4.9), by (4.5), we conclude that
\[
\begin{align*}
\|(1 - a_n - b_n)\|x_n + a_n(y_n - gy_n + J_\rho^{A\phi_n}(y_n))(Ey_n)\| + b_n p_n - x^*\| \\
&\leq (1 - (1 - \theta)a_n)\|x_n - x^*\| + a_n[a_n'(2d_n + Lb_n'') + Lb_n' + d_n] + b_n L \\
&\leq (1 - (1 - \theta)A)\|x_n - x^*\| + a_n'(2d_n + Lb_n'') \\
&+ Lb_n' + d_n + b_n L, \quad n \geq 0.
\end{align*}
\]
Suppose that $\lim_{n \to \infty} x_n = x^*$. By virtue of (3.4), (4.2), (4.8), (4.13) and one of (3.5) and (3.6), we obtain that

$$
\varepsilon \leq ||x_{n+1} - x^*|| + ||(1 - a_n - b_n)x_n + a_n(y_n - gy_n + J^\phi_n(y_n)(Ey_n)) + b_np_n - x^*||
$$

$$
\leq ||x_{n+1} - x^*|| + (1 - (1 - \theta)A)||x_n - x^*|| + a_n'(2d_n + Lb_n'' + d_n + b_nL \to 0 \text{ as } n \to \infty.
$$

That is, $\lim_{n \to \infty} \varepsilon_n = 0$. Conversely, suppose that $\lim_{n \to \infty} \varepsilon_n = 0$. In light of (4.2) and (4.13), we know that

$$
\|x_{n+1} - x^*\|
\leq \|((1 - a_n - b_n)x_n + a_n(y_n - gy_n + J^\phi_n(y_n)(Ey_n)) + b_np_n - x^*\| + \varepsilon_n
\leq (1 - (1 - \theta)A)||x_n - x^*|| + a_n'(2d_n + Lb_n'')
+ Lb_n' + d_n + b_nL + \varepsilon_n, \quad n \geq 0.
$$

It follows from Lemma 2.1, (4.8), (4.12), (4.14) and one of (3.5) and (3.6) that $\lim_{n \to \infty} x_n = x^*$. This completes the proof. \ □

**Remark 4.1.** If $h$ is the identity mapping in $H$ and

$$
N(x, y) = x - y, M(x, y) = 0 \text{ for all } x, y \in H
$$

in Theorem 4.1, then Theorem 4.1 generalizes Theorems 3.6~3.8 in [6]. Furthermore, if $\eta(x, y) = x - y$ for all $x, y \in H$, then Theorem 4.1 extends Theorem 3.3 in [4].

**Theorem 4.2.** Let $a, b, g, h, k, \eta, \phi_n, \{\phi_n\}_{n \geq 0}, \{\varepsilon_n\}_{n \geq 0}$ and $\{x_n\}_{n \geq 0}$ be as in Theorem 4.1. Suppose that $c$ and $d : H \to H$ are Lipschitz continuous with constants $\lambda_c$ and $\lambda_d$, respectively, $N : H \times H \to H$ is relaxed coercive with respect to $a$ in the first argument with constants $\gamma > 0$ and $r > 0$, and Lipschitz continuous in the first and second arguments with constants $\xi$ and $\zeta > 0$, respectively. Assume that $M : H \times H \to H$ is generalized pseudocontractive with respect to $c$ in the first argument with constant $\nu$ and Lipschitz continuous in the first and second arguments with constants $\beta > 0$ and $l > 0$, respectively. Let

$$
K = \left(1 + \frac{\tau}{\delta}\right)\sqrt{1 - 2\sigma + \lambda^2} + \frac{\tau^2}{\delta}\sqrt{1 - 2\gamma + \lambda^2} + \mu,
$$

$$
P_l = (\lambda_c \xi + \lambda_c \beta)^2\tau^2 - (\lambda_c \xi + \lambda_c \beta)^2\delta^2,
$$

$$
Q_l = \tau^2[(r - \gamma ln)\lambda^2] - \lambda_c \nu].
$$
If there exists a constant $\rho > 0$ satisfying (4.1), (4.2) and one of the following conditions:

\[ P_1 > 0, \]
\[ \left( 1 - K \right) \left( \lambda_b \xi + \lambda_d \lambda \right) \delta^2 - Q_1 > P_1 \left( \tau^2 - \left( 1 - K \right)^2 \delta^2 \right), \]
\[ (4.15) \]
\[ \left| \rho - \frac{Q_1 - \left( 1 - K \right) \left( \lambda_b \xi + \lambda_d \lambda \right) \delta^2}{P_1} \right| < \sqrt{Q_1 - \left( 1 - K \right) \left( \lambda_b \xi + \lambda_d \lambda \right) \delta^2} - P_1 \left( \tau^2 - \left( 1 - K \right)^2 \delta^2 \right), \]
\[ P_1 < 0, \]
\[ \left| \rho - \frac{Q_1 - \left( 1 - K \right) \left( \lambda_b \xi + \lambda_d \lambda \right) \delta^2}{P_1} \right| > \sqrt{Q_1 - \left( 1 - K \right) \left( \lambda_b \xi + \lambda_d \lambda \right) \delta^2} - P_1 \left( \tau^2 - \left( 1 - K \right)^2 \delta^2 \right), \]
\[ (4.16) \]

then the problem (2.1) has a unique solution $x^* \in H$ and the iterative sequence $\{x_n\}_{n \geq 0}$ defined by Algorithm 3.1 converges strongly to $x^*$. Moreover, if there exists a constant $A > 0$ satisfying (4.5), then $\lim_{n \to \infty} x_n = x^*$ if and only if $\lim_{n \to \infty} \varepsilon_n = 0$.

**Proof.** As in the proof of Theorem 4.1, for any $x, y \in H$, we obtain that

\[ \|F x - F y\| \]
\[ \leq \|x - y - (gx - gy)\| \]
\[ + \|J^\Delta \phi(x, x) E x - J^\Delta \phi(x, x) E y\| \]
\[ + \|J^\Delta \phi(x, y) E y - J^\Delta \phi(x, y) E y\| \]
\[ \leq \left( 1 + \frac{\tau}{\delta} \right) \|x - y - (gx - gy)\| \]
\[ + \frac{\tau}{\delta} \|gx - gy - (hx - hy)\| \]
\[ + \frac{\tau}{\delta} \|x - y - \rho(N(ax, bx) - N(ay, by)) \]
\[ + \rho(M(cx, dx) - M(cy, dy))\|\| + \mu \|x - y\| \]
\[ \leq \left( 1 + \frac{\tau}{\delta} \right) \sqrt{1 - 2\sigma + \lambda^2} \|x - y\| \]
\[ + \frac{\tau}{\delta} \lambda \sqrt{1 - 2\gamma + \lambda^2} \|x - y\| \]
\[ + \frac{\tau}{\delta} \left( \sqrt{1 - 2\rho \left( r - \gamma \lambda^2 \right) - \nu} + \rho^2 \left( \xi \lambda + \lambda \lambda \right)^2 \right) \]
\[ + \rho \lambda \xi + \rho \lambda \lambda \|x - y\| + \mu \|x - y\| \]
\[ = \theta_1 \|x - y\|, \]
where
\[ \theta_1 = \left(1 + \frac{\tau}{\delta}\right) \sqrt{1 - 2\lambda + \sigma^2} + \frac{\tau}{\delta} \lambda \sqrt{1 - 2\gamma + \lambda^2} \]
\[ + \frac{\tau}{\delta} \left(\sqrt{1 - 2\rho(r - \gamma \lambda a - \lambda c \nu) + \rho^2(\xi \lambda a + \lambda c \beta)^2} \right) \]
\[ + \rho \lambda \zeta + \rho \lambda d + \mu. \]

It follows from one of (4.15) and (4.16) that \( \theta_1 < 1 \). From (4.17) we infer that \( F \) is a contraction mapping in \( H \) and it has a unique fixed point \( x^* \in H \), which is a unique solution of the problem (2.1). Similarly, we can show that
\[ \|u_{n+1} - x^*\| \leq (1 - (1 - \theta_1)a_n)\|u_n - x^*\| \]
\[ + a_n[a_n^3(2d_n + Lb_n') + Lb_n' + d_n] + b_n L, \quad n \geq 0. \]

The rest of argument follows as in the proof of Theorem 4.1 and is therefore omitted. This completes the proof. □

**Remark 4.2.** Theorems 4.1 and 4.2 provide the convergence and stability of iterative sequence generated by Algorithm 3.1 under certain conditions.

**References**


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