A CERTAIN PROPERTY OF POLYNOMIALS AND
THE CI-STABILITY OF TANGENT BUNDLE
OVER PROJECTIVE SPACES

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Abstract. We determine the largest integer $i$ such that $0 < i \leq n$ and the coefficient of $t^i$ is odd in the polynomial $(1 + t + t^2 + \cdots + t^n)^{n+1}$. We apply this to prove that the co-index of the tangent bundle over $FP^n$ is stable if $2^r \leq n < 2^r + \frac{3}{4}(2^r - 2)$ for some integer $r$.

1. Introduction

Let $\alpha$ be a finite-dimensional real vector bundle over a finite complex $B$, and let $S(\alpha)$ be its sphere bundle equipped with a $\mathbb{Z}_2$-action by the antipodal map on each fibre. The co-index of $\alpha$, denoted co-ind $\alpha$, is defined to be the smallest integer $k$ for which there exists a $\mathbb{Z}_2$-map from $S(\alpha)$ to $S^{k-1}$ [1, 2, 4]. Here, $S^{k-1}$ also is equipped with a $\mathbb{Z}_2$-action by the antipodal map. By the Borsuk-Ulam theorem, co-ind $\alpha$ is equal to dim $\alpha$ if $\alpha$ is a trivial bundle. We have the inequality co-ind $\alpha \leq$ co-ind($\alpha \oplus 1$) $\leq$ co-ind $\alpha + 1$. We describe $\alpha$ as CI-stable if the equality co-ind($\alpha \oplus k$) = co-ind $\alpha + k$ holds for any positive integer $k$. Here, we abuse notation and denote the $k$-dimensional trivial bundle simply by $k$. Our definition of the stability is slightly different from that in [1] in the sense that we consider the fibrewise suspension. It is obvious that a trivial bundle is CI-stable.

In this paper, we prove the following theorem.

Theorem 1.1. Let $\tau_n$ be the tangent bundle over the projective space $FP^n$ ($F = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$) and let $n = 2^r + k$ with $0 \leq k < 2^r$. If $0 \leq k < \frac{1}{3}(2^r - 2)$, then $\tau_n$ is CI-stable.

The proof of this theorem is given by using the Stiefel-Whitney classes and the following theorem.

Theorem 1.2. In $\mathbb{Z}_2[t]/(t^{n+1})$, the truncated polynomial algebra over $\mathbb{Z}_2$, the degree of $(1 + t + t^2 + \cdots + t^n)^{n+1}$ is equal to $2^r - 1$ where $n = 2^r + k$ and $0 \leq k < 2^r$.

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2. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. First, for \( i = 0, 1, 2, \ldots \), we define power series \( f_i \in \mathbb{Z}_2[[t]] \) as follows.

\[
f_0 \equiv f_0(t) := 1 + t + t^2 + \cdots + t^n + \cdots
\]
\[
f_i \equiv f_i(t) := f_0(t^2) \quad (i = 1, 2, 3, \ldots).
\]

**Lemma 2.1.** We have the following equalities.

1. \( f_{i+1} = f_i^2 \).
2. \( f_i = f_0 2^i \).

**Proof.** In \( \mathbb{Z}_2[[t]] \), we have the following relation in general.

\[
(1 + u + u^2 + \cdots + u^n + \cdots)^2 = 1 + u^2 + u^4 + \cdots + u^{2n} + \cdots.
\]

Putting \( u = t^2 \), we obtain \( f_i^2 = f_{i+1} \), which is the equality (1). The equality (2) follows immediately from (1). \( \square \)

Next, we define polynomials \( G(i, j) \in \mathbb{Z}_2[t] \) by

\[
G(i, j) := 1 + t^2 + (t^2)^2 + (t^2)^3 + \cdots + (t^2)^{2j-i-1}
\]

for non-negative integers \( i \) and \( j \) with \( i \leq j \). We note that \( \deg G(i, j) = 2^j - 2^i \) and also note that \( G(i, i) = 1 \).

**Lemma 2.2.** In \( \mathbb{Z}_2[[t]] \), we have the following formulas for all \( i \leq j \).

1. \( f_i = G(i, j)f_j \).
2. \( f_if_j = G(i, j)f_{j+1} \).

**Proof.** In \( \mathbb{Z}_2[[t]] \), we have the following relation in general.

\[
(1 + u + u^2 + \cdots + u^{r-1})(1 + u^2 + u^4 + \cdots + u^{2r} + \cdots) = 1 + u + u^2 + \cdots + u^n + \cdots.
\]

Putting \( u = t^2 \) and \( \ell = 2^{j-i} \), we obtain \( G(i, j)f_j = f_i \), which is the formula (1). Using this and Lemma 2.1, we have \( f_i \cdot f_j = G(i, j)f_j \cdot f_j = G(i, j)f_j^2 = G(i, j)f_{j+1} \) and the formula (2) follows. \( \square \)

**Proof of Theorem 1.2.** Let \( n = 2^r + k \) with \( 0 \leq k < 2^r \). We express \( k+1 \) as \( k+1 = 2^i_1 + 2^i_2 + 2^i_3 + \cdots + 2^i_s \) where \( 0 \leq i_1 < i_2 < i_3 < \cdots < i_s \leq r \). Then, in \( \mathbb{Z}_2[[t]] \), we have

\[
(1 + t + t^2 + \cdots + t^n + \cdots)^{n+1} = f_0^{2^i_1 + 2^i_2 + 2^i_3 + \cdots + 2^i_s + 2^r} = f_{i_1}f_{i_2}f_{i_3} \cdots f_{i_s}f_r
\]

from the formula (2) of Lemma 2.1. This is equal to \( G(i_1, i_2)f_{i_2+1}f_{i_3} \cdots f_{i_s}f_r \) from the formula (2) of Lemma 2.2. And so, the repeated use of the formula (2) of Lemma 2.2 yields:

\[
(1 + t + t^2 + \cdots + t^n + \cdots)^{n+1} = G(i_1, i_2)G(i_2 + 1, i_3)G(i_3 + 1, i_4) \cdots G(i_s + 1, r)f_{r+1}.
\]
Here, the degree of $G(i_1,i_2)G(i_2+1,i_3)\cdots G(i_s+1,r)$ is equal to:

$$(2^{i_1} - 2^{i_2}) + (2^{i_3} - 2^{i_4+1}) + \cdots + (2^{i_r} - 2^{i_{r+1}})$$

$$= -2^{i_1} - 2^{i_2} - 2^{i_3} - \cdots - 2^{i_s} + 2^{r}$$

$$= 2^r - k - 1.$$  

On the other hand, $f_{r+1}$ is of the form

$$f_{r+1} = 1 + t^{2^{r+1}} + \text{higher terms}$$

and $n < 2^{r+1}$. Hence, in $\mathbb{Z}_2[t]/(t^{n+1})$, we have

$$(1 + t + t^2 + \cdots + t^n)^{n+1} = G(i_1,i_2)G(i_2+1,i_3)\cdots G(i_s+1,r),$$

the degree of which is equal to $2^r - k - 1$. This completes the proof. □

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Let $\alpha$ be a finite-dimensional real vector bundle over a finite complex $B$. For a virtual vector bundle $\gamma$, we write $g\dim \gamma$ for its geometric dimension, that is, the smallest integer $\ell$ such that there exists an $\ell$-dimensional vector bundle $\beta$ which represents the same stable class as $\gamma$. Now, let $\dim \alpha = m$ and suppose $m \geq \dim B$.

First, we put $g\dim(-\alpha) = \ell$ and assume $\ell > 0$. By the definition of geometric dimension, there exists an $\ell$-dimensional vector bundle $\beta$ such that $\alpha \oplus \beta$ is stably trivial. Then, by the stability theorem, we have $\alpha \oplus \beta \cong m \oplus \ell$ since $m + \ell \geq \dim B + 1$. The composite of the inclusion $\alpha \hookrightarrow \alpha \oplus \beta$ with this isomorphism gives rise to a $\mathbb{Z}_2$-map $S(\alpha) \to S^{m+\ell-1}$. Hence, we have $h\dim \alpha \leq m + \ell$.

Next, we put $\coind \alpha = k$. By the definition of co-index, there exists a $\mathbb{Z}_2$-map $f : S(\alpha) \to S^{k-1}$. If $\dim B \leq 2(k - m) - 1$, that is, if $k > m + \frac{1}{2} \dim B$, the map $f$ becomes fibrewise-homotopic to a restriction of some fibrewise-monomorphism $g : \alpha \to \mathbb{R}^k$ [3, Theorem 1.2]. We consider $g$ as the bundle map $g : \alpha \to B \times \mathbb{R}^k$ and let $\beta$ be the cokernel of $g$. Then, we have $\alpha \oplus \beta \cong k$. Hence, we have $h\dim(-\alpha) \equiv \ell \leq \dim \beta = k - m$, that is, $k \geq m + \ell$.

From the above two paragraphs, we obtain the following.

**Lemma 3.1.** [4, Proposition 3.3] Let $\alpha$ be a real vector bundle over $B$ with $\dim \alpha \geq \dim B$ and suppose that $\alpha$ is not stably trivial. Then, if $\coind \alpha > \dim \alpha + \frac{1}{2} \dim B$, we have $\coind \alpha = m + g\dim(-\alpha)$.

For a virtual vector bundle $\alpha$, we denote by $\omega\dim \alpha$ the largest integer $k$ for which the $k$th Stiefel-Whitney class $w_k(\alpha)$ is not zero. It is clear that $\omega\dim \alpha \leq g\dim \alpha$. As shown in the proof of [5, Theorem 2.5], we have the following.

**Lemma 3.2.** $\coind \alpha \geq \dim \alpha + \omega\dim(-\alpha)$.

Combining these two lemmas, we obtain the following proposition.
Proposition 3.3. Let $\alpha$ be a vector bundle over a finite complex $B$ with $\dim \alpha \geq \dim B$, and suppose that the inequality $\omega\dim(-\alpha) > \frac{1}{2} \dim B$ holds. Then:

1. $\text{co-ind} \alpha = \dim \alpha + \text{g-dim}(-\alpha)$.
2. $\alpha$ is CI-stable.

In fact, (2) follows from (1) since $\omega\dim(-\alpha \oplus k) = \omega\dim(-\alpha)$ and $\text{g-dim}(-\alpha \oplus k) = \text{g-dim}(-\alpha)$ for any positive integer $k$.

Proof of Theorem 1.1. Let $\tau_n$ be the tangent bundle over the projective space $\mathbb{P}^n$ with $F = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. The total Stiefel-Whitney class of $\tau_n$ is given by $(1 + t)^{n+1}$, where $t$ is the generator of $H^*(\mathbb{P}^n; \mathbb{Z}_2) = \mathbb{Z}_2[t]/(t^{n+1})$. Let $n = 2^r + k$ with $0 \leq k < 2^r$. Then, Theorem 1.2 states that $\omega\dim(-\tau_n)$ is equal to $d(2^r - k - 1)$, where $d = 1, 2, 4$ according as $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$. Therefore, by Proposition 3.3, $\tau_n$ is CI-stable if $d(2^r - k - 1) > \frac{1}{2} \dim \mathbb{P}^n$. This last inequality is equivalent to $0 \leq k < \frac{1}{3}(2^r - 2)$ and the proof is completed. 

References