SOLUTION OF A VECTOR VARIABLE BI-ADDITIVE FUNCTIONAL EQUATION

Won-Gil Park and Jae-Hyeong Bae
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Abstract. We investigate the relation between the vector variable bi-additive functional equation
\[ f\left(\sum_{i=1}^{n} x_i, \sum_{j=1}^{n} y_j\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} f(x_i, y_j) \]
and the multi-variable quadratic functional equation
\[ g\left(\sum_{i=1}^{n} x_i\right) + \sum_{1 \leq i < j \leq n} g(x_i - x_j) = n \sum_{i=1}^{n} g(x_i). \]
Furthermore, we find out the general solution of the above two functional equations.

1. Introduction

Throughout this paper, let \( n \) be a positive integer greater than 1 and let \( X \) and \( Y \) be vector spaces.

Definition 1. A mapping \( f : X \times X \to Y \) is called bi-additive if \( f \) satisfies the system of equations

\[
\begin{align*}
  f(x + y, z) &= f(x, z) + f(y, z), \\
  f(x, y + z) &= f(x, y) + f(x, z)
\end{align*}
\]

for all \( x, y, z \in X \).

When \( X = Y = \mathbb{R} \), the function \( f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) given by \( f(x, y) := cxy \) is a solution of (1). In particular, letting \( x = y \), we get a quadratic function \( g : \mathbb{R} \to \mathbb{R} \) in one variable given by \( g(x) := f(x, x) = cx^2 \).

For a mapping \( f : X \times X \to Y \), consider the bi-additive functional equation:

\[ f\left(\sum_{i=1}^{n} x_i, \sum_{j=1}^{n} y_j\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} f(x_i, y_j). \]

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For a mapping \( g : X \to Y \), consider the quadratic functional equation:

\[
g \left( \sum_{i=1}^{n} x_i \right) + \sum_{1 \leq i < j \leq n} g \left( x_i - x_j \right) = n \sum_{i=1}^{n} g(x_i).
\]

J.-H. Bae and K.-W. Jun [2] proved the stability in Banach spaces of the equation (3). Recently, J.-H. Bae and W.-G. Park [3] proved the stability in Banach Modules over a \( C^*\)-algebra of the same equation. There are numerous results about various functional equations ([1, 4, 5, 6, 7]).

In this paper, we investigate the relation between (2) and (3). And we find out the general solution of (2) and (3).

2. Results

**Theorem 2.1.** Let \( f : X \times X \to Y \) be a mapping satisfying (2) and let \( g : X \to Y \) be the mapping given by

\[
g(x) := f(x, x)
\]

for all \( x \in X \). If

\[
f(x, y) = \frac{1}{4} [g(x + y) - g(x - y)]
\]

for all \( x, y \in X \), then \( g \) satisfies (3).

**Proof.** Letting \( x_1 = \cdots = x_n = y_1 = \cdots = y_n = 0 \) in (2) and then using (4), we have \( g(0) = 0 \). Putting \( y = x \) in (5) and then using (4), we get

\[
g(2x) = 4g(x)
\]

for all \( x \in X \). Setting \( y_1 = \cdots = y_n = 0 \) in (2), we have

\[
f \left( \sum_{i=1}^{n} x_i, 0 \right) = n \sum_{i=1}^{n} f(x_i, 0)
\]

for all \( x_1, \ldots, x_n \in X \). Taking \( x_2 = \cdots = x_n = 0 \) in the above equality, we get \( f(x_1, 0) = 0 \) for all \( x_1 \in X \). Similarly, \( f(0, y_1) = 0 \) for all \( y_1 \in X \). Letting \( x_1 = x, x_2 = y, x_3 = \cdots = x_n = 0 \) and \( y_1 = z, y_2 = w, y_3 = \cdots = y_n = 0 \) in (2), we have

\[
f(x + y, z + w) = f(x, z) + f(x, w) + f(y, z) + f(y, w)
\]

for all \( x, y, z, w \in X \). By (7) and (5), we obtain

\[
g(x + y + z + w) - g(x + y - z - w)
\]

\[
= g(x + z) - g(x - z) + g(x + w) - g(x - w)
\]

\[
+ g(y + z) - g(y - z) + g(y + w) - g(y - w)
\]

for all \( x, y, z, w \in X \). Putting \( x = y = z = 0 \) and then replacing \( w \) by \( x \) in (8), we see that

\[
g(-x) = g(x)
\]
Letting $Y$ be the mapping given by

\[ \text{Proof.} \]

that using (6) and (9), we see that for all $x, y \in X$, for all

\[ g(x + y) + g(x - y) = 2g(x) + 2g(y) \]

for all $x, y \in X$. By (2) and (5), we obtain

\[ g \left( \sum_{i=1}^{n} x_i + \sum_{j=1}^{n} y_j \right) - g \left( \sum_{i=1}^{n} x_i - \sum_{j=1}^{n} y_j \right) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ g(x_i + y_j) - g(x_i - y_j) \right] \]

for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$. Taking $y_1 = x_1, \ldots, y_n = x_n$ in (11) and then using (6) and (9), we see that

\[ 2g \left( \sum_{i=1}^{n} x_i \right) = 2 \sum_{i=1}^{n} g(x_i) + \sum_{1 \leq i < j \leq n} g(x_i + x_j) - \sum_{1 \leq i < j \leq n} g(x_i - x_j) \]

for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$. By (10) and the above equality, we obtain that

\[ 2g \left( \sum_{i=1}^{n} x_i \right) = 2 \sum_{i=1}^{n} g(x_i) + \sum_{1 \leq i < j \leq n} [2g(x_i) + 2g(x_j) - g(x_i - x_j)] - \sum_{1 \leq i < j \leq n} g(x_i - x_j) \]

and thus

\[ g \left( \sum_{i=1}^{n} x_i \right) + \sum_{1 \leq i < j \leq n} g(x_i - x_j) = \sum_{i=1}^{n} g(x_i) + \sum_{1 \leq i < j \leq n} [g(x_i) + g(x_j)] \]

for all $x_1, \ldots, x_n \in X$. Hence $g$ satisfies (3).

**Theorem 2.2.** Let $g : X \to Y$ be a mapping satisfying (3) and let $f : X \times X \to Y$ be the mapping given by (5) for all $x, y \in X$. Then $f$ satisfies (2) and (4).

**Proof.** Letting $x_1 = \cdots = x_n = 0$ in (3), we have $g(0) = 0$. Putting $x_1 = x$, $x_2 = y$ and $x_3 = \cdots = x_n = 0$ in (3), we obtain that $g$ satisfies (10) and so satisfies (6) and (9). Setting $y = x$ in (5) and then using (6), the equality (4) holds. By (3), we see that

\[ g \left[ \sum_{i=1}^{n} (x_i + y_i) \right] = n \sum_{i=1}^{n} g(x_i + y_i) - \sum_{1 \leq i < j \leq n} g[(x_i + y_i) - (x_j + y_j)] \]

for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$. By (10), we have that

\[ g(x + y) - g(x - y) = 2g(x + y) - g(x) - g(y) \]
for all \(x, y \in X\). By (10), (12) and (13),

\[
\begin{align*}
(14) \quad & g \left( \sum_{i=1}^{n} x_i + \sum_{j=1}^{n} y_j \right) - g \left( \sum_{i=1}^{n} x_i - \sum_{j=1}^{n} y_j \right) \\
& = g \left( \sum_{i=1}^{n} (x_i + y_i) \right) - g \left( \sum_{i=1}^{n} (x_i - y_i) \right) \\
& = n \sum_{i=1}^{n} [g(x_i + y_i) - g(x_i - y_i)] \\
& \quad - \sum_{1 \leq i < j \leq n} \left( g[(x_i + y_i) - (x_j + y_j)] - g[(x_i - y_i) - (x_j - y_j)] \right) \\
& = n \sum_{i=1}^{n} [g(x_i + y_i) - g(x_i - y_i)] \\
& \quad - \sum_{1 \leq i < j \leq n} \left( g[(x_i + y_i) - (x_j + y_j)] - g[(x_i + y_j) - (x_j + y_i)] \right) \\
& = 2n \sum_{i=1}^{n} [g(x_i + y_i) - g(x_i) - g(y_i)] \\
& \quad - \sum_{1 \leq i < j \leq n} \left[ \left( 2g(x_i + y_i) + 2g(x_j + y_j) - g[(x_i + y_i) + (x_j + y_j)] \right) \\
& \quad - \left( 2g(x_i + y_j) + 2g(x_j + y_i) - g[(x_i + y_j) + (x_j + y_i)] \right) \right] \\
& = 2n \sum_{i=1}^{n} [g(x_i + y_i) - g(x_i) - g(y_i)] \\
& \quad - 2 \sum_{1 \leq i < j \leq n} \left[ g(x_i + y_i) + g(x_j + y_j) - g(x_i + y_j) - g(x_j + y_i) \right]
\end{align*}
\]

for all \(x_1, \ldots, x_n, y_1, \ldots, y_n \in X\). Note that

\[
(15) \quad \sum_{1 \leq i < j \leq n} (a_i + a_j) = (n - 1) \sum_{i=1}^{n} a_i
\]

for all \(a_1, \ldots, a_n \in Y\). By (10), (13) and (15),

\[
(16) \quad 2n \sum_{i=1}^{n} [g(x_i + y_i) - g(x_i) - g(y_i)] \\
& \quad - 2 \sum_{1 \leq i < j \leq n} \left[ g(x_i + y_i) + g(x_j + y_j) - g(x_i + y_j) - g(x_j + y_i) \right]
\]
\[= 2n \sum_{i=1}^{n} [g(x_i + y_i) - g(x_i) - g(y_i)]
\]
\[+ 2 \sum_{1 \leq i < j \leq n} [g(x_i + y_j) + g(x_j + y_i)] - 2(n - 1) \sum_{i=1}^{n} g(x_i + y_i)
\]
\[= n \sum_{i=1}^{n} [g(x_i + y_i) - g(x_i - y_i)]
\]
\[+ 2 \sum_{1 \leq i < j \leq n} [g(x_i + y_j) + g(x_j + y_i)] - 2(n - 1) \sum_{i=1}^{n} g(x_i + y_i)
\]
\[= n \sum_{i=1}^{n} [g(x_i + y_i) - g(x_i - y_i)] + (n - 1) \sum_{i=1}^{n} [g(x_i + y_i) - g(x_i - y_i)]
\]
\[+ 2 \sum_{1 \leq i < j \leq n} [g(x_i + y_j) + g(x_j + y_i)] - 2(n - 1) \sum_{i=1}^{n} g(x_i + y_i)
\]
\[= n \sum_{i=1}^{n} [g(x_i + y_i) - g(x_i - y_i)] - (n - 1) \sum_{i=1}^{n} [g(x_i + y_i) + g(x_i - y_i)]
\]
\[+ 2 \sum_{1 \leq i < j \leq n} [g(x_i + y_j) + g(x_j + y_i)]
\]
\[= n \sum_{i=1}^{n} [g(x_i + y_i) - g(x_i - y_i)]
\]
\[+ 2 \sum_{1 \leq i < j \leq n} \left( [g(x_i + y_j) - g(x_i) - g(y_j)] + [g(x_j + y_i) - g(x_i) - g(y_j)] \right)
\]
\[= n \sum_{i=1}^{n} [g(x_i + y_i) - g(x_i - y_i)]
\]
\[+ \sum_{1 \leq i < j \leq n} \left( [g(x_i + y_j) - g(x_i - y_j)] + [g(x_j + y_i) - g(x_j - y_i)] \right)
\]
\[= \sum_{i=1}^{n} g(x_i + y_i) + \sum_{1 \leq i < j \leq n} g(x_i + y_j) + \sum_{1 \leq i < j \leq n} g(x_j + y_i)
\]
\[- \sum_{i=1}^{n} g(x_i - y_i) - \sum_{1 \leq i < j \leq n} g(x_i - y_j) - \sum_{1 \leq i < j \leq n} g(x_j - y_i)\]
\[
\begin{align*}
&= \sum_{i=1}^{n} \sum_{j=1}^{n} g(x_i + y_j) - \sum_{i=1}^{n} \sum_{j=1}^{n} g(x_i - y_j) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \left( g(x_i + y_j) - g(x_i - y_j) \right)
\end{align*}
\]
for all \(x_1, \ldots, x_n, y_1, \ldots, y_n \in X\). By (14) and (16), we obtain that \(g\) satisfies (11). By (5) and (11), we see that \(f\) satisfies (2). \(\square\)

Next we obtain the solutions of the equations (2) and (3).

**Theorem 2.3.** A mapping \(f : X \times X \to Y\) satisfies (1) if and only if it satisfies (2).

**Proof.** If \(f\) satisfies (1), then
\[
f \left( \sum_{i=1}^{n} x_i, \sum_{j=1}^{n} y_j \right) = \sum_{i=1}^{n} f(x_i, \sum_{j=1}^{n} y_j) = \sum_{i=1}^{n} \sum_{j=1}^{n} f(x_i, y_j)
\]
for all \(x_1, \ldots, x_n \in X\).

Conversely, assume that \(f\) satisfies (2). Choosing \(x_1 = \cdots = x_n = y_1 = \cdots = y_n = 0\) in (2), \(f(0, 0) = 0\). Letting \(x_1 = x\) and \(x_2 = \cdots = x_n = y_1 = \cdots = y_n = 0\) in (2), we have \(f(x, 0) = 0\) for all \(x \in X\). Putting \(x_1 = x, x_2 = y, y_1 = z\) and \(x_3 = \cdots = x_n = y_2 = \cdots = y_n = 0\) in (2), we get
\[
f(x + y, z) = f(x, z) + f(y, z)
\]
for all \(x, y, z \in X\). Setting \(y_1 = y\) and \(x_1 = \cdots = x_n = y_2 = \cdots = y_n = 0\) in (2), we obtain \(f(0, y) = 0\) for all \(y \in X\). Taking \(x_1 = x, y_1 = y, y_2 = z\) and \(x_2 = \cdots = x_n = y_3 = \cdots = y_n = 0\) in (2), we see that
\[
f(x, y + z) = f(x, y) + f(x, z)
\]
for all \(x, y, z \in X\). \(\square\)

**Theorem 2.4.** A function \(g : X \to Y\) satisfies (3) if and only if there exists a symmetric bi-additive function \(S : X \times X \to Y\) such that \(g(x) = S(x, x)\) for all \(x \in X\).

**Proof.** Define \(f : X \times X \to Y\) by (5) for all \(x, y \in X\). By Theorem 2.2, we obtain that \(f\) satisfies (2) and (4). Using Theorem 2.3, we see that \(f\) also satisfies (1). So \(f\) is bi-additive. Define \(S : X \times X \to Y\) by
\[
S(x, y) := \frac{1}{2} [f(x, y) + f(y, x)]
\]
for all \(x, y \in X\). Then \(S\) is symmetric and bi-additive. By (4), we obtain that \(g(x) = S(x, x)\) for all \(x \in X\).
Conversely, assume that there exists a symmetric bi-additive function \( S : X \times X \to Y \) such that \( g(x) = S(x, x) \) for all \( x \in X \). Note that

\[
\sum_{1 \leq i < j \leq n} (a_i + b_j) = \sum_{i=1}^{n-1} (n - i)a_i + \sum_{j=2}^{n} (j - 1)b_j
\]

for all \( a_1, \ldots, a_{n-1}, b_2, \ldots, b_n \in Y \). Thus

\[
g\left( \sum_{i=1}^{n} x_i \right) + \sum_{1 \leq i < j \leq n} g(x_i - x_j) = S \left( \sum_{i=1}^{n} x_i, \sum_{i=1}^{n} x_i \right) + \sum_{1 \leq i < j \leq n} \left[ S(x_i, x_i) - 2S(x_i, x_j) + S(x_j, x_j) \right]
\]

\[
= \sum_{i=1}^{n} S(x_i, x_i) + 2 \sum_{1 \leq i < j \leq n} S(x_i, x_j)
\]

\[
= \left[ \sum_{i=1}^{n-1} (n - i)S(x_i, x_i) - 2 \sum_{1 \leq i < j \leq n} S(x_i, x_j) + \sum_{j=2}^{n} (j - 1)S(x_j, x_j) \right]
\]

\[
= S(x_1, x_n) + \sum_{i=1}^{n-1} (1 + n - i)S(x_i, x_i) + \sum_{j=2}^{n} (j - 1)S(x_j, x_j)
\]

\[
= S(x_1, x_n) + \sum_{i=2}^{n-1} [(1 + n - i) + (i - 1)]S(x_i, x_i)
\]

\[
+ nS(x_1, x_1) + (n - 1)S(x_n, x_n)
\]

\[
= n \sum_{i=1}^{n} S(x_i, x_i) = n \sum_{i=1}^{n} g(x_i)
\]

for all \( x_1, \ldots, x_n \in X \).

Let \( Y \) be complete and \( \varphi : X \times X \times X \to [0, \infty) \) and \( \psi : X \times X \times X \to [0, \infty) \) be two functions satisfying

\[
\tilde{\varphi}(x, y, z) := \sum_{j=0}^{\infty} \left[ \frac{1}{2^{j+1}} \varphi(2^j x, 2^j y, z) + \frac{1}{2^j} \varphi(x, 2^j z) \right] < \infty
\]

and

\[
\tilde{\psi}(x, y, z) := \sum_{j=0}^{\infty} \left[ \frac{1}{2^{j+1}} \psi(2^j x, 2^j y, z) + \frac{1}{2^j} \psi(2^j x, y, z) \right] < \infty
\]

for all \( x, y, z \in X \).
Theorem 2.5. Let $f : X \times X \rightarrow Y$ be a mapping such that

\begin{align}
\|f(x + y, z) - f(x, z) - f(y, z)\| &\leq \varphi(x, y, z) \\
\|f(x, y + z) - f(x, y) - f(x, z)\| &\leq \psi(x, y, z)
\end{align}

for all $x, y, z \in X$, and let $f(x, 0) = 0$ and $f(0, y) = 0$ for all $x, y \in X$. Then there exist two bi-additive mappings $F_1, F_2 : X \times X \rightarrow Y$ such that

\begin{align}
\|f(x, y) - F_1(x, y)\| &\leq \tilde{\varphi}(x, x, y) \\
\|f(x, y) - F_2(x, y)\| &\leq \tilde{\psi}(x, x, y)
\end{align}

for all $x, y \in X$. The mappings $F_1, F_2 : X \times X \rightarrow Y$ are given by

$$
F_1(x, y) := \lim_{j \rightarrow \infty} \frac{1}{2^j} f(2^j x, y), \quad F_2(x, y) := \lim_{j \rightarrow \infty} \frac{1}{2^j} f(x, 2^j y)
$$

for all $x, y \in X$.

Proof. Letting $y = x$ in (19), we get

$$
\left\| f(x, z) - \frac{1}{2} f(2x, z) \right\| \leq \frac{1}{2} \varphi(x, x, z)
$$

for all $x, z \in X$. Thus

$$
\left\| \frac{1}{2^j} f(2^j x, z) - \frac{1}{2^{j+1}} f(2^{j+1} x, z) \right\| \leq \frac{1}{2^{j+1}} \varphi(2^j x, 2^j x, z)
$$

for all $x, z \in X$ and all $j$. Replacing $z$ by $y$, we have

$$
\left\| \frac{1}{2^j} f(2^j x, y) - \frac{1}{2^{j+1}} f(2^{j+1} x, y) \right\| \leq \frac{1}{2^{j+1}} \varphi(2^j x, 2^j x, y)
$$

for all $x, y \in X$ and all $j$. For given integers $l, m (0 \leq l < m)$, we obtain

$$
\left\| \frac{1}{2^j} f(2^j x, y) - \frac{1}{2^m} f(2^m x, y) \right\| \leq \sum_{j=l}^{m-1} \frac{1}{2^{j+1}} \varphi(2^j x, 2^j x, y)
$$

for all $x, y \in X$. By (17), the sequence $\left\{ \frac{1}{2^j} f(2^j x, y) \right\}$ is a Cauchy sequence for all $x, y \in X$. Since $Y$ is complete, the sequence $\left\{ \frac{1}{2^j} f(2^j x, y) \right\}$ converges for all $x, y \in X$. Define $F_1 : X \times X \rightarrow Y$ by

$$
F_1(x, y) := \lim_{j \rightarrow \infty} \frac{1}{2^j} f(2^j x, y)
$$

for all $x, y \in X$. Putting $l = 0$ and taking $m \rightarrow \infty$ in (24), one can obtain the inequality (21). By (19) and (20), we see that

$$
\left\| \frac{1}{2^j} f(2^j x + 2^j y, z) - \frac{1}{2^j} f(2^j x, z) - \frac{1}{2^j} f(2^j y, z) \right\| \leq \frac{1}{2^j} \varphi(2^j x, 2^j y, z)
$$

and

$$
\left\| \frac{1}{2^j} f(2^j x, y + z) - \frac{1}{2^j} f(2^j x, y) - \frac{1}{2^j} f(2^j x, z) \right\| \leq \frac{1}{2^j} \psi(2^j x, y, z)
$$
for all $x, y, z \in X$ and all $j$. Letting $j \to \infty$ in the above two inequalities and using (18), we obtain that $F_1$ is bi-additive.

Next, setting $y = z$ in (20), we get

$$
\left\| f(x, y) - \frac{1}{2} f(x, 2y) \right\| \leq \frac{1}{2} \psi(x, y, y)
$$

(25)

for all $x, y \in X$. By the same method as above, $F_2$ is bi-additive which satisfies (22), where $F_2(x, y) := \lim_{j \to \infty} \frac{1}{2^j} f(x, 2^j y)$ for all $x, y \in X$. \hfill \Box

References


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