OSCULATORY WFI-ALGEBRAS

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ABSTRACT. The notions of mote, beam and osculatory WFI-algebra are introduced, and several properties are investigated. Relations between osculatory WFI-algebra and associative WFI-algebra are provided. Characterizations of osculatory WFI-algebra are given.

1. Introduction

In 1990, W. M. Wu [7] introduced the notion of fuzzy implication algebra (FI-algebra, for short), and investigated several properties. In [6], Z. Li and C. Zheng introduced the notion of distributive (resp. regular, commutative) FI-algebra, and investigated the relations between such FI-algebra and MV-algebra. In [1], Y. B. Jun discussed several aspects of WFI-algebra. He introduced the notion of associative (resp. normal, medial) WFI-algebra, and investigated several properties. He gave conditions for a WFI-algebra to be associative/medial, and provided characterizations of associative/medial WFI-algebra, and showed that every associative WFI-algebra is a group in which every element is an involution. He also verified that the class of all medial WFI-algebras is a variety. Y. B. Jun and S. Z. Song [5] introduced the notions of simulative and/or mutant WFI-algebra and investigated some properties. They established characterizations of a simulative WFI-algebra, and gave a relation between an associative WFI-algebra and a simulative WFI-algebra. They also found some types for a simulative WFI-algebra to be mutant. Jun et al. [4] introduced the concept of ideals of WFI-algebra, and gave relations between a filter and an ideal. Moreover, they provided characterizations of an ideal, and established an extension property for an ideal. In [2] and [3], the present author discussed perfect (resp. weak and concrete) filters of WFI-algebra. In this paper, we introduce the notions of mote, beam and osculatory WFI-algebra, and investigate several properties. We give relations between osculatory WFI-algebra and associative WFI-algebra. We provide characterizations of osculatory WFI-algebra.

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41
2. Preliminaries

Let $K(\tau)$ be the class of all algebras of type $\tau = (2,0)$. By a WFI-algebra we mean a system $X := (X, \ominus, 1) \in K(\tau)$ in which the following axioms hold:

(a1) $(x \in X) \ (x \ominus x = 1)$,
(a2) $(x, y \in X) \ (x \ominus y = y \ominus x = 1 \Rightarrow x = y)$,
(a3) $(x, y, z \in X) \ (x \ominus (y \ominus z) = y \ominus (x \ominus z))$,
(a4) $(x, y, z \in X) \ (((x \ominus y) \ominus ((y \ominus z) \ominus (x \ominus z))) = 1)$.

We call the special element 1 the unit. For the convenience of notation, we shall write $[x, y_1, y_2, \ldots, y_n]$ for $(\cdots ((x \ominus y_1) \ominus y_2) \ominus \cdots ) \ominus y_n$.

We define $[x, y]^0 = x$, and for $n > 0, [x, y]^n = [x, y, y, \ldots, y]$, where $y$ occurs $n$-times. We use the notation $x^n \ominus y$ instead of $x \ominus (\cdots (x \ominus (x \ominus y)) \cdots )$ in which $x$ occurs $n$-times.

Proposition 2.1 ([1]). In a WFI-algebra $X$, the following are true:

(b1) $x \ominus [x, y]^2 = 1$,
(b2) $1 \ominus x = 1 \Rightarrow x = 1$,
(b3) $1 \ominus x = x$,
(b4) $x \ominus y = 1 \Rightarrow [y, z, x \ominus z] = 1 \& [z, x, z \ominus y] = 1$,
(b5) $[x, y, 1] = [x, 1, y \ominus 1]$,
(b6) $[x, y]^3 = x \ominus y$.

A nonempty subset $S$ of a WFI-algebra $X$ is called a subalgebra of $X$ if $x \ominus y \in S$ whenever $x, y \in S$. A nonempty subset $F$ of a WFI-algebra $X$ is called a filter of $X$ if it satisfies:

(c1) $1 \in F$,
(c2) $(\forall x \in F) (\forall y \in X) \ (x \ominus y \in F \Rightarrow y \in F)$.

A filter $F$ of a WFI-algebra $X$ is said to be closed [1] if $F$ is also a subalgebra of $X$.

Proposition 2.2 ([1]). Let $F$ be a filter of a WFI-algebra $X$. Then $F$ is closed if and only if $x \ominus 1 \in F$ for all $x \in F$.

Proposition 2.3 ([1]). In a finite WFI-algebra, every filter is closed.

We now define a relation “$\preceq$” on $X$ by $x \preceq y$ if and only if $x \ominus y = 1$. It is easy to verify that a WFI-algebra is a partially ordered set with respect to $\preceq$. A WFI-algebra $X$ is said to be associative [1] if it satisfies $[x, y, z] = x \ominus (y \ominus z)$ for all $x, y, z \in X$. For a WFI-algebra $X$, the set

$S(X) := \{ x \in X \mid x \preceq 1 \}$

is called the simulative part of $X$. A WFI-algebra $X$ is said to be simulative [5] if it satisfies

(S) $x \preceq 1 \Rightarrow x = 1$. 

Note that the condition (S) is equivalent to $S(X) = \{1\}$.

**Proposition 2.4 ([5]).** The simulative part $S(X)$ of a WFI-algebra $X$ is a filter of $X$.

### 3. Osculatory WFI-algebra

We begin with the following definition.

**Definition 3.1.** A WFI-algebra $X$ is said to be osculatory if it satisfies:

$$(\forall x, y \in X) (y \odot x = 1 \Rightarrow [x, y]^2 = x).$$

**Example 3.2.** Let $X = \{1, a, b, c\}$ be a set with the following Cayley table.

$$
\begin{array}{|c|cccc|}
\hline
\oplus & 1 & a & b & c \\
\hline
1 & 1 & a & b & c \\
a & 1 & 1 & c & c \\
b & c & c & 1 & 1 \\
c & c & b & a & 1 \\
\hline
\end{array}
$$

Then $X := (X, \ominus, 1)$ is an osculatory WFI-algebra.

Let $X$ be a WFI-algebra. Consider the following equation:

$$[x, y]^2 = [y, x, [y, x]^2].$$

**Example 3.3.** Let $X = \{1, a, b\}$ be a set with the following Cayley table.

$$
\begin{array}{|c|cc|}
\hline
\ominus & 1 & a & b \\
\hline
1 & 1 & a & b \\
a & 1 & 1 & b \\
b & b & b & 1 \\
\hline
\end{array}
$$

Then $X := (X, \ominus, 1)$ is a WFI-algebra which satisfies the equation (3.2). But $X$ is not associative since $[a, a]^2 = a \neq 1 = a^2 \odot a$.

**Example 3.4.** Let $X = \{1, a, b\}$ be a set with the following Cayley table.

$$
\begin{array}{|c|cc|}
\hline
\ominus & 1 & a & b \\
\hline
1 & 1 & a & b \\
a & b & 1 & a \\
b & a & b & 1 \\
\hline
\end{array}
$$

Then $X := (X, \ominus, 1)$ is a WFI-algebra which does not satisfy the equation (3.2) since $[a, b]^2 = a \neq 1 = [b, a, [b, a]^2]$.

**Lemma 3.5 ([1]).** Let $X$ be a WFI-algebra. Then the following are equivalent.

(i) $X$ is associative.

(ii) $(\forall x \in X) (x \odot 1 = x)$.

(iii) $(\forall x, y \in X) (x \odot y = y \odot x)$.

**Proposition 3.6.** Every associative WFI-algebra satisfies the equation (3.2).
Proof. Let $X$ be an associative WFI-algebra and let $x, y \in X$. Using the associativity of $X$, (a1), (b3) and Lemma 3.5, we have
\[
[y, x, [y, x]^2] = [y, x, y \odot x, x] = 1 \odot x = x
= x \odot 1 = x \odot (y \odot y) = [x, y]^2.
\]
This completes the proof. □

**Proposition 3.7.** If a WFI-algebra $X$ satisfies the equation (3.2), then $X$ is osculatory.

**Proof.** Let $x, y \in X$ be such that $y \odot x = 1$. Using (b3), it is straightforward. □

**Corollary 3.8.** Every associative WFI-algebra is osculatory.

**Definition 3.9.** If an element $m$ of a WFI-algebra $X$ is maximal in $(X, \preceq)$, we say that $m$ is a mote of $X$.

Denote by $M(X)$ the set of all motes of $X$. For any $m \in M(X)$, the set
\[
B(m) := \{x \in X \mid x \odot m = 1\}
\]
is called a beam of $X$ with respect to $m$ (briefly, $m$-beam of $X$). Obviously, $1 \in M(X)$ and $B(1) = S(X)$.

**Example 3.10.** Let $X = \{1, a, b, c, d\}$ be a set with the following Cayley table.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
</tr>
<tr>
<td>a</td>
<td>1</td>
<td>1</td>
<td>b</td>
<td>c</td>
<td>d</td>
</tr>
<tr>
<td>b</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>c</td>
<td>c</td>
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<td>c</td>
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<td>c</td>
<td>d</td>
<td>1</td>
<td>b</td>
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<td>d</td>
<td>c</td>
<td>c</td>
<td>c</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Then $X := (X, \odot, 1)$ is a WFI-algebra and $M(X) = \{1, c\}$. Hence $B(1) = \{1, a, b\}$ and $B(c) = \{c, d\}$.

**Proposition 3.11.** Every mote $m$ of a WFI-algebra $X$ is represented by the following identity:
\[
(\forall x \in X) (m = [m, x]^2).
\]

**Proof.** Since $m \odot [m, x]^2 = 1$ for all $x \in X$, we have $m = [m, x]^2$ for all $x \in X$. □

We give a condition for an element of $X$ to be a mote of $X$.

**Theorem 3.12.** If an element $m$ of a WFI-algebra $X$ satisfies the following equation:
\[
(\forall x, y \in X) (x \odot m = [x, m, y, y])
\]
then $m$ is a mote of $X$. 
Proof. Let \( x \in X \) be such that \( m \odot x = 1 \). Then
\[
m = 1 \odot m = [1, m, x, x] = [m, x]^2 = 1 \odot x = x,
\]
and so \( m \) is a mote of \( X \). \( \square \)

The following is more simple condition for an element to be a mote.

**Theorem 3.13.** If an element \( m \) of a WFI-algebra \( X \) satisfies the following identity:
\[
[m, 1]^2 = m,
\]
then \( m \) is a mote of \( X \).

**Proof.** Using (a3), (b1), (b4), (b5) and (3.5), we have
\[
[m, x, 1] = [m, 1, x \odot 1] = x \odot [m, 1]^2 = x \odot m
\]
for all \( x \in X \), and
\[
[x, m, 1, 1] = [x, 1, m \odot 1] \odot 1 = [x, 1]^2 \odot [m, 1]^2 = [x, 1]^2 \odot m \preceq x \odot m.
\]
for all \( x \in X \). Using (b1), we have \( x \odot m \preceq [x, m, 1, 1] \).

\[
x \odot m = [x, m, 1, 1],
\]
and so
\[
[x, m, y, y] \preceq [y, 1, [x, m, y, 1]] = [y, [x, m, y], 1] = [x, m, y \odot y, 1] = [x, m, 1, 1] = x \odot m
\]
for all \( x, y \in X \). Combining (a2), (b1) and (3.9), we get
\[
x \odot m = [x, m, y, y]
\]
for all \( x, y \in X \). It follows from Theorem 3.12 that \( m \) is a mote of \( X \). \( \square \)

In the following theorem, we show that every mote \( m \) of a WFI-algebra \( X \) satisfies the condition (3.5).

**Theorem 3.14.** Every mote \( m \) of a WFI-algebra \( X \) satisfies the condition (3.5).

**Proof.** Let \( m \) be a mote of a WFI-algebra \( X \). Using (a3) and (3.3), we obtain
\[
[m, x, y \odot x] = y \odot [m, x]^2 = y \odot m
\]
for all \( x, y \in X \). Using (b1), (b4) and (3.11), we have
\[
[m, x, z] \preceq [m, x, [z, x]^2] = [z, x, m].
\]
It follows from (a3) and (b4) that
\[
[m, x, y \odot z] = y \odot [m, x, z] \preceq y \odot [z, x, m] = [z, x, y \odot m].
\]
Using (a1), (b5) and (3.13), we get
\[
[m, 1, x \odot 1] = [m, x, 1] = [m, x, 1 \odot 1] \preceq [1, x, 1 \odot m] = x \odot m.
\]
Obviously, \( x \ominus m \preceq [m, 1, x \ominus 1] \), and hence
\[
(3.14) \quad [m, 1, x \ominus 1] = x \ominus m.
\]
This implies that \([m, 1]^2 = [m, 1, 1 \ominus 1] = 1 \ominus m = m\). This completes the proof. \( \Box \)

**Corollary 3.15.** If \( m \) is a mote of a WFI-algebra \( \mathfrak{X} \), then
\[
(3.15) \quad (\forall x \in X) ([x \ominus m, 1]^2 = [x, 1]^2 \ominus m).
\]
*Proof.* Using (b5) and Theorem 3.14, we have
\[
[x \ominus m, 1]^2 = [x, 1]^2 \ominus [m, 1]^2 = [x, 1]^2 \ominus m.
\]
This completes the proof. \( \Box \)

**Corollary 3.16.** If \( p \) and \( q \) are motes of a WFI-algebra \( \mathfrak{X} \), then so is \( p \ominus q \).
*Proof.* Let \( p \) and \( q \) be motes of a WFI-algebra \( \mathfrak{X} \). Then \([p, 1]^2 = p \) and \([q, 1]^2 = q\). Hence
\[
[p \ominus q, 1]^2 = [p, 1]^2 \ominus [q, 1]^2 = p \ominus q,
\]
and so \( p \ominus q \) is a mote of \( \mathfrak{X} \) by Theorem 3.13. \( \Box \)

**Proposition 3.17.** For any element \( x \) of a WFI-algebra \( \mathfrak{X} \), the element \([x, 1]^2\)
is a mote of \( \mathfrak{X} \).
*Proof.* Let \( x \in X \) and \( m = [x, 1]^2 \). Then
\[
[m, 1]^2 = [x, 1]^2 \ominus 1 = [x, 1]^2 \ominus m.
\]
It follows from Theorem 3.13 that \([x, 1]^2\) is a mote of \( \mathfrak{X} \). \( \Box \)

**Theorem 3.18.** A WFI-algebra \( \mathfrak{X} \) is osculatory if and only if it satisfies the following identity:
\[
(3.16) \quad (\forall x, y \in X) ([x, y]^2, x]^2 = [x, y]^2).
\]
*Proof.* Assume that a WFI-algebra \( \mathfrak{X} \) is osculatory. Since \( x \ominus [x, y]^2 = 1 \), it follows from (3.1) that
\[
[x, y]^2 = [[x, y]^2, x]^2
\]
which proves (3.16). Now let \( \mathfrak{X} \) be a WFI-algebra in which the identity (3.16) is valid. Let \( x, y \in X \) be such that \( y \ominus x = 1 \). Using (b3) and (3.16), we have
\[
[x, y]^2 = [1 \ominus x, y]^2 = [[y, x]^2, y]^2 = [y, x]^2 = 1 \ominus x = x.
\]
Hence \( \mathfrak{X} \) is osculatory. \( \Box \)

**Lemma 3.19.** For any motes \( p \) and \( q \) of \( \mathfrak{X} \), we have
\begin{enumerate}
  \item [(i)] \( (\forall x, y \in X) \ (x \in B(p) \ & y \in B(q) \Rightarrow x \ominus y \in B(p \ominus q)) \).
  \item [(ii)] \( (\forall x, y \in B(p)) \ ([x, y, 1] = 1) \).
\end{enumerate}
Proof. (i) Let \( x \in B(p) \) and \( y \in B(q) \). Then \( x \uplus p = 1 \) and \( y \ominus q = 1 \). Hence
\[
[x, y, p \ominus q] = [x, y, [p \ominus q, 1]^2] = [p, q, 1] \ominus [x, y, 1]
\]
\[
= [p, q, 1] \ominus [x, 1, y \ominus 1] = [x, 1, y \ominus [p \ominus q, 1]^2]
\]
\[
= [x, 1, y \ominus (p \ominus q)] = [x, 1, p \ominus (y \ominus q)]
\]
\[
= [x, 1, p \ominus 1] = [x, p, 1] = 1 \ominus 1 = 1,
\]
and so \( x \ominus y \in B(p \ominus q) \).

(ii) It is direct consequence of (i).

Proposition 3.20. Let \( X \) be an osculatory WFI-algebra. Then
\[
(3.17) \quad (\forall m \in M(X)) (\forall x, y \in X) (x, y \in B(m) \Rightarrow [x, y]^2 = [y, x]^2).
\]

Proof. Let \( m \in M(X) \) and \( x, y \in B(m) \). Then \( [x, y, 1] = 1 \) by Lemma 3.19(ii).
Using Theorem 3.18, (a3) and (b6), we have
\[
[y, x]^2 \ominus [x, y]^2 = [y, x]^2 \ominus [[x, y]^2, x]^2 = [[x, y]^2, x, [y, x]^3]
\]
\[
= [[x, y]^2, x, y \ominus x] = y \ominus [[x, y]^2, x^2]
\]
\[
= y \ominus [x, y]^2 = [x, y, y \ominus y]
\]
\[
= [x, y, 1] = 1.
\]
Similarly, \( [x, y]^2 \ominus [y, x]^2 = 1 \). This completes the proof.

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