HOLOMORPHIC FUNCTIONS ON THE MIXED NORM SPACES ON THE POLYDISC

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Abstract. We generalize several integral inequalities for analytic functions on the open unit polydisc \( U_n = \{ z \in \mathbb{C}^n \mid |z_j| < 1, \ j = 1, \ldots, n \} \). It is shown that if a holomorphic function on \( U_n \) belongs to the mixed norm space \( A_{p,q}^{\vec{\omega}}(U_n) \), where \( \omega_j(\cdot), \ j = 1, \ldots, n, \) are admissible weights, then all weighted derivations of order \(|k|\) (with positive orders of derivations) belong to a related mixed norm space. The converse of the result is proved when, \( p, q \in [1, \infty) \) and when the order is equal to one. The equivalence of these conditions is given for all \( p, q \in (0, \infty) \) if \( \omega_j(z_j) = (1 - |z_j|^2)^{\alpha_j}, \ \alpha_j > -1, \ j = 1, \ldots, n \) (the classical weights.) The main results here improve our results in Z. Anal. Anwendungen 23 (3) (2004), no. 3, 577–587 and Z. Anal. Anwendungen 23 (2004), no. 4, 775–782.

1. Introduction

Let \( U^1 = U \) be the unit disk in the complex plane \( \mathbb{C} \), \( dm(\cdot) = \frac{1}{\pi} r dr d\theta \) the normalized area measure on \( U \), \( D(a, r_0) \) the disk in \( \mathbb{C} \) centered at \( a \) with radius \( r_0 \), \( U^n \) the unit polydisc in the complex vector space \( \mathbb{C}^n \), \( r, \rho, \delta \in (0, \infty)^n \) and \( \alpha \in (-1, \infty)^n \). If we write \( 0 \leq r < 1 \), where \( r = (r_1, \ldots, r_n) \) it means \( 0 \leq r_j < 1 \) for \( j = 1, \ldots, n \), and \( r + 2 \) stands for \( (r_1 + 2, \ldots, r_n + 2) \). For \( z, w \in \mathbb{C}^n \) we write \( z \cdot w = (z_1 w_1, \ldots, z_n w_n) \); \( e^{i\theta} \) is an abbreviation for \( (e^{i\theta_1}, \ldots, e^{i\theta_n}) \); \( dt = dt_1 \cdots dt_n; \ d\theta = d\theta_1 \cdots d\theta_n \). Let \( \gamma = (\gamma_1, \ldots, \gamma_n) \) be a multi-index, \( \gamma_k \) being nonnegative integers, we write

\[
|\gamma| = \gamma_1 + \cdots + \gamma_n, \quad \gamma! = \gamma_1! \cdots \gamma_n!, \quad z^\gamma = z_1^{\gamma_1} \cdots z_n^{\gamma_n}.
\]

For a holomorphic function \( f \) we denote

\[
D^\gamma f = \frac{\partial^{|\gamma|} f}{\partial z_1^{\gamma_1} \cdots \partial z_n^{\gamma_n}}.
\]

Let

\[
P^n(w, r) = \{ z \in \mathbb{C}^n \mid |z_j - w_j| < r_j, j = 1, \ldots, n \}
\]

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be a polydisc in $\mathbb{C}^n$ and let $H(P^n(w, r))$ be the class of all holomorphic functions $f$ defined on $P^n(w, r)$.

For $f \in H(U^n)$ and $p \in (0, \infty)$ we usually write

$$M_p(f, r) = \left( \frac{1}{(2\pi)^n} \int_{|z|<1} |f(r \cdot e^{i\theta})|^p \, d\theta \right)^{1/p},$$

$0 \leq r < 1$, for the integral means of $f$.

Let $\omega(s), 0 \leq s < 1, be a weight function which is positive and integrable on $(0, 1)$. We extend $\omega$ on $U$ by setting $\omega(z) = \omega(|z|)$. We may assume that our weights are normalized so that $\int_0^1 \omega(s) \, ds = 1$.

Let $L^p_\omega = L^p_\omega(U^n)$ denotes the class of all measurable functions defined on $U^n$ such that

$$(1) \quad ||f||_{L^p_\omega}^p = \int_{U^n} |f(z)|^p \prod_{j=1}^n \omega_j(z_j) \, dm(z_j) < \infty,$$

where $\omega_j(z_j), j = 1, \ldots, n, are admissible weights (see, Definition 1) on the unit disk $U$. The weighted Bergman space $A^p_\omega = A^p_\omega(U^n)$ is the intersection of $L^p_\omega$ and $H(U^n)$. For $\omega_j(z_j) = (\alpha_j + 1)(1 - |z_j|^2)^{\alpha_j}, \alpha_j > -1, j = 1, \ldots, n$, we obtain the classical Bergman space $A^p_\omega$, see [1, p.33] and Lebesgue space $L^p_\omega$.

Let $L^{p,q}_{\omega} = L^{p,q}_{\omega}(U^n)$ denotes the class of all measurable functions defined on $U^n$ such that

$$(2) \quad ||f||_{L^{p,q}_{\omega}}^q = \int_{(0,1)^n} M_q^p(f, r) \prod_{j=1}^n \omega_j(r_j) \, dr_j < \infty,$$

and $A^{p,q}_{\omega} = A^{p,q}_{\omega}(U^n)$ be the intersection of $L^{p,q}_{\omega}$ and $H(U^n)$. When $p = q$ we denote $A^{p,q}_{\omega}$ by $A^\omega_{p,q}$. This space is called the mixed norm space. If $\omega_j(z_j) = (\alpha_j + 1)(1 - |z_j|^2)^{\alpha_j}, \alpha_j > -1, j = 1, \ldots, n$, then the space will be denoted by $A^{p,q}_{\omega}(U^n)$ (the classical mixed norm space).

Using polar coordinates and by some elementary calculations it is easy to see that in the case $p = q$, norms (1) and (2) are equivalent on the space $H(U^n)$.

Recently there is a huge interest in studying the weighted Bergman spaces of analytic functions of one variable see, for example, [4, 5, 6, 7, 8, 15, 16, 24], and the weighted Bergman spaces of analytic and harmonic functions on the unit ball $B \subset \mathbb{C}^n$ see, for example, in [1, 3, 9, 10, 12, 13, 14, 22, 23] (see, also the references therein).

In [1] and [18] the authors proved the following theorem.

**Theorem A.** Let $p \in (0, \infty), \alpha = (\alpha_1, \ldots, \alpha_n), with \alpha_j > -1 for j = 1, \ldots, n, m be a fixed positive integer and let $k = (k_1, \ldots, k_n) \in (\mathbb{Z}_+)^n$. Let $f \in H(U^n)$, then $f \in A^\alpha_{p}(U^n)$ if and only if

$$I_k = \left[ \prod_{j=1}^n (1 - |z_j|^2)^{k_j} \frac{\partial^{|k|} f}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}}(z) \right] \in L^p_\alpha, \quad for \ every \ k, \ |k| = m.$$
Moreover,
\[ \|f\|_{A_p^\alpha} \approx \sum_{|k|=0}^{m-1} \|D^k f(0)\| + \sum_{|k|=m} \|I_k\|_{L_p^\alpha}. \]

The expression \( A \asymp B \) means that there are finite positive constants \( C \) and \( C' \) such that \( CA \leq B \leq C'A \).

In the proof of Theorem A, when \( p \in [1, \infty) \), G. Benke and D. C. Chang used the weighted Bergman projection \( B_\alpha : L_2^\alpha \rightarrow A_2^\alpha \), which can be extended as a bounded operator from \( L_p^\alpha \) onto \( A_p^\alpha \). Case \( p \in (0, 1] \) was considered by a quite different method in [18] by the author of this paper. Closely related results on the unit disc and the unit ball in \( \mathbb{C}^n \) or \( \mathbb{R}^n \) can be found in [1, 2, 4, 5, 12, 14, 15, 16, 17, 21, 22, 24].

Motivated by paper [22], in [19] we proved the following result:

**Theorem B.** Let \( p \in (0, \infty) \), \( \alpha = (\alpha_1, \ldots, \alpha_n) \), with \( \alpha_j > -1 \) for \( j = 1, \ldots, n \), and \( f \in H(U^n) \). Then \( f \in A_p^\alpha(U^n) \) if and only if the functions
\[ T_S f = \prod_{j \in S} (1 - |z_j|^2) \prod_{j \in S} \frac{\partial |S|}{\partial z_j} (\chi_S(1)z_1, \chi_S(2)z_2, \ldots, \chi_S(n)z_n), \]
belong to the space \( L_p^\alpha(U^n) \), for every \( S \subseteq \{1, 2, \ldots, n\} \), where \( \chi_S(\cdot) \) is the characteristic function of \( S \), \( |S| \) is the cardinal number of \( S \), and \( \prod_{j \in S} \partial z_j = \partial z_{j_1} \cdots \partial z_{j_2} \), where \( j_k \in S, k = 1, \ldots, |S| \).

Moreover, \( \| \cdot \|_{A_p^\alpha} \) and the following norm
\[ \|f\|_* = |f(\vec{0})| + \sum_{S \subseteq \{1, \ldots, n\}, S \neq \emptyset} \|T_S f\|_{L_p^\alpha}, \]
\( \| \cdot \|_* \) are equivalent on \( A_p^\alpha(U^n) \).

From now on \( \|f\|_* \) will denote the following quantity
\[ |f(\vec{0})| + \sum_{S \subseteq \{1, \ldots, n\}, S \neq \emptyset} \|T_S f\|_{L_p^\alpha}. \]

Note that Theorems A and B are both characterizations for a function \( f \) to belong to \( A_p^\alpha(U^n) \). The main purpose of this paper is to generalize Theorems A and B in the case of the mixed norm space.

For a given weight \( \omega \) the function
\[ \psi(r) = \psi_\omega(r) \triangleq \frac{1}{\omega(r)} \int_r^1 \omega(u)du, \quad 0 \leq r < 1, \]
is called the distortion function of \( \omega \). We put \( \psi(z) = \psi(|z|) \) for \( z \in B \).

**Definition 1 ([15]).** We say that a weight \( \omega \) is admissible if it satisfies the following conditions:
(a) There is a positive constant $A = A(\omega)$ such that
\[ \omega(r) \geq \frac{A}{1-r} \int_r^1 \omega(u) \, du \quad \text{for} \quad 0 \leq r < 1; \]

(b) $\omega$ is differentiable and there is a positive constant $B = B(\omega)$ such that
\[ \omega'(r) \leq \frac{B}{1-r} \omega(r) \quad \text{for} \quad 0 \leq r < 1; \]

(c) For each sufficiently small positive $\delta$ there is a positive constant $C = C(\delta, \omega)$ such that
\[ \sup_{0 \leq r < 1} \frac{\omega(r)}{\omega(r + \delta \psi(r))} \leq C. \]

Observe that (a) implies $\Delta \psi(r) \leq 1 - r$ thus for sufficiently small positive $\delta$ we have $r + \delta \psi(r) < 1$ and the quantity in the denominator of the fraction in (c) is well defined. It is easy to see that the classical weight $\omega(r) = (1 - r^2)^{\alpha}$, $\alpha > -1$ is admissible. Some other examples of admissible weights can be found in [15, pp.660–663].

In this paper we prove the following results.

**Theorem 1.** Let $k = (k_1, \ldots, k_n) \in (\mathbb{Z}_+)^n$, $f$ be a holomorphic function defined on $U^n$ in $\mathbb{C}^n$ and $\omega_j(z_j)$, $j = 1, \ldots, n$ are admissible weights on the unit disk $U$, with distortion functions $\psi_j(z_j)$, $j = 1, \ldots, n$.

(a) If $f \in A_{p,q}^n(U^n)$ with $p, q > 0$, then
\[ I_{k,\omega} = \left[ \prod_{j=1}^n \psi_j^{k_j}(z_j) \right] \frac{\partial |f|}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}}(z) \in L_{p,q}^n(U^n). \]

Moreover, let $m$ be a fixed positive integer. Then there is a positive constant $C = C(p, q, \omega_j, n)$ such that
\[ \| f \|_{A_{p,q}^n} \geq C \left( \sum_{|k| = 0}^{m-1} \| D^k f(\bar{0}) \| + \sum_{|k| = m} \| I_{k,\omega} \|_{L_{p,q}^n} \right). \]

(b) If $p, q \in [1, \infty)$ and for all $j = 1, \ldots, n$, $\psi_j(z_j) \frac{\partial f}{\partial z_j}(z) \in L_{p,q}^n$, then $f \in A_{p,q}^n$ and there is a positive constant $C = C(p, q, \omega_j, n)$ such that
\[ \| f \|_{A_{p,q}^n} \leq C \left( |f(\bar{0})| + \sum_{j=1}^n \left\| \psi_j \frac{\partial f}{\partial z_j} \right\|_{L_{p,q}^n} \right). \]

Theorem 1 (b) was proved in [20] so that we give here only a sketch of the proof for the benefit of the reader. It is an open problem whether Theorem 1 (b) holds if $p$ or $q$ belong to the interval $(0, 1]$. A partial answer to the question gives the following main result of this paper, which concerns the classical weight case.
Theorem 2. Let \( p, q \in (0, \infty) \), \( \alpha = (\alpha_1, \ldots, \alpha_n) \), with \( \alpha_j > -1 \) for \( j = 1, \ldots, n \), \( m \) be a fixed positive integer and let \( k = (k_1, \ldots, k_n) \in (\mathbb{Z}_+)^n \). Let \( f \in H(U^n) \), then the following conditions are equivalent

\[(a) \quad f \in A^{p,q}_\alpha(U^n); \]

\[(b) \quad I_k = \prod_{j=1}^n (1 - |z_j|^2)^{k_j} \frac{\partial^{|k|} f}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}}(z) \in \mathcal{L}^{p,q}_\alpha \quad \text{for all } k, \ |k| = m; \]

\[(c) \quad \text{The functions} \]

\[T_S f = \prod_{j \in S} (1 - |z_j|^2) \frac{\partial^{|S|} f}{\partial z_j^{S}} \chi_S(1)z_1, \chi_S(2)z_2, \ldots, \chi_S(n)z_n, \]

\[\text{for every } S \subseteq \{1, 2, \ldots, n\}, \text{ are in } \mathcal{L}^{p,q}_\alpha(U^n). \]

Moreover,

\[||f||_{A^{p,q}_\alpha} \leq \sum_{|k|=0}^{m-1} |D^k f(\vec{0})| + \sum_{|k|=m} \|I_k\|_{\mathcal{L}^{p,q}_\alpha} ||f||_{\ast}. \]

We would like to point out that Theorem 2 cannot be easily obtained from the results in our papers [18] and [19].

The organization of the paper is as follows: In Section 2 we prove several auxiliary results, which we use in the proofs of the main results. The main results of the paper, i.e., Theorems 1 and 2 are proved in Section 3.

We have to say that throughout the rest of the paper \( C \) will denote a constant not necessarily the same at each occurrence.

2. Auxiliary results

In this section we prove several auxiliary results which we use in proving Theorems 1 and 2 in the subsequent section.

Lemma 1 ([18, p.579]). Let \( f \in H(U^n) \), \( \gamma \) be a multi-index and \( p > 0 \). Then

\[|D^\gamma f(w)|^p \leq \frac{C}{r^{n\gamma}} \prod_{j=1}^n r_j \int_{P^n(w,r)} |f|^p \prod_{j=1}^n dm(z_j), \]

whenever \( P^n(w,r) \subset U^n \), where \( C \) is a constant depending only on \( p, \gamma \) and \( n \).

Lemma 2. Let \( \beta \) be a multi-index and \( a \in U^n \). Then the point evaluations \( \Lambda_{a,\beta}(f) = D^\beta f(a) \) are bounded linear functionals on \( A^{p,q}_\alpha(U^n) \) for all \( p, q \in (0, \infty) \).
Proof. Choose $P^n(a, \delta) \subset U^n$. Let $m = \min_{z \in P^n(a, \delta)} \prod_{j=1}^n \omega_j(z_j) > 0$ and $d = \max_{j \in \{1, \ldots, n\}} ([|a_j| + \delta_j])$. Note that $d < 1$. By Lemma 1, using polar coordinates and the monotonicity of the integral means $M_p(f, r)$, we have

$$|D^\beta f(a)|^p \leq \frac{C}{\delta^{\beta p}} \prod_{j=1}^n \int_{P^n(a, \delta)} |f(z)|^p \prod_{j=1}^n dm(z_j)$$

$$\leq \frac{C}{m\delta^{\beta p + 2}} \int_{P^n(0, d)} |f(z)|^p \prod_{j=1}^n \omega_j(z_j) dm(z_j)$$

(6) $$\leq \frac{C2^n}{m\delta^{\beta p + 2}} \int_0^d \frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} |f(r \cdot e^{i\theta})|^p d\theta \prod_{j=1}^n \omega_j(r_j) r_j dr_j$$

$$\leq \frac{C}{m\delta^{\beta p + 2}} M_p(f, d) \int_0^n \prod_{j=1}^n \omega_j(r_j) r_j dr_j$$

for $r \in [d, 1)$.

Raising (6) to the $q/p$th power, then multiplying obtained inequality by $\prod_{j=1}^n \omega(r_j)$ and integrating over $r \in [d, 1)^n$, we obtain

$$|D^\beta f(a)|^q \int_{[d, 1)^n} \prod_{j=1}^n \omega(r_j) dr_j \leq C \int_{[d, 1)^n} M_p(f, r) \prod_{j=1}^n \omega(r_j) dr_j,$$

from which the result follows.

Lemma 3. Let $f \in H(U^n)$, and

$$M_p^n(f, r_k) = \frac{1}{2\pi} \int_0^{2\pi} |f(\ldots, r_k e^{i\theta_k}, \ldots)|^p d\theta_k.$$

Then there are positive constants $C_1$ and $C_2$ independent of $f$, $z_j$, $j \neq k$, $\rho_k$ and $r_k$, such that

(a) If $p \in (0, 1]$, then

$$M_p^n(f, \rho_k) - M_p^n(f, r_k) \leq C_1 (\rho_k - r_k)^p M_p^n\left(\frac{\partial f}{\partial z_k}, \rho_k\right).$$

(b) If $p \geq 1$, then

$$M_p(f, \rho_k) - M_p(f, r_k) \leq C_2 (\rho_k - r_k) M_p\left(\frac{\partial f}{\partial z_k}, \rho_k\right).$$

Proof. Let $l = \min\{1, p\}$. Using Minkowski’s inequality in the case $p \geq 1$ or the following elementary inequality $(x + y)^p \leq x^p + y^p$, $x, y \geq 0$, when $p \in (0, 1]$,
we have

\[ M_p^j(f, \rho_k) - M_p^j(f, r_k) \]
\[ \leq \left( \frac{1}{2\pi} \int_0^{2\pi} |f(\ldots, r_ke^{i\theta_k}, \ldots) - f(\ldots, r_ke^{i\theta_k}, \ldots)|^p d\theta_k \right)^{1/p} \]
\[ \leq \frac{(\rho_k - r_k)^l}{2\pi} \left( \int_0^{2\pi} \sup_{r < t < \rho_k} |\frac{\partial f}{\partial z_k}(\ldots, t e^{i\theta_k}, \ldots)|^p d\theta_k \right)^{1/p} \]
\[ \leq C(\rho_k - r_k)^l M_p^l \left( \frac{\partial f}{\partial z_k}, \rho_k \right). \]

In the last inequality we have used Hardy-Littlewood maximal theorem, see, for example, \cite[Theorem 1.9]{4}.

**Lemma 4.** Let \( f \in H(U^n) \), \( p, q \in (0, \infty) \) and \( \beta_{k,j} \in \mathbb{R}, k, j = 1, \ldots, n \). Then there is a constant \( C = C(p, q, \beta_{k,j}, n) \) such that

\[ \max_{z \in D^n(0,1/2)} |f(z)|^q \leq C \left( |f(0)|^q + \sum_{k=1}^n \int_0^1 M_p^q \left( \frac{\partial f}{\partial z_k}, r \right) \prod_{j=1}^n (1 - r_2^2)^{\beta_{k,j}} dr \right), \]

for all \( k = 1, \ldots, n \).

**Proof.** Without loss of generality we may assume that \( n = 2 \). The case \( n \geq 3 \) is only technically complicated. Since

\[ f(z_1, z_2) - f(0, 0) = \int_0^1 \frac{d}{dt} f(tz_1, z_2) dt + \int_0^1 \frac{d}{dt} f(0, tz_2) dt, \]

by some well-known inequalities we obtain

\[ |f(z_1, z_2)|^p \leq c_p \left( |f(0, 0)|^p + \max_{|\zeta_1| \leq 1/2} \left| \frac{\partial f}{\partial z_1}(\zeta_1, z_2) \right|^p + \max_{|\zeta_2| \leq 1/2} \left| \frac{\partial f}{\partial z_2}(0, \zeta_2) \right|^p \right), \]

for all \( z_1, z_2 \in D(0,1/2) \), where \( c_p = 1 \) for \( 0 < p < 1 \) and \( c_p = 3^{p-1} \) for \( p \geq 1 \).

On the other hand, from (5), (7), by polar coordinates and the monotonicity of \( M_p^q \left( \frac{\partial f}{\partial z_k}, r_1, r_2 \right) \), we obtain

\[ |f(z_1, z_2)|^p \leq C \left( |f(0, 0)|^p + \sum_{k=1}^2 M_p^q \left( \frac{\partial f}{\partial z_k}, r_1, r_2 \right) \right) \]

for all \( z_1, z_2 \in D(0,1/2) \), and \( r_1, r_2 \in [3/4, 1] \).

Let \( m_j = \min_{r \in [0,7/8]} \prod_{j=1}^2 (1 - r_j^2)^{\beta_{k,j}}, j = 1, 2 \). Raising (8) to the \( q/p \)th power and by some simple calculation, it follows that

\[ |f(z_1, z_2)|^q \leq C \left( |f(0, 0)|^q + \sum_{k=1}^2 \frac{\prod_{j=1}^2 (1 - r_j^2)^{\beta_{k,j}}}{m_j} M_p^q \left( \frac{\partial f}{\partial z_k}, r_1, r_2 \right) \right). \]
Integrating this inequality over $[3/4, 7/8]^2$ with respect to $r_1$ and $r_2$ and dividing by the obtained constant standing nearby $|f(z_1, z_2)|^q$, it follows that

$$|f(z_1, z_2)|^q \leq C \left( |f(0, 0)|^q + \sum_{k=1}^{2} \int_{[3/4, 7/8]^2} M_p^q \left( \frac{\partial f}{\partial z_k}, r_1, r_2 \right) \prod_{j=1}^{2} (1 - r_j^2)^{\beta_{k,j}} dr_j \right)$$

for every $z_1, z_2 \in D(0, 1/2)$, from which the result follows. □

Using the change $r \to (1+r)/2$ and some well known elementary inequalities the following lemma can be proved (see [11]).

**Lemma 5.** Let $g(r)$ be a nonnegative continuous function on the interval $[0, 1)$, $b > 0$ and let $a > -1$. Then there is a constant $C = C(a, b)$ such that

$$\int_0^1 g^b(r)(1 - r)^a dr \leq C \left( \max_{r \in [0,1/2]} g^b(r) + \int_0^1 |g\left(\frac{1+r}{2}\right) - g(r)|^b (1 - r)^a dr \right).$$

**Lemma 6.** Suppose $p, q \in [1, \infty)$ and $f \in H(U^n)$. Then

$$d\left( M_p^q(f, tr) \right) \leq q M_p^{q-1}(f, tr) \sum_{i=1}^{n} r_i M_p\left( \frac{\partial f}{\partial z_i}, tr \right),$$

almost everywhere.

**Proof.** Let first $p = q$. For $f \equiv 0$ the result is obvious. If $f \not\equiv 0$, at points where $f$ is not zero, it is easy to see that

$$d\left( |f(tr \cdot e^{i\theta})|^p \right) \leq p |f(tr \cdot e^{i\theta})|^{p-1} \sum_{i=1}^{n} r_i \left| \frac{\partial f}{\partial z_i}(tr \cdot e^{i\theta}) \right|.$$

From (10) and by the dominated convergence theorem we obtain

$$d\left( M_p^p(f, tr) \right) \leq \frac{p}{(2\pi)^n} \sum_{i=1}^{n} r_i \int_{[0,2\pi]^n} |f(tr \cdot e^{i\theta})|^{p-1} \left| \frac{\partial f}{\partial z_i}(tr \cdot e^{i\theta}) \right| d\theta.$$

If $p = 1$ the assertion is clear. If $p > 1$, applying on the last integral Hölder’s inequality with exponents $p/(p-1)$ and $p$ we obtain the result.

If $p \neq q$, computing $d\left( M_p^q(f, tr) \right)$ and then using the case $p = q$, the result follows. □

3. **Proof of the main results**

In this section we prove the main results in this paper.
Proof of Theorem 1. (a) Let $\gamma$ be a multi-index, such that $|\gamma| = m$. Let $f \in H(U^n)$ and $z = (z_1, \ldots, z_n) \in U^n$. Applying Lemma 1 to the functions $f(z \cdot e^{i\theta})$, where $\theta_j \in [0, 2\pi)$, $j = 1, \ldots, n$, when $\rho > r$, we get
\begin{equation}
|D^\gamma f(r \cdot e^{i\theta})|^p \leq \frac{C}{(\rho - r)^{\gamma p}} \prod_{j=1}^n (\rho_j - r_j)^2 \int_{P^n(r, \rho - r)} |f(\omega \cdot e^{i\theta})|^p \prod_{j=1}^n dm(\omega_j).
\end{equation}

Integrating (11) over $[0, 2\pi]^n$ and then using Fubini's theorem, we obtain
\begin{equation}
M_p^r(D^\gamma f, r) \leq \frac{C}{(\rho - r)^{\gamma q}2} \int_{P^n(r, \rho - r)} \left( \int_{[0, 2\pi]^n} |f(\omega \cdot e^{i\theta})|^p d\theta \right)^{q/p} \prod_{j=1}^n dm(\omega_j).
\end{equation}

Assume first that $q \geq p$. Raising both sides of inequality (12) to the $q/p$th power and applying Jensen's inequality, it follows that
\begin{equation}
M_q^r(D^\gamma f, r) \leq \frac{C}{(\rho - r)^{\gamma q}2} \int_{P^n(r, \rho - r)} \left( \int_{[0, 2\pi]^n} |f(\omega \cdot e^{i\theta})|^p d\theta \right)^{q/p} \prod_{j=1}^n dm(\omega_j).
\end{equation}

If $p \geq q$, then using Minkowski’s inequality to inequality (11), where instead of $p$ stands $q$, we also obtain inequality (13). By the monotonicity of the integral means, $2\pi$-periodicity of the function $|f(r \cdot e^{i\theta})|$ in each variable $\theta_j$, $j \in \{1, \ldots, n\}$ and (13), it follows that
\begin{equation}
(p - r)^\gamma M_p^r(D^\gamma f, r) \leq C \left( \int_{[0, 2\pi]^n} |f(\rho_1 e^{i\theta_1}, \ldots, \rho_n e^{i\theta_n})|^p d\theta \right)^{q/p}.
\end{equation}

Put $\rho_j = \rho_j(r_j) = r_j + \delta_j \psi_j(r_j)$, $0 \leq r_j < 1$, in (14), where $\delta_j$ are chosen in the following way. First note that, if $\delta \in (0, A)$ we have $r_j < \rho_j(r_j) < 1$ for $r_j \in [0, 1)$. On the other hand by conditions (b) and (c) of Definition 1 we obtain
\[ \rho'_j(r_j) = 1 - \delta_j - \delta_j \omega_j(r_j) \psi_j(r_j) \geq 1 - \delta_j \left(1 + \frac{B}{A}\right). \]

We choose $\delta_j \in (0, A)$ such that $\rho'_j(r_j) > c_0 > 0$ for $r_j \in [0, 1)$. Putting $\rho_j = \rho_j(r_j)$ in (14), then multiplying obtained inequality by $\prod_{j=1}^n \omega_j(r_j)$, using condition (c) in Definition 1 and the fact that $\rho'_j(r_j) > c_0 > 0$ for $r_j \in [0, 1)$ and every $j \in \{1, \ldots, n\}$, we obtain
\begin{align*}
\prod_{j=1}^n (\delta_j \psi_j(r_j))^{\gamma q} \omega_j(r_j) M_q^r(D^\gamma f, r) \\
\leq C \left( \int_{[0, 2\pi]^n} |f(r \cdot e^{i\theta})|^p d\theta \right)^{q/p} \prod_{j=1}^n \omega_j(\rho_j(r_j)) \rho'_j(r_j).
\end{align*}
Integrating this inequality over \([0,1]^n\) and making the changes \(t_j = \rho_j(r_j), \ j = 1, \ldots, n\), it follows that

\[
\prod_{j=1}^{n} \frac{q_j}{q_j} \int_{[0,1]^n} M_p(q_j(D^j f, r)) \prod_{j=1}^{n} \psi_j^q(r_j) \omega_j(t_j) dt_j \\
\leq C \int_{[\rho_j(0,1)]^n} \left( \int_{[0,2\pi]^n} |f(t_1 e^{i\theta_1}, \ldots, t_n e^{i\theta_n})|^p d\theta \right)^{q/p} \prod_{j=1}^{n} \omega_j(t_j) dt_j \\
\leq C \int_{(0,1)^n} M_p^q(f, t) \prod_{j=1}^{n} \omega_j(t_j) dt_j,
\]

from which inequality (3) follows.

Let \(\beta\) be a multi-index. By Lemma 2 we know that the linear functional \(L(f) = D^\beta f(\vec{0})\), is bounded. Hence \(|D^\beta f(\vec{0})|^q \leq C ||f||_{A_p}^q\) for all \(f \in H(U^n)\) and for some \(C = C(p, q, \beta, \vec{a}) > 0\). Hence inequality (4) holds.

(b) Without loss of generality, we may assume that \(n = 2\), and \(f(0, 0) = 0\). Also we assume that \(f\) is not constant and all integrals are finite. In order to avoid some complicated notations we use \(M_p^q(r_1 t, r_2 t)\) instead of \(M_p^q(f, r_1 t, r_2 t)\).

We have

\[
\|(f)\|_{A_p}^q = \int_0^1 \int_0^1 \left( \int_0^1 \frac{d}{dt} M_p^q(r_1 t, r_2 t) dt \right) \omega_1(r_1) \omega_2(r_2) dr_1 dr_2 \\
\leq q \int_0^1 \int_0^1 \left( \int_0^1 M_p^{q-1}(r_1 t, r_2 t) M_p \left( \frac{\partial f}{\partial z_1}, r_1 t, r_2 t \right) \right) \omega_1(r_1) \omega_2(r_2) dr_1 dr_2 \\
+ q \int_0^1 \int_0^1 \left( \int_0^1 M_p^{q-1}(r_1, r_2 t) M_p \left( \frac{\partial f}{\partial z_2}, r_1, r_2 t \right) \right) \omega_1(r_1) \omega_2(r_2) dr_1 dr_2 \\
\leq q \int_0^1 \int_0^1 \left( \int_0^{r_1} M_p^{q-1}(s, r_2) M_p \left( \frac{\partial f}{\partial z_1}, s, r_2 \right) ds \right) \omega_1(r_1) \omega_2(r_2) dr_1 dr_2 \\
+ q \int_0^1 \int_0^1 \left( \int_0^{r_2} M_p^{q-1}(r_1, \tau) M_p \left( \frac{\partial f}{\partial z_2}, r_1, \tau \right) d\tau \right) \omega_1(r_1) \omega_2(r_2) dr_1 dr_2 \\
\leq q \int_0^1 \int_0^1 \left( \int_0^{r_1} M_p^{q-1}(s, r_2) M_p \left( \frac{\partial f}{\partial z_1}, s, r_2 \right) ps_1(s) \omega_1(s_1) \omega_2(r_2) ds dr_2 \right) \\
+ q \int_0^1 \int_0^1 \left( \int_0^{r_2} M_p^{q-1}(r_1, \tau) M_p \left( \frac{\partial f}{\partial z_2}, r_1, \tau \right) p_2(\tau) \omega_2(\tau) \omega_1(r_1) d\tau dr_1 \right.
= I_1 + I_2.
\]

If \(q > 1\), by Hölder inequality with the exponents \(q/(q - 1)\) and \(q\), we get

\[
I_1 \leq ||f||_{A_p}^{q-1} \left( \int_0^1 \int_0^1 M_p^q \left( \frac{\partial f}{\partial z_1}, s, r_2 \right) p_1(s) \omega_1(s_1) \omega_2(r_2) ds dr_2 \right)^{1/q}.
\]
Similar inequality holds for $I_2$. From the inequality, (15) and (16) we obtain the result in this case. For $q = 1$ the result follows from (15). If $f$ is constant the result is clear. To remove the restriction of the finiteness of the integrals we consider holomorphic functions $f_\rho(z) = f(\rho z), \rho \in (0, 1)$ and use the Monotone Convergence Theorem, when $\rho \to 1$.

Proof of Theorem 2. (a)$\Rightarrow$(b), (c). Implication (a)$\Rightarrow$(b) is a consequence of Theorem 1, when $\omega_j(z_j) = (1 - |z_j|^2)^{\alpha_j}, j = 1, \ldots, n$. Indeed, in this case $\psi_j(z_j) \approx (1 - |z_j|^2)^{\alpha_j}, j = 1, \ldots, n$. (a)$\Rightarrow$(c) follows if we take the points $(\chi_S(1)z_1, \ldots, \chi_S(n)z_n), S \subseteq \{1, \ldots, n\},$ into the functions

$$\left[\prod_{j=1}^n(1 - |z_j|^2)\chi_S(j)\right] \frac{\partial^{|S|}f}{\partial z_1^{\chi_S(1)} \cdots \partial z_n^{\chi_S(n)}}.$$

(b)$\Rightarrow$(a). Without loss of generality, we may assume that $n = 2$. By some simple calculation, it is easy to see that

$$||f||_{A_{p,q}^\alpha}^q \approx \int_0^1 (1 - r_2)^{\alpha_2} \int_0^1 M_{q}^\alpha(f, r_1, r_2)(1 - r_1)^{\alpha_1} dr_1 dr_2.$$

Let $l = \min\{1, p\}$. By Lemmas 3 and 5, and since $M_p^l(f, r_1, r_2)$ is nondecreasing in $r_1$, we obtain

$$\int_0^1 M_{p}^l(f, r_1, r_2)(1 - r_1)^{\alpha_1} dr_1$$

$$= \int_0^1 (M_{p}^l(f, r_1, r_2))^{\eta/l}(1 - r_1)^{\alpha_1} dr_1$$

$$\leq C \left((M_{p}^l(f, 1/2, r_2))^{\eta/l} \right.$$

$$+ \int_0^1 \left|M_p^l\left(f, \frac{r_1 + r_2}{2}, r_2\right) - M_p^l\left(f, r_1, r_2\right)\right|^{\eta/l} (1 - r_1)^{\alpha_1} dr_1$$

$$\leq C \left(M_{p}^l(f, 1/2, r_2) + \int_0^1 M_{p}^l\left(\frac{\partial f}{\partial z_1}, \frac{1 + r_1}{2}, r_2\right)(1 - r_1)^{\alpha_1 + q} dr_1\right).$$

Hence

$$||f||_{A_{p,q}^\alpha}^q$$

$$\leq C \left(\int_0^1 (1 - r_2)^{\alpha_2} M_{p}^\alpha(f, 1/2, r_2) dr_2 + \int_0^1 (1 - r_2)^{\alpha_2} \int_0^1 M_{p}^\alpha\left(\frac{\partial f}{\partial z_1}, \frac{1 + r_1}{2}, r_2\right)(1 - r_1)^{\alpha_1 + q} dr_1 dr_2\right).$$
Since $M^q_p\left(\frac{\partial f}{\partial z_j}, \frac{1+r_1}{2}, r_2\right)$ is nondecreasing in $r_2$ and applying the changes $\frac{1+r_j}{2} \to r_j, j = 1, 2$, we obtain

$$\int_0^1 (1 - r_2)^{\alpha_2} \int_0^1 M^q_p\left(\frac{\partial f}{\partial z_1}, \frac{1+r_1}{2}, r_2\right)(1 - r_1)^{\alpha_1+q} dr_1 dr_2 \leq C \int_0^1 \int_0^1 M^q_p\left(\frac{\partial f}{\partial z_1}, r_1, r_2\right)(1 - r_1)^{\alpha_1+q}r_1dr_1(1 - r_2)^{\alpha_2}dr_2. \tag{18}$$

Using again Lemmas 3 and 5, and since $M^q_p(f, 1/2, r_2)$ is nondecreasing in $r_2$ we get

$$\int_0^1 (1 - r_2)^{\alpha_2}M^q_p(f, 1/2, r_2)dr_2 \leq C \left(M^q_p(f, 1/2, 1/2) + \int_0^1 (1 - r_2)^{\alpha_2} \left|M^q_p\left(f, \frac{1}{2}, \frac{1+r_2}{2}\right) - M^q_p\left(f, \frac{1}{2}, r_2\right)\right|^{q/2} dr_2 \right)$$

$$= C \left(\max_{|z_1|\leq 1/2, |z_2|\leq 1/2} |f(z_1, z_2)|^q + \int_0^1 (1 - r_2)^{\alpha_2+q}M^q_p\left(\frac{\partial f}{\partial z_2}, \frac{1}{2}, \frac{1+r_2}{2}\right)dr_2 \right).$$

It is clear that there is a constant $C$ independent of $f$ such that

$$\int_0^{3/4} (1 - r_2)^{\alpha_2+q}M^q_p\left(\frac{\partial f}{\partial z_2}, \frac{1}{2}, \frac{1+r_2}{2}\right)dr_2 \leq C \max_{z \in D(0,7/8)} \left|\frac{\partial f}{\partial z_2}(z_1, z_2)\right|^q. \tag{20}$$

Similar to Lemma 4 we can prove the following inequality

$$\max_{z \in D(0,7/8)} \left|\frac{\partial f}{\partial z_2}(z_1, z_2)\right|^q \leq C \int_{[0,1]^2} M^q_p\left(\frac{\partial f}{\partial z_2}, r_1, r_2\right)(1 - r_1)^{\alpha_1}(1 - r_2)^{\alpha_2+q}dr_1dr_2. \tag{21}$$

On the other hand, using the change $(1 + r_2)/2 \to r_2$ we obtain

$$\int_{3/4}^1 (1 - r_2)^{\alpha_2+q}M^q_p\left(\frac{\partial f}{\partial z_2}, \frac{1}{2}, \frac{1+r_2}{2}\right)dr_2 = C_1 \int_{7/8}^1 (1 - r_2)^{\alpha_2+q}M^q_p\left(\frac{\partial f}{\partial z_2}, \frac{1}{2}, r_2\right)dr_2 = J_1$$

for some $C_1 > 0$. 
Using again the monotonicity of the integral means, we obtain that there is a constant $C$ independent of $f$ such that

\[
J_1 = \frac{1}{C_1(\alpha_1 + 1)2^{\alpha_1 + 1}} \int_{\frac{1}{8}}^{1} (1 - r_2)^{\alpha_2 + q} M_f^q \left( \frac{\partial f}{\partial z_2}, \frac{1}{2}, r_2 \right) dr_2
\]

\[
= \frac{1}{C_1} \int_{\frac{1}{1/2}}^{1} (1 - r_1)^{\alpha_1} r_1 \int_{\frac{1}{8}}^{1} (1 - r_2)^{\alpha_2 + q} M_f^q \left( \frac{\partial f}{\partial z_2}, \frac{1}{2}, r_2 \right) dr_2
\]

\[
\leq \frac{1}{C_1} \int_{\frac{1}{1/2}}^{1} (1 - r_1)^{\alpha_1} \int_{\frac{1}{8}}^{1} (1 - r_2)^{\alpha_2 + p} M_f^q \left( \frac{\partial f}{\partial z_2}, r_1, r_2 \right) dr_2 dr_1
\]

\[
(22) \leq C \int_{0}^{1} \int_{0}^{1} (1 - r_1)^{\alpha_1} (1 - r_2)^{\alpha_2 + p} M_f^q \left( \frac{\partial f}{\partial z_2}, r_1, r_2 \right) dr_2 dr_1.
\]

From (17)-(22) the result and the asymptotics in Theorem 2 follow, for $m = 1$. Using induction we obtain the result for $m \geq 2$.

(c)$\Rightarrow$(a). As in the previous case, we may assume that $n = 2$ and $f(0, 0) = 0$.

From (17)-(19), it follows that we should estimate the following quantities

\[
I_1 = \int_{0}^{1} \int_{0}^{1} M_f^q \left( \frac{\partial f}{\partial z_1}, r_1, r_2 \right) (1 - r_1^{2})^{\alpha_1 + q} r_1 dr_1 (1 - r_2^{2})^{\alpha_2 + q} r_2 dr_2,
\]

\[
I_2 = \max_{|z_1| \leq 1/2; |z_2| \leq 1/2} |f(z_1, z_2)|^q
\]

and

\[
I_3 = \int_{0}^{1} (1 - r_2)^{\alpha_2 + q} M_f^q \left( \frac{\partial f}{\partial z_2}, \frac{1}{2}, \frac{1 + r_2}{2} \right) dr_2.
\]

Using the inequality

\[
\max_{z \in B^{n}(0, 1/2)} \left| \frac{\partial f}{\partial z} (z_1, z_2) \right|^q \leq CM_f^q \left( \frac{\partial f}{\partial z_j}, r_1, r_2 \right), \quad j \in \{1, 2\}, \quad r_1, r_2 \in [3/4, 1),
\]

which can be proved similar to (8), taking $r_2 = 3/4$ for $j = 1$ and $r_1 = 3/4$ for $j = 2$, it follows that

\[
(23) \max_{z \in B^{2}(0, 1/2)} \left| \frac{\partial f}{\partial z_1} (z_1, z_2) \right|^q \leq CM_f^q \left( \frac{\partial f}{\partial z_1}, r_1, \frac{3}{4} \right)
\]

when $r_1 \in [3/4, 1)$, and

\[
(24) \max_{z \in B^{2}(0, 1/2)} \left| \frac{\partial f}{\partial z_2} (z_1, z_2) \right|^q \leq CM_f^q \left( \frac{\partial f}{\partial z_2}, \frac{3}{4}, r_2 \right),
\]

when $r_2 \in [3/4, 1)$.

Multiplying (23) by $(1 - r_1)^{\alpha_1 + q}$, then integrating obtained inequality from $3/4$ to $1$ with respect to $r_1$, and multiplying (24) by $(1 - r_2)^{\alpha_2 + q}$, then integrating from $3/4$ to $1$ with respect to $r_2$, and using inequality (7) with $p = q$,
By Fubini's theorem, Lemma 5, Lemma 3 and the monotonicity of \( M_p \), we get

\[
I_2 \leq C \left( \int_{3/4}^1 (1 - r_1)^{\alpha_1+q} M_p^q \left( \frac{\partial f}{\partial z_1}, r_1, \frac{3}{4} \right) dr_1 \right.
\]
\[
+ \left. \int_{3/4}^1 (1 - r_2)^{\alpha_2+q} M_p^q \left( \frac{\partial f}{\partial z_2}, \frac{3}{4}, r_2 \right) dr_2 \right).
\]

By Lemma 3 and the inequality \((x + y)^p \leq c_p (x^p + y^p), x, y \geq 0, \) where \( c_p = 1 \) when \( p \in (0, 1), \) and \( c_p = 2^{p-1} \) when \( p \geq 1, \) we obtain that there is a positive constant \( C \) such that

\[
I_2 \leq C \left( \int_{3/4}^1 (1 - r_1)^{\alpha_1+q} M_p^q \left( \frac{\partial f}{\partial z_1}, 0, r_2 \right) + M_p^q \left( \frac{\partial f}{\partial z_1}, \frac{3}{4}, r_2 \right) \right)
\]
and

\[
M_p^q \left( \frac{\partial f}{\partial z_1}, r_1, \frac{3}{4} \right) \leq C \left( M_p^q \left( \frac{\partial f}{\partial z_1}, r_1, 0 \right) + M_p^q \left( \frac{\partial f}{\partial z_1}, r_1, \frac{3}{4} \right) \right).
\]

Multiplying (26) by \((1 - r_2)^{\alpha_2+q}, \) then integrating obtained inequality from 0 to 1 with respect to \( r_2, \) and multiplying (27) by \((1 - r_1)^{\alpha_1+q}, \) then integrating obtained inequality from 0 to 1 with respect to \( r_1, \) using the monotonicity of the integral means \( M_p(\cdot, r_1, r_2) \) in each variable and (25), we get

\[
I_2 \leq C \left( \int_0^1 (1 - r_2)^{\alpha_2+q} M_p^q \left( \frac{\partial f}{\partial z_2}, 0, r_2 \right) dr_2 \right.
\]
\[
+ \left. \int_0^1 (1 - r_1)^{\alpha_1+q} M_p^q \left( \frac{\partial f}{\partial z_1}, r_1, 0 \right) dr_1 \right.
\]
\[
+ \left. \int_0^1 \int_0^1 (1 - r_1)^{\alpha_1+q} (1 - r_2)^{\alpha_2+q} M_p^q \left( \frac{\partial f}{\partial z_1}, r_1, r_2 \right) M_p^q \left( \frac{\partial f}{\partial z_2}, r_1, r_2 \right) dr_1 dr_2 \right) = CI_4.
\]

Further, using the change \( \frac{1 + r_1}{2} \to r_2, \) the monotonicity of \( M_p(\cdot, r_1, r_2), \) (27) and (28), we have that

\[
I_3 = C \int_{1/2}^1 (1 - r_2)^{\alpha_2+q} M_p^q \left( \frac{\partial f}{\partial z_2}, \frac{1}{2}, r_2 \right) dr_2 \leq CI_4.
\]
By Fubini's theorem, Lemma 5, Lemma 3 and the monotonicity of \( M_p(\cdot, r_1, r_2), \) it follows that

\[
I_1 \leq C \left( \int_0^1 M_p^q \left( \frac{\partial f}{\partial z_1}, r_1, \frac{3}{4} \right)(1 - r_1)^{\alpha_1+q} dr_1 \right.
\]
\[
+ \left. \int_0^1 \int_0^1 M_p^q \left( \frac{\partial f}{\partial z_1}, r_1, r_2 \right) (1 - r_1)^{\alpha_1+q} (1 - r_2)^{\alpha_2+q} dr_2 (1 - r_2)^{\alpha_2+q} dr_1 \right).
\]
From (23)-(30) we see that for the case \( n = 2 \), the quantities \( I_1, I_2 \) and \( I_3 \) are estimated by a linear combination of the terms

\[
\int_{[0,1]^2} M_p^S \left( \frac{D|S|}{\prod_{j \in S} \partial z_j} \chi_S(1) r_1, \chi_S(2) r_2 \right) \prod_{j \in S} (1 - r_j^2)^{\alpha_j + q} dr_j, \quad S \subseteq \{1, 2\},
\]

from which the implication follows. For the case \( n \geq 3 \) we can use the induction. \( \square \)

References


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