GALOIS GROUPS OF MODULES AND INVERSE POLYNOMIAL MODULES

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ABSTRACT. Given an injective envelope $E$ of a left $R$-module $M$, there is an associative Galois group $Gal(\phi)$. Let $R$ be a left noetherian ring and $E$ be an injective envelope of $M$, then there is an injective envelope $E[x^{-1}]$ of an inverse polynomial module $M[x^{-1}]$ as a left $R[x]$-module and we can define an associative Galois group $Gal(\phi[x^{-1}])$. In this paper we describe the relations between $Gal(\phi)$ and $Gal(\phi[x^{-1}])$. Then we extend the Galois group of inverse polynomial module and can get $Gal(\phi[x^{-s}])$, where $S$ is a submonoid of $N$ (the set of all natural numbers).

1. Introduction

Given an injective envelope $M \subset E$, by the Galois group of this envelope we mean all $f \in Hom_R(E, E)$ such that $f(x) = x$ for all $x \in M$ or equivalently such that

$$
\begin{array}{ccc}
M & \longrightarrow & E \\
\downarrow f & & \downarrow f \\
E & \longrightarrow & E
\end{array}
$$

is a commutative diagram. Any such $f$ is an automorphism of $E$ and we also see that

$$
\begin{array}{ccc}
M & \longrightarrow & E \\
\downarrow f^{-1} & & \downarrow f^{-1} \\
E & \longrightarrow & E
\end{array}
$$

is commutative. So we easily see that the set of $f$ form a group (using the composition of functions as operation). If $\phi : M \longrightarrow E$ denotes the canonical injection then the group is denoted $Gal(\phi)$. Northcott ([4]) defined inverse polynomial modules and used inverse polynomial modules to study the properties of injective modules and he studied $K[x^{-1}]$ as $K[x]$-module on field $K$. And

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McKerraw ([2]) showed that if $R$ is a left noetherian ring and $E$ is an injective left $R$-module, then $E[x^{-1}]$ is an injective envelope of $M[x^{-1}]$ as $R[x]$-module. Inverse polynomial modules were studied in ([5]), ([6]) and recently in ([1]), ([7]), ([8]), ([9]).

**Definition 1.1** ([5]). Let $R$ be a ring and $M$ be a left $R$-module, then $M[x^{-1}]$ is a left $R[x]$-module defined by

$$x(m_0 + m_1 x^{-1} + \cdots + m_n x^{-n}) = m_1 + m_2 x^{-1} + \cdots + m_n x^{-n+1}$$

and such that

$$r(m_0 + m_1 x^{-1} + \cdots + m_n x^{-n}) = rm_0 + rm_1 x^{-1} + \cdots + rm_n x^{-n}$$

where $r \in R$. We call $M[x^{-1}]$ as an inverse polynomial module.

If $R$ is left noetherian and if $M \subseteq E$ as above, then $M[x^{-1}] \subseteq E[x^{-1}]$ is an injective envelope over $R[x]$. If $\phi[x^{-1}] : M[x^{-1}] \rightarrow E[x^{-1}]$ denotes the canonical injection, then the group is denoted $Gal(\phi[x^{-1}])$.

**Lemma 1.2** ([5]). Let $M$ and $N$ be left $R$-modules, then

$$\text{Hom}_{R[x]}(M[x^{-1}], N[x^{-1}]) \cong \text{Hom}_R(M, N)[[x]].$$

**Theorem 1.3.** There is a ring isomorphism

$$\text{Hom}_{R[x]}(M[x^{-1}], N[x^{-1}]) \cong \text{Hom}_R(M, N)[[x]].$$

**Proof.** By the Lemma 1.2., we know that two groups are isomorphic. Let $\sigma, \tau \in \text{Hom}_{R[x]}(M[x^{-1}], N[x^{-1}])$, then $\sigma$ corresponds to $f_0 + f_1 x + f_2 x^2 + \cdots \in \text{Hom}_R(M, N)[[x]]$ and $\tau$ corresponds to $g_0 + g_1 x + g_2 x^2 + \cdots \in \text{Hom}_R(M, N)[[x]]$. Then $\sigma \circ \tau$ corresponds to

$$\sum_{n=0}^{\infty} (\sum_{i+j=n} f_i \circ g_j)x^n.$$

Hence, $\text{Hom}_{R[x]}(M[x^{-1}], N[x^{-1}]) \cong \text{Hom}_R(M, N)[[x]].$ \hfill $\Box$

**2. $Gal(\phi)$ and $Gal(\phi[x^{-1}])$**

**Theorem 2.1.** If $R$ is a left noetherian ring and if $M \subseteq E$ is an injective envelope of $R$-module, then $f = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + \cdots \in \text{End}_R(E)[[x]]$ is in $Gal(\phi[x^{-1}])$ if and only if $f_0 \in Gal(\phi)$ and $f_i(M) = 0$ for all $i \geq 1$.

**Proof.** Let $m \in M$ and $f \in Gal(\phi[x^{-1}])$, then

$$f(m + 0 x^{-1} + 0 x^{-2} + \cdots + 0 x^{-i})$$

$$= (f_0 + f_1 x + f_2 x^2 + \cdots)(m + 0 x^{-1} + 0 x^{-2} + \cdots + 0 x^{-i})$$

$$= (f_0 + f_1 x + f_2 x^2 + \cdots)(m)$$

$$= f_0(m) + f_1(m)x + f_2(m)x^2 + \cdots$$

$$= m.$$
Thus \( f_0(m) = m \) for all \( m \in M \), so that \( f_0 \in Gal(\phi) \). And
\[
f(m + mx^{-1}) = (f_0 + f_1x + f_2x^2 + \cdots)(m + mx^{-1})
\]
\[
= f_0(m) + f_0(m)x^{-1} + f_1(m)x + f_1(m) + f_2(m)x + f_2(m)x + \cdots
\]
\[
= (f_0(m) + f_1(m)) + f_0(m)x^{-1} + (f_1(m) + f_2(m))x + \cdots
\]
\[
= m + mx^{-1}.
\]
Since \( f_0(m) = m, m + f_1(m) = m \) implies \( f_1(m) = 0 \). Thus \( f_1(m) = 0 \). And
\[
f(m + mx^{-1} + mx^{-2}) = (f_0 + f_1x + f_2x^2 + \cdots)(m + mx^{-1} + mx^{-2})
\]
\[
= f_0(m) + f_0(m)x^{-1} + f_1(m)x + f_1(m) + f_1(m)x^{-1} + f_2(m)x^2
\]
\[
+ f_2(m)x + f_2(m) + \cdots
\]
\[
= (f_0(m) + f_1(m) + f_2(m)) + (f_0(m) + f_1(m))x^{-1} + f_0(m)x^{-2}
\]
\[
+ (f_1(m) + f_2(m))x + \cdots
\]
\[
= m + mx^{-1} + mx^{-2}.
\]
Since \( f_0(m) = m, f_1(m) = 0, f_0(m) + f_1(m) + f_2(m) = m \) implies \( f_2(m) = 0 \).
Thus \( f_2(m) = 0 \). By the same process we can get \( f_i(M) = 0 \) for all \( i \geq 1 \).

Conversely, let \( f = f_0 + f_1x + f_2x^2 + f_3x^3 + \cdots \) with \( f \in Gal(\phi[x^{-1}]) \) and \( f_i(M) = 0, i \geq 1 \). Let \( m_0 + m_1x^{-1} + m_2x^{-2} + \cdots + m_ix^{-i} \in M[x^{-1}] \). We want to show
\[
f(m_0 + m_1x^{-1} + m_2x^{-2} + \cdots + m_ix^{-i}) = m_0 + m_1x^{-1} + m_2x^{-2} + \cdots + m_ix^{-i}.
\]
Then
\[
f(m_0 + m_1x^{-1} + m_2x^{-2} + \cdots + m_ix^{-i}) = (f_0 + f_1x + f_2x^2 + \cdots)(m_0 + m_1x^{-1} + m_2x^{-2} + \cdots + m_ix^{-i})
\]
\[
= f_0(m_0) + f_0(m_1)x^{-1} + f_0(m_2)x^{-2} + \cdots + f_0(m_i)x^{-i}
\]
\[
+ f_1(m_0)x + f_1(m_1)x^{-1} + f_1(m_2)x^{-2} + \cdots + f_1(m_i)x^{-i+1} + f_2(m_0)x^2
\]
\[
+ f_2(m_1)x + f_2(m_2)x^{-1} + f_2(m_3)x^{-2} + \cdots + f_2(m_i)x^{-i+2} + \cdots + f_i(m_i)
\]
\[
= m_0 + m_1x^{-1} + m_2x^{-2} + \cdots + m_ix^{-i},
\]
since \( f_0 \in Gal(\phi) \) and \( f_i(M) = 0 \) for all \( i \geq 1 \). Therefore, \( f = f_0 + f_1x + f_2x^2 + f_3x^3 + \cdots \in Gal(\phi[x^{-1}]) \).

There are natural group homomorphisms \( Gal(\phi) \to Gal(\phi[x^{-1}]) \) by \( g \mapsto g + 0x + 0x^2 + \cdots \) and \( Gal(\phi[x^{-1}]) \to Gal(\phi) \) by \( f_0 + f_1x + f_2x^2 + \cdots \mapsto f_0 \). The composition \( Gal(\phi) \to Gal(\phi[x^{-1}]) \to Gal(\phi) \) is the identity map on \( Gal(\phi) \).

The kernel of \( Gal(\phi[x^{-1}]) \to Gal(\phi) \) consists of all \( \text{id}_E + f_1x + f_2x^2 + \cdots \), where \( f_i \in Hom_R(E, E) \) and \( f_i(M) = 0 \), for all \( i \geq 1 \).
Lemma 2.2. Let \( \psi : \text{Gal}(\phi) \to \text{Gal}(\phi[x^{-1}]) \) be defined by \( \psi(f) = f + 0x + 0x^2 + \cdots \). If \( \text{End}(E) \) is a commutative ring, then \( \text{Im}(\psi) \) is a normal subgroup of \( \text{Gal}(\phi[x^{-1}]) \).

Proof. Let \( f_0 + 0x + 0x^2 + \cdots \in \text{Im}(\psi) \), and \( g_0 + g_1x + g_2x^2 + \cdots \in \text{Gal}(\phi[x^{-1}]) \). Let \( (g_0 + g_1x + g_2x^2 + \cdots)^{-1} = h_0 + h_1x + h_2x^2 + \cdots \). Then

\[
(g_0 + g_1x + g_2x^2 + \cdots) \circ (h_0 + h_1x + h_2x^2 + \cdots) = id_E + 0x + 0x^2 + \cdots,
\]

implies \( g_0 \circ h_0 = id_E \) so that \( h_0 = g_0^{-1} \) and \( \sum i+j=n g_i \circ h_j = 0, n \geq 1 \). Thus

\[
(g_0 + g_1x + \cdots) \circ (f_0 + 0x + \cdots) \circ (h_0 + h_1x + \cdots) = ((g_0 \circ f_0) + (g_1 \circ f_0)x + (g_2 \circ f_0)x^2 + \cdots) \circ (h_0 + h_1x + h_2x^2 + \cdots)
\]

\[
= (g_0 \circ f_0 \circ h_0) + (g_0 \circ f_0 \circ h_1 + g_1 \circ f_0 \circ h_0)x + (g_0 \circ f_0 \circ h_2 + g_1 \circ f_0 \circ h_1 + g_2 \circ f_0 \circ h_0)x^2 + \cdots = f_0,
\]

since \( \text{End}(E) \) is a commutative ring. Hence, \( \text{Im}(\psi) \) is a normal subgroup of \( \text{Gal}(\phi[x^{-1}]) \). \( \square \)

We note that \( \text{Im}(\psi) \) is not a normal subgroup of \( \text{Gal}(\phi[x^{-1}]) \), in general. So \( \text{Gal}(\phi[x^{-1}]) \) is the semidirect product of \( \text{Gal}(\phi) \) and \( K = \ker(\text{Gal}(\phi[x^{-1}]) \to \text{Gal}(\phi)) \).

Lemma 2.3. \( \text{Gal}(\phi) \) is commutative if and only if \( g \circ g' = g' \circ g \) for all \( g, g' \in \text{Hom}_R(E, E) \) with \( g(M) = 0, g'(M) = 0 \).

Proof. If \( f \in \text{Gal}(\phi) \), then \( f = f - id_E \in \text{Hom}_R(E, E) \) with \( g(M) = 0 \). And given \( g \in \text{Gal}(\phi[x^{-1}]) \) with \( g(M) = 0 \), \( f = g + id_E \in \text{Gal}(\phi) \). Therefore, there is one to one correspondence between \( \text{Gal}(\phi) \) and the set of \( g \in \text{Hom}_R(E, E) \) with \( g(M) = 0 \). So, given \( f, f' \in \text{Gal}(\phi) \) choose \( g = f - id_E, g' = f' - id_E \in \text{Hom}_R(E, E) \) with \( g(M) = 0, g'(M) = 0 \). Then \( g \circ g' = g' \circ g \).

Conversely, given \( g, g' \in \text{Hom}_R(E, E) \) with \( g(M) = 0, g'(M) = 0 \) choose \( f = g + id_E, f' = g' + id_E \in \text{Gal}(\phi) \). Then \( f \circ f' = f' \circ f \). Thus, \( \text{Gal}(\phi) \) is commutative. \( \square \)

Theorem 2.4. \( \text{Gal}(\phi[x^{-1}]) \) is commutative if and only if \( \text{Gal}(\phi) \) is commutative.

Proof. Since \( \text{Gal}(\phi) \) is a subgroup of \( \text{Gal}(\phi[x^{-1}]) \), \( \text{Gal}(\phi) \) is commutative. Conversely, let \( f_0 + f_1x + f_2x^2 + \cdots, g_0 + g_1x + g_2x^2 + \cdots \in \text{Gal}(\phi[x^{-1}]) \). Then by the Theorem 2.1., \( f_0, g_0 \in \text{Gal}(\phi), f_i(M) = 0, g_j(M) = 0, \) for all \( i, j \geq 1 \).

And by the Lemma 2.3., \( f_1 \circ g_j = g_j \circ f_i, i, j \geq 1 \). Given \( f_i \in \text{Gal}(\phi) \) choose \( g_i = f_i - id_E \in \text{Hom}(E, E) \) with \( g_i(M) = 0 \). Then

\[
f_0 \circ g_i = f_0 \circ (f_i - id_E) = f_0 \circ f_i - f_0 = f_i \circ f_0 - f_0
\]

\[
= (f_i - id_E) \circ f_0 = g_i \circ f_0.
\]
So
\[(f_0 + f_1x + f_2x^2 + \cdots) \circ (g_0 + g_1x + g_2x^2 + \cdots)\]
\[= (f_0 \circ g_0) + (f_0 \circ g_1 + f_1 \circ g_0)x + (f_0 \circ g_2 + f_1 \circ g_1 + f_2 \circ g_0)x^2 + \cdots\]
\[= (g_0 \circ f_0) + (g_1 \circ f_0 + g_0 \circ f_1)x + (g_2 \circ f_0 + g_1 \circ f_1 + g_0 \circ f_2)x^2 + \cdots\]
\[= (g_0 + g_1x + g_2x^2 + \cdots) \circ (f_0 + f_1x + f_2x^2 + \cdots).\]

Therefore, \(\text{Gal}(\phi(x^{-1}))\) is commutative. \(\square\)

**Theorem 2.5.** Let \(\varphi : \text{Gal}(\phi(x^{-1})) \to \text{Gal}(\phi)\) be defined by \(\varphi(f_0 + f_1x + f_2x^2 + \cdots) = f_0\). Then \(\text{Gal}(\phi(x^{-1}))\) is the direct product of \(K\) and \(\text{Gal}(\phi)\) if and only if \(\text{Gal}(\phi)\) is commutative, where \(K = \ker(\varphi)\).

**Proof.** Let \(g, g' \in \text{Gal}(\phi)\). Then \(id_E + g \in \text{Gal}(\phi)\) and \((id_E + g'x)^{-1} \circ (id_E + g) \circ (id_E + g'x) \in \text{Gal}(\phi)\). So let \((id_E + g'x)^{-1} = id_E - g'x + \text{etc.}\), then
\[
(id_E + g'x)^{-1} \circ (id_E + g) \circ (id_E + g'x)
= (id_E - g'x + \text{etc.}) \circ (id_E + g) \circ (id_E + g'x)
= id_E + (-g' \circ g + g \circ g')x + \text{etc.} \in \text{Gal}(\phi)
\]
implies \(-g' \circ g + g \circ g' = 0\) so that \(g' \circ g = g \circ g'\).

Therefore, \(\text{Gal}(\phi)\) is commutative.

Conversely, by the Theorem 2.4., if \(\text{Gal}(\phi)\) is commutative then \(\text{Gal}(\phi(x^{-1}))\) is commutative. Therefore, \(\text{Gal}(\phi(x^{-1}))\) is the direct product of \(K\) and \(\text{Gal}(\phi)\). \(\square\)

### 3. Generalization of Galois group

**Definition 3.1.** ([8]) Let \(R\) be a ring and \(M\) be a left \(R\)-module, and \(S = \{0, k_1, k_2, \ldots\}\) be a submonoid of \(\mathbb{N}\) (the set of all natural numbers). Then \(M[x^{-s}]\) is a left \(R[x^{-s}]\)-module such that
\[
x^{k_i}(m_0 + m_1x^{-k_1} + m_2x^{-k_2} + \cdots + m_nx^{-k_n})
= m_i^{-k_1+k_2} + m_2x^{-k_2+k_3} + \cdots + m_nx^{-k_n+k_i},
\]
where
\[
x^{-k_j+k_i} = \begin{cases} x^{-k_j+k_i} & \text{if } k_j - k_i \in S \\ 0 & \text{if } k_j - k_i \notin S. \end{cases}
\]

For example, if \(S = \{0, 2, 3, \ldots\}\), then \(m_0 + m_2x^{-2} + m_3x^{-3} + \cdots + m_4x^{-4} \in M[x^{-s}]\) and if \(S = \{0, 1, 2, 3, \ldots\}\), then \(M[x^{-1}] = M[x^{-s}]\).

Similarly, we define \(M[[x^{-s}], M[x^s, x^{-s}], M[x^s, x^{-s}], M[x^s, x^{-s}]]\) and \(M[[x^s, x^{-s}]]\) as left \(R[x^{-s}]\)-modules.

**Definition 3.2.** Given any module \(M\) and \(f \in \text{End}(E)\) we say \(f\) is locally nilpotent on \(M\) if for every \(x \in M\), there exist \(n \geq 1\) such that \(f^n(x) = 0\).
Theorem 3.3 (Matlis and Gabriel). If $R$ is a left noetherian ring and $E$ is an injective left $R$-module and $f \in \text{End}(E)$ is such that $E$ is an essential extension of $\ker (f)$ then $f$ is locally nilpotent on $E$.

Theorem 3.4. Let $R$ be a commutative noetherian ring and $S$ be a submonoid, and $E$ be an injective left $R$-module. Then $E[x^{-s}]$ is an injective left $R[x^s]$-module.

Proof. Let $S = \{0, k_1, k_2, \ldots \}$ be a submonoid. Then $\text{Hom}_R(R[x^s], E) \cong E[[x^{-s}]]$ is an injective left $R[x^s]$-module. Define $\phi : E[[x^{-s}]] \rightarrow E[[x^{-s}]]$ by $\phi(f) = x^{k_1}f$ for $f \in E[[x^{-s}]]$. Then $\phi$ is not locally nilpotent on $E[[x^{-s}]]$. So $E[[x^{-s}]]$ is not an essential extension of $\ker (\phi)$. Let $\bar{E}$ be an injective envelope of $\ker (\phi)$. Then $\ker (\phi) \subset \bar{E} \subset E[[x^{-s}]]$.

Then $\phi : \bar{E} \rightarrow \bar{E}$ defined by

$$\phi(f) = x^{k_1}f,$$

for $f \in \bar{E}$ is locally nilpotent on $\bar{E}$. So $\bar{E} \subset E[x^{-s}]$. But $E[x^{-s}]$ is an essential extension of $\ker (\phi)$, so that $E[x^{-s}]$ is an essential extension of $\bar{E}$. Therefore, $\bar{E} = E[x^{-s}]$. Hence, $E[x^{-s}]$ is an injective left $R[x^s]$-module. \hfill \square

We can generalize the Theorem 1.3. and get

$$\text{Hom}_R(R[x^s](M[x^{-s}], N[x^{-s}])) \cong \text{Hom}_R(M, N)[[x^s]].$$

If $\phi[x^{-s}] : M[x^{-s}] \rightarrow E[x^{-s}]$ denotes the canonical injection, then the group is denoted $\text{Gal}(\phi[x^{-s}])$.

Theorem 2.1. can be extended to the following remark.

Remark 1. If $R$ is a left noetherian ring and if $M \subset E$ is an injective envelope of $R$-module, then $f = f_{k_0} + f_{k_1}x^{k_1} + f_{k_2}x^{k_2} + f_{k_3}x^{k_3} + \cdots \in \text{End}_R(E)[[x^s]]$ is in $\text{Gal}(\phi[x^{-s}])$ if and only if $f_{k_0} \in \text{Gal}(\phi)$ and $f_{k_i}(M) = 0$, $k_i \in S, k_i \neq k_0$.

Lemma 2.2. can be extended to the following remark.

Remark 2. Let $\psi : \text{Gal}(\phi) \rightarrow \text{Gal}(\phi[x^{-s}])$ be defined by $\psi(f) = f + 0x^{k_1} + 0x^{k_2} + \cdots$. If $\text{End}(E)$ is a commutative ring, then $\text{Im}(\psi)$ is a normal subgroup of $\text{Gal}(\phi[x^{-s}])$.

Theorem 2.4. can be extended to the following remark.

Remark 3. $\text{Gal}(\phi[x^{-s}])$ is commutative if and only if $\text{Gal}(\phi)$ is commutative.

Theorem 2.5. can be extended to the following remark.

Remark 4. Let $\varphi : \text{Gal}(\phi[x^{-s}]) \rightarrow \text{Gal}(\phi)$ be defined by $\varphi(f_{k_0} + f_{k_1}x^{k_1} + f_{k_2}x^{k_2} + \cdots) = f_{k_0}$. Then $\text{Gal}(\phi[x^{-s}])$ is the direct product of $K$ and $\text{Gal}(\phi)$ if and only if $\text{Gal}(\phi)$ is commutative, where $K = \ker(\text{Gal}(\phi[x^{-s}]) \rightarrow \text{Gal}(\phi))$. 
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