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Abstract. In this paper we consider the problem of whether certain homogeneous or non-homogeneous differential polynomials in \( f(z) \) necessarily have infinitely many zeros. Particularly, this extends a result of Gopalakrishna and Bhoosnurmath [3, Theorem 2] for a general differential polynomial of degree \( d(P) \) and lower degree \( d(P) \).

1. Introduction

Let \( f(z) \) be a transcendental meromorphic function in the complex plane. It is assumed that the reader is familiar with the usual notations of Nevanlinna theory (See e.g. [4, 9]). We denote by \( S(r, f) \) any quantity satisfying \( S(r, f) = o(T(r, f)) \) as \( r \to +\infty \), possibly outside a set of finite linear measure \( E \). Throughout this paper we denote by \( a_j(z) \) any small meromorphic function satisfying \( T(r, a_j) = S(r, f) \), \( j = 1, 2, \ldots, n \).

Many mathematicians were interested in the value distribution of different expressions of a meromorphic function \( f(z) \) and obtained a lot of fruitful results. In [5], Hayman discussed Picard’s values of a meromorphic function \( f(z) \) and its derivatives. In particular, he showed that

**Theorem A.** Let \( f(z) \) be a transcendental entire function. Then

(a) for \( n \geq 3 \) and \( a \neq 0 \), \( \Psi(z) = f'(z) - a[f(z)]^n \) assumes all finite values infinitely often;

(b) for \( n \geq 2 \), \( \Psi(z) = f'(z)[f(z)]^n \) assumes all finite values except possibly zero infinitely often.

Later in 1964, Hayman showed further in his monograph [4] that

**Theorem B.** If \( f(z) \) is meromorphic and transcendental in the plane and has only a finite number of poles and zeros, then every meromorphic function \( \Psi(z) \) of the form \( \Psi(z) = \sum_{i=1}^{n} a_i f^{(i)}(z) \) assumes every finite complex value except possibly zero infinitely often, or else \( \Psi(z) \) is constant.
In 1967, Clunie [2] proved Theorem A(b) for \( n \geq 1 \) and later on Sons [6] generalized Theorem A(b) and in fact, he proved the following result on a monomial in \( f(z) \)

**Theorem C.** If \( f(z) \) is a transcendental entire function and

\[
\Psi(z) = [f(z)]^{n_0}[f'(z)]^{n_1} \cdots [f^{(k)}(z)]^{n_k},
\]

where \( n_0 \geq 2, n_k \geq 1 \) and \( n_i \geq 0 \) for \( i \neq 0, k \), then \( \delta(a, \Psi) < 1 \) for \( a \neq 0, \infty \).

Moreover if \( N_{(1)}(r, \frac{1}{f}) = S(r, f) \), then for \( n_0 \geq 1 \) the same conclusion holds good, where in \( N_{(1)}(r, \frac{1}{f}) \) we count only simple zeros of \( f(z) \).

Regarding the deficiencies of a monomial in \( f(z) \), Yang [7, 8] further generalized Theorem C to meromorphic functions as follows

**Theorem D.** Let \( f \) be transcendental meromorphic with

\[
(1) \quad N(r, f) + N \left(r, \frac{1}{f}\right) = S(r, f)
\]

and \( \Psi(z) = \sum a(z)[f(z)]^{p_0}[f'(z)]^{p_1} \cdots [f^{(k)}(z)]^{p_k} \) with no constant term. If the degree \( n \) of the homogeneous differential polynomial \( \Psi(z) \) is greater than one and \( p_0 < n, 0 \leq p_i \leq n \) for \( i \neq 0 \), then \( \delta(a, \Psi) < 1 \) for all \( a \neq 0, \infty \).

**Theorem E.** Let \( f(z) \) and \( \Psi(z) \) be as in Theorem D and all the terms of \( \Psi(z) \) have different degrees at least two, i.e., \( \Psi(z) \) is non-homogeneous. Then we have \( \delta(a, \Psi) \leq 1 - \frac{1}{2n} \) for \( a \neq \infty \).

Independently, by generalizing Theorem B as Gopalakrishna and Bhosnurmath’s goal, they actually obtained a result which was a generalization of Theorem D above and the argument they used is much simpler and elegant than that of Yang applied. In fact, they proved the following

**Theorem F** ([3, Theorem 2]). Let \( f(z) \) be a transcendental meromorphic function satisfying (1) and let \( P[f] \) be a homogeneous differential polynomial in \( f(z) \). If \( P[f] \) does not reduce to a constant, then \( \delta(a, P[f]) = 0 \) for \( a \neq 0, \) i.e., \( P[f] \) assumes all finite complex values except possibly zero infinitely often.

In this paper, two results are proved. In Theorem 1, we try to obtain bounds for

\[
\lim_{r \to +\infty} \frac{T(r, P[f])}{T(r, f)} \quad \text{and} \quad \lim_{r \to +\infty} \frac{T(r, P[f])}{T(r, f)},
\]

where \( P[f] \) is a differential polynomial in \( f(z) \). Then as a consequence we can obtain the result of Theorem F as a special case of Theorem 2.
2. Definitions and lemmas

For a positive integer \(j\), by a monomial in \(f(z)\) we mean an expression of the type
\[
M_j[f] = a_j(z)[f(z)]^{n_{0j}}[f'(z)]^{n_{1j}} \cdots [f^{(k)}(z)]^{n_{kj}},
\]
where \(n_{0j}, n_{1j}, \ldots, n_{kj}\) are non-negative integers. We define
\[
d(M_j) = \sum_{i=0}^{k} i n_{ij}
\]
as the degree of \(M_j[f]\) and
\[
\Gamma_{M_j} = \sum_{i=0}^{k} (i+1) n_{ij}
\]
as the weight of \(M_j[f]\).

Next, a differential polynomial in \(f(z)\) is a finite sum of such monomials, i.e.,
\[
P[f] = \sum_{j=1}^{n} a_j(z) M_j[f].
\]

We define
\[
\overline{d}(P) = \max_{1 \leq j \leq n} \{d(M_j)\}, \quad \underline{d}(P) = \min_{1 \leq j \leq n} \{d(M_j)\} \quad \text{and} \quad \Gamma_P = \max_{1 \leq j \leq n} \{\Gamma_{M_j}\}
\]
as the degree, the lower degree and the weight of \(P[f]\) respectively. If, in particular, \(\overline{d}(P) = \underline{d}(P)\), then \(P[f]\) is called homogeneous and non-homogeneous otherwise.

**Lemma 1** ([1]). Let \(f(z)\) be a meromorphic function and \(P[f]\) be a differential polynomial with coefficient \(a_j(z)\) and degree \(\overline{d}(P)\) and lower degree \(\underline{d}(P)\). Then
\[
m \left( r, \frac{P[f]}{\overline{d}(P)} \right) \leq [\overline{d}(P) - \underline{d}(P)] m \left( r, \frac{1}{r} \right) + S(r, f).
\]

**Lemma 2** ([4, Lemma of the logarithmic derivatives]). Let \(f(z)\) be meromorphic and non-constant in the plane. Then there are positive constants \(C_1\) and \(C_2\) such that
\[
m \left( r, \frac{f'}{f} \right) \leq C_1 \log r + C_2 \log T(r, f)
\]
as \(r\) tends to infinity outside possibly a set \(E\) of finite measure.

Consequently, Lemma 2 implies the famous result
\[
m \left( r, \frac{f^{(k)}}{f} \right) = S(r, f)
\]
for any positive integer \(k\) (See [4, Theorem 3.1]).

**Lemma 3.** Let \(f(z)\) be a meromorphic function with a pole of order \(p \geq 1\) at \(z_0\). If \(P[f]\) is a differential polynomial in \(f(z)\) whose coefficient are analytic at \(z_0\), then \(P[f]\) has a pole at \(z_0\) of order at most \(p \overline{d}(P) + \Gamma_P - \overline{d}(P)\).
Proof. Now $P[f]$ is a sum of terms of the form $a_jf^{n_{0j}}(f')^{n_{1j}}\cdots(f^{(k)})^{n_{kj}}$ where $a_j$ is analytic at $z_0$. If this term has a pole at $z_0$, then its order is at most

$$\max_{1 \leq j \leq n} \left\{ \sum_{s=0}^{k} (p+s)n_{sjj} \right\}$$

$$= \max_{1 \leq j \leq n} \left\{ (p-1) \sum_{s=0}^{k} n_{sjj} + (n_{0j} + 2n_{1j} + \cdots + (k+1)n_{kj}) \right\}$$

$$\leq (p-1)\overline{d}(P) + \Gamma P$$

$$\leq \overline{d}(P) + \Gamma P - \overline{d}(P),$$

completing the proof of the lemma. □

3. Our main results

**Theorem 1.** Let $f(z)$ be a transcendental meromorphic function satisfying condition (1) and let $P[f]$ be a differential polynomial in $f(z)$ of degree $\overline{d}(P)$ and lower degree $d(P)$. Then

$$d(P) \leq \lim_{r \to +\infty} \frac{T(r,P[f])}{T(r,f)} \leq \lim_{r \to +\infty} \frac{T(r,P[f])}{T(r,f)} \leq 2\overline{d}(P) - d(P).$$

Proof. The poles of $P[f]$ can occur only at the poles of $f$ or at the poles of the coefficients $a_j$ of $P[f]$. As $T(r,a_j) = S(r,f)$, we can ignore the poles of the coefficients $a_j$.

At $z_0$, a pole of $f$ of order $p$, it is easily seen from Lemma 3 that $P[f]$ has a pole $z_0$ of order at most $p\overline{d}(P) + \Gamma P - \overline{d}(P)$. Hence we have

$$N(r,P[f]) \leq \overline{d}(P)N(r,f) + [\Gamma P - \overline{d}(P)]N(r,f) + S(r,f)$$

and then this and the assumption (1) give

$$N\left(r, \frac{P[f]}{\overline{d}(P)}\right) \leq N(r,P[f]) + N\left(r, \frac{1}{\overline{f}(P)}\right)$$

$$\leq [\Gamma P - \overline{d}(P)]N(r,f) + \overline{d}(P)\left[N(r,f) + N\left(r, \frac{1}{f}\right)\right] + S(r,f)$$

(3)

On the one hand, it follows from (3), Lemma 1 and then the first fundamental theorem that

$$T\left(r, \frac{P[f]}{\overline{d}(P)}\right) \leq T\left(r, \frac{P[f]}{\overline{d}(P)}\right) + T\left(r, \overline{d}(P)\right)$$

$$\leq m\left(r, \frac{P[f]}{\overline{d}(P)}\right) + \overline{d}(P)T(r,f)$$

$$\leq [\overline{d}(P) - d(P)] m\left(r, \frac{1}{f}\right) + \overline{d}(P)T(r,f) + S(r,f)$$

(4)
\[
\leq [\overline{d}(P) - \underline{d}(P)] T\left(r, \frac{1}{f}\right) + \overline{d}(P) T(r, f) + S(r, f)
\]
\[
= [2\overline{d}(P) - \underline{d}(P)] T(r, f) + S(r, f).
\]

Thus inequality (4) implies that
\[
(5) \quad \lim_{r \to +\infty} \frac{T(r, P[f])}{T(r, f)} \leq 2\overline{d}(P) - \underline{d}(P).
\]

On the other hand, we also have from the first fundamental theorem, (3) and then Lemma 1 the following
\[
\overline{d}(P) T(r, f) \leq T\left(r, \frac{\overline{d}(P)}{P[f]}\right) + T(r, P[f])
\]
\[
\leq T\left(r, \frac{P[f]}{\overline{d}(P)}\right) + T(r, P[f]) + O(1)
\]
\[
\leq T(r, P[f]) + [\overline{d}(P) - \underline{d}(P)] m\left(r, \frac{1}{f}\right) + S(r, f)
\]
\[
\leq T(r, P[f]) + [\overline{d}(P) - \underline{d}(P)] T(r, f) + S(r, f)
\]
\[
(6) \quad \underline{d}(P) T(r, f) \leq T(r, P[f]) + S(r, f).
\]

Thus inequality (6) implies that
\[
(7) \quad \underline{d}(P) \leq \lim_{r \to +\infty} \frac{T(r, P[f])}{T(r, f)}.
\]

Hence by inequalities (5) and (7) we get
\[
\underline{d}(P) \leq \lim_{r \to +\infty} \frac{T(r, P[f])}{T(r, f)} \leq \lim_{r \to +\infty} \frac{T(r, P[f])}{T(r, f)} \leq 2\overline{d}(P) - \underline{d}(P),
\]
completing the proof of the theorem. \(\square\)

**Remark 1.** In particular, if the given differential polynomial is homogenous, i.e., \(\overline{d}(P) = \underline{d}(P) = n\) for some positive integer \(n\), then we obtain
\[
n \leq \lim_{r \to +\infty} \frac{T(r, P[f])}{T(r, f)} \leq \lim_{r \to +\infty} \frac{T(r, P[f])}{T(r, f)} \leq n,
\]
so that
\[
\lim_{r \to +\infty} \frac{T(r, P[f])}{T(r, f)} = n,
\]
outside possibly a set \(E\) of finite linear measure. In other words, we have
\[
(8) \quad T(r, P[f]) = nT(r, f) + O(1)
\]
as \(r \to +\infty\) outside possibly a set \(E\) of finite linear measure in this case.
Theorem 2. Let $f(z)$ be a transcendental meromorphic function satisfying the assumption (1) and let $P[f]$ be a differential polynomial in $f(z)$ of degree $\overline{d}(P)$ and lower degree $\underline{d}(P)$. Suppose that $P[f]$ does not reduce to a constant.

(a) If $P[f]$ is a homogeneous differential polynomial, then we have

$$\delta(a, P[f]) = 0$$

for any $a \neq 0$, i.e., $P[f]$ assumes all finite complex values except possibly zero infinitely often.

(b) If $P[f]$ is a non-homogeneous differential polynomial with $2\overline{d}(P) > \underline{d}(P)$, then we have

$$\delta(a, P[f]) \leq 1 - \frac{2\overline{d}(P) - \underline{d}(P)}{\overline{d}(P)} < 1$$

for any $a \neq 0$, i.e., $P[f]$ assumes all finite complex values except possibly zero infinitely often.

Proof. By Theorem 1, we see that small functions of $f$ are small functions of $P[f]$ and small functions of $P[f]$ are also small functions of $f$, i.e.,

$$S(r, f) = S(r, P[f]).$$

By (9), it follows from assumption (1) and inequality (2) that

$$N(r, P[f]) = S(r, P[f]).$$

We also have

$$\overline{N} \left( r, \frac{1}{P[f]} \right) \leq \overline{N} \left( r, \frac{1}{f^{\overline{d}(P)}} \right) + \overline{N} \left( r, \frac{f^{\underline{d}(P)}}{P[f]} \right)$$

$$\leq \overline{N} \left( r, \frac{1}{f} \right) + T \left( r, \frac{f^{\underline{d}(P)}}{P[f]} \right)$$

$$\leq \overline{N} \left( r, \frac{1}{f} \right) + T \left( r, \frac{P[f]}{f^{\underline{d}(P)}} \right) + O(1).$$

Now Lemma 1, inequalities (3) and (9) imply that

$$T \left( r, \frac{P[f]}{f^{\underline{d}(P)}} \right) = m \left( r, \frac{P[f]}{f^{\underline{d}(P)}} \right) + N \left( r, \frac{P[f]}{f^{\underline{d}(P)}} \right)$$

$$\leq [\overline{d}(P) - \underline{d}(P)] m \left( r, \frac{1}{f} \right) + S(r, f)$$

$$= [\overline{d}(P) - \underline{d}(P)] m \left( r, \frac{1}{f} \right) + S(r, P[f]).$$

Hence using (11), inequality (10) can be written as

$$\overline{N} \left( r, \frac{1}{P[f]} \right) \leq \overline{N} \left( r, \frac{1}{f} \right) + [\overline{d}(P) - \underline{d}(P)] m \left( r, \frac{1}{f} \right) + S(r, P[f]).$$
and by hypothesis (1) and (9), we get

\[ N \left( r, \frac{1}{P[f]} \right) \leq \overline{d}(P) - \underline{d}(P) m \left( r, \frac{1}{f} \right) + S \left( r, P[f] \right). \]

If \( b \neq 0 \), then the second fundamental theorem and inequality (12) imply that

\[ T \left( r, P[f] \right) \leq N \left( r, P[f] \right) + N \left( r, \frac{1}{P[f] - b} \right) + S \left( r, P[f] \right). \]

We have the following two cases.

**Case (a):** If \( P[f] \) is a homogeneous differential polynomial, i.e., \( \overline{d}(P) = \underline{d}(P) \) then by the above inequality (13) we obtain

\[ T \left( r, P[f] \right) \leq N \left( r, \frac{1}{P[f] - b} \right) + S \left( r, P[f] \right), \]

but it follows from (8) that \( P[f] \) is a transcendental meromorphic function and then this relation and inequality (14) imply (a).

**Case (b):** By Theorem 1, we still have \( \overline{d}(P) T(r, f) \leq T(r, P[f]) + S(r, f) \) for all sequences of \( r \) tending to \(+\infty\) outside possibly a set \( E \) of finite linear measure. If \( P[f] \) is a non-homogeneous differential polynomial with \( 2\overline{d}(P) > \underline{d}(P) \), then we obtain from inequality (13) that

\[ T \left( r, P[f] \right) \leq \left[ \overline{d}(P) - \underline{d}(P) \right] m \left( r, \frac{1}{f} \right) + N \left( r, \frac{1}{P[f] - b} \right) + S \left( r, P[f] \right). \]

Since \( 2\overline{d}(P) > \overline{d}(P) \), the desired result follows and thus we complete the proof of Theorem 2.

\[ \square \]

4. Further remarks

In this section, a few remarks will be given concerning the question we consider in this paper.

**Remark 2.** Our Theorem 2 is much more general than that of Gopalakrishna and Bhoosnurmath [3, pp. 334–335] because they obtained the inequality (14) for homogeneous \( P[f] \) only, but the main inequality we obtain here is (13) which works for any, homogeneous or non-homogeneous, differential polynomial \( P[f] \).
Remark 3. The following example shows that the condition $2d(P) > \bar{d}(P)$ cannot be dropped from Theorem 2(b).

**Example 1.** Let

$$f(z) = e^z \quad \text{and} \quad P[f] = f^2(z) + af(z) - af'(z) + 1$$

for any complex number $a$. Then we have $\bar{d}(P) = 2$ and $d(P) = 0$. However, $P[f] - 1 = e^z \neq 0$ for any $z$ and hence

$$\delta(1, P[f]) = 1.$$

Remark 4. We note that the condition (1) was used heavily in the proofs of Theorems D to F, and our two theorems here. In the remark made in [7, p. 201], Yang noted that Theorem D is also valid when the condition (1) is replaced by the weaker condition

$$(15) \quad N_1(r, f) + N_1\left(r, \frac{1}{f}\right) = S(r, f),$$

where $N_1(r, f)$ and $N_1\left(r, \frac{1}{f}\right)$ denote the counting functions of simple poles and simple zeros of $f(z)$ in $|z| \leq r$ respectively.

However, Yang [8] and Gopalakrishna and Bhoosnurmath [3] did not say whether Theorems E and F were still valid under the condition (15). Hence it is natural to conjecture that

**Conjecture.** Theorems 1 and 2 hold good even under the weaker condition (15).

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