ON UNIQUENESS OF MEROMORPHIC FUNCTIONS WHEN TWO DIFFERENTIAL MONOMIALS SHARE ONE VALUE

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Abstract. We prove four theorems on the uniqueness of non linear differential polynomials sharing one value which improve a result of Yang and Hua, and supplements some results of Lahiri, Xu and Qiu and Banerjee.

1. Introduction definitions and results

Let \( f \) and \( g \) be two nonconstant meromorphic functions defined in the open complex plane \( \mathbb{C} \). If for some \( a \in \mathbb{C} \cup \{\infty\} \), \( f - a \) and \( g - a \) have the same set of zeros with the same multiplicities, we say that \( f \) and \( g \) share the value \( a \) CM (counting multiplicities), and if we do not consider the multiplicities then \( f \) and \( g \) are said to share the value \( a \) IM (ignoring multiplicities). Let \( m \) be a positive integer or infinity and \( a \in \mathbb{C} \cup \{\infty\} \). We denote by \( E_m(a; f) \) the set of all \( a \)-points of \( f \) with multiplicities not exceeding \( m \), where an \( a \)-point is counted according to its multiplicity. Also we denote by \( E_m(a; f) \) the set of distinct \( a \)-points of \( f \) with multiplicities not greater than \( m \). If for some \( a \in \mathbb{C} \cup \{\infty\} \), \( E_\infty(a; f) = E_\infty(a; g) \) we say that \( f, g \) share the value \( a \) CM.

We denote by \( T(r) \) the maximum of \( T(r, f) \) and \( T(r, g) \). The notation \( S(r) \) denotes any quantity satisfying \( S(r) = o(T(r)) \) as \( r \to \infty \), outside of a possible exceptional set of finite linear measure.

We use \( I \) to denote any set of infinite linear measure of \( 0 < r < \infty \).

Yang and Hua [12] studied the problem of non linear differential polynomials when they share the same value \( a \) CM. They proved the following result.

**Theorem A** ([12]). Let \( f \) and \( g \) be two nonconstant meromorphic functions, \( n \geq 11 \) an integer and \( a \in \mathbb{C} - \{0\} \). If \( f^n f' \) and \( g^n g' \) share the value \( a \) CM, then either \( f = d g \) for some \((n + 1)\)th root of unity \( d \) or \( g(z) = c_1 e^{cz} \) and \( f(z) = c_2 e^{-cz} \), where \( c, c_1 \) and \( c_2 \) are constants satisfying \((c_1 c_2)^{n+1} c_2^2 = -a^2 \).

Corresponding to entire functions Xu and Qiu proved the following result.

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Theorem B ([10]). Let $f$ and $g$ be two nonconstant entire functions, $n \geq 12$ an integer and $a \in \mathbb{C} - \{0\}$. If $f^n f'$ and $g^n g'$ share the value $a$ IM, then either $f = dg$ for some $(n + 1)$th root of unity $d$ or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$, where $c, c_1$ and $c_2$ are constants satisfying $(c_1 c_2)^{n+1} c^2 = -a^2$.

To state the next result we require the following definition.

Definition 1.1 ([4, 5]). Let $k$ be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all $a$-points of $f$, where an $a$-point of multiplicity $m$ is counted $m$ times if $m \leq k$ and $k + 1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that $f, g$ share the value $a$ with weight $k$.

The definition implies that if $f, g$ share a value $a$ with weight $k$ then $z_0$ is an $a$-point of $f$ with multiplicity $m$ if and only if it is an $a$-point of $g$ with multiplicity $m$ if and only if it is an $a$-point of $f$ with multiplicity $m$ if and only if it is an $a$-point of $g$ with multiplicity $n$. If for $m$ is not necessarily equal to $n$.

We write $f, g$ share $(a, k)$ to mean that $f, g$ share the value $a$ with weight $k$. Clearly if $f, g$ share $(a, k)$, then $f, g$ share $(a, p)$ for any integer $p$, $0 \leq p < k$. Also write note that $f, g$ share a value $a$ IM or CM if and only if $f, g$ share $(a, 0)$ or $(a, \infty)$ respectively.

With the notion of weighted sharing of values improving Theorem A the following result is proved in [5].

Theorem C ([5]). Let $f$ and $g$ be two nonconstant meromorphic functions, $n \geq 12$ an integer and $a \in \mathbb{C} - \{0\}$. If $f^n f'$ and $g^n g'$ share $(a, 2)$, then either $f = dg$ for some $(n + 1)$th root of unity $d$ or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$, where $c, c_1$ and $c_2$ are constants satisfying $(c_1 c_2)^{n+1} c^2 = -a^2$.

In the same direction recently the present author improved Theorem B and supplement Theorem C. The present author proved the following two theorems.

Theorem D ([11]). Let $f$ and $g$ be two nonconstant meromorphic functions such that $n > 22 - \left[5 \Theta(\infty; f) + 5 \Theta(\infty; g) + \min\{\Theta(\infty; f), \Theta(\infty; g)\}\right]$, where $n$ is an integer. If for $a \in \mathbb{C} - \{0\}$, $f^n f'$ and $g^n g'$ share $(a, 0)$, then conclusion of the Theorem C holds.

Theorem E ([11]). Let $f$ and $g$ be two nonconstant meromorphic functions and $n > \max\{8, 12 - 3 \Theta(\infty; f) + 3 \Theta(\infty; g)\}$, an integer. If for $a \in \mathbb{C} - \{0\}$, $f^n f'$ and $g^n g'$ share $(a, 1)$, then conclusion of the Theorem C holds.

Now one may ask the following questions which are the motivations of the paper.

(i) In Theorem A can the nature of sharing the value $a$ be further relaxed other than the concept of weighted sharing?

(ii) Can one obtain the same result as in Theorem D and Theorem E with a different nature of sharing the value $a$?
In this paper we shall investigate the possible solutions of the above questions. We now state the following theorems which are the main results of the paper.

**Theorem 1.1.** Let \( f \) and \( g \) be two nonconstant meromorphic functions and \( n > 22 - \left[ 5 \Theta(\infty; f) + 5 \Theta(\infty; g) + \min\{ \Theta(\infty; f), \Theta(\infty; g) \} \right] \), an integer. If for \( a \in \mathbb{C} - \{0\} \), \( E_2)(a; f^n f') = E_2)(a; g^n g') \) then conclusion of the Theorem C holds.

**Theorem 1.2.** Let \( f \) and \( g \) be two nonconstant meromorphic functions and \( n > \max\{8, \frac{22}{3} - \{3 \Theta(\infty; f) + 3 \Theta(\infty; g) + \frac{1}{3} \min\{ \Theta(\infty; f), \Theta(\infty; g) \} \} \} \), an integer. If for \( a \in \mathbb{C} - \{0\} \), \( E_3)(a; f^n f') = E_3)(a; g^n g') \) and \( E_1)(a; f^n f') = E_1)(a; g^n g') \) then conclusion of the Theorem C holds.

**Theorem 1.3.** Let \( f \) and \( g \) be two nonconstant meromorphic functions and \( n \geq 11 \), an integer. If for \( a \in \mathbb{C} - \{0\} \), \( E_3)(a; f^n f') = E_3)(a; g^n g') \) and \( E_1)(a; f^n f') = E_1)(a; g^n g') \) then conclusion of the Theorem C holds.

**Theorem 1.4.** Let \( f \) and \( g \) be two nonconstant meromorphic functions and \( n \geq 11 \), an integer. If for \( a \in \mathbb{C} - \{0\} \), \( E_3)(a; f^n f') = E_3)(a; g^n g') \) and \( E_2)(a; f^n f') = E_2)(a; g^n g') \) then conclusion of the Theorem C holds.

**Remark 1.1.** Theorem 1.4 improves Theorem A.

Though the standard definitions and notations of the value distribution theory are available in [2], we explain some definitions and notations which are used in the paper.

**Definition 1.2** ([3]). For \( a \in \mathbb{C} \cup \{ \infty \} \) we denote by \( N(r,a; f) \) the counting function of simple \( a \) points of \( f \). For a positive integer \( m \) we denote by \( N(r,a; f | \leq m) \) the counting function of those \( a \) points of \( f \) whose multiplicities are not greater(less) than \( m \), where each \( a \) point is counted according to its multiplicity. \( \overline{N}(r,a; f | \leq m) \) are defined similarly, where in counting the \( a \)-points of \( f \) we ignore the multiplicities. Also \( N(r,a; f | < m) \), \( N(r,a; f | > m) \), \( \overline{N}(r,a; f | < m) \) and \( \overline{N}(r,a; f | > m) \) are defined analogously.

**Definition 1.3** ([5], [14]). We denote by \( N_2(r,a; f) \) the sum \( \overline{N}(r,a; f) + \overline{N}(r,a; f | \geq 2) \).

**Definition 1.4.** Let \( m \) and \( r \) be two positive integers such that \( 1 \leq r < m - 1 \) and for \( a \in \mathbb{C} \), \( E_m)(a; f) = E_m)(a; g) \), \( E_1)(a; f) = E_1)(a; g) \). Let \( z_0 \) be a zero of \( f(z) - a \) of multiplicity \( p \) and a zero of \( g(z) - a \) of multiplicity \( q \). We denote by \( \overline{N}_L(r,a; f)(\overline{N}_L(r,a; g)) \) the reduced counting function of those \( a \)-points of \( f \) and \( g \) for which \( p > q \geq r + 1 \) (\( q > p \geq r + 1 \)), by \( \overline{N}_E(r,a; f) \) the reduced counting function of those \( a \)-points of \( f \) and \( g \) for which \( p = q \geq r + 1 \), by \( \overline{N}_f(r,a; f \mid g \neq a) \) the reduced counting
functions of those \(a\)-points of \(f\) and \(g\) for which \(p \geq m + 1\) and \(q = 0\) \((q \geq m + 1\) and \(p = 0\)).

**Definition 1.5.** If \(r = 0\) in definition 1.4 then we use the same notations as in definition 1.4 except by \(N_E^1(r; a; f)\) we mean the common simple \(a\)-points of \(f\) and \(g\) and by \(N_E^2(r; a; f)\) we mean the reduced counting functions of those \(a\)-points of \(f\) and \(g\) for which \(p \geq m + 1\) and \(q = 0\) \((q \geq m + 1\) and \(p = 0\)).

**Definition 1.6.** Let \(E_m(a; f) = E_m(a; g)\) for \(a \in \mathbb{C}\). Also let \(z_0\) be a zero of \(f - a\) of multiplicity \(p\) and a zero of \(g - a\) of multiplicity \(q\). We denote by \(N(f > s(r; a; g))\) the reduced counting functions of those \(a\)-points of \(f\) and \(g\) for which \(p = q \geq 2\).

**Definition 1.7** ([6]). Let \(a, b \in \mathbb{C} \cup \{\infty\}\). We denote by \(N(r, a; f \mid g = b)\) the counting function of those \(a\)-points of \(f\), counted according to multiplicity, which are \(b\)-points of \(g\).

**Definition 1.8** ([6]). Let \(a, b \in \mathbb{C} \cup \{\infty\}\). We denote by \(N(r, a; f \mid g \neq b)\) the counting function of those \(a\)-points of \(f\), counted according to multiplicity, which are not the \(b\)-points of \(g\).

## 2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let \(f, g, F, G\) be four nonconstant meromorphic functions. Henceforth we shall denote by \(h\) and \(H\) the following two functions.

\[
h = \left( \frac{f''}{f'} - \frac{2f'}{f - 1} \right) - \left( \frac{g''}{g'} - \frac{2g'}{g - 1} \right)
\]

and

\[
H = \left( \frac{F''}{F'} - \frac{2F'}{F - 1} \right) - \left( \frac{G''}{G'} - \frac{2G'}{G - 1} \right).
\]

**Lemma 2.1.** If \(f, g\) be two nonconstant meromorphic functions such that \(E_{11}(1; f) = E_{11}(1; g)\) and \(h \neq 0\) then

\[
N(r, 1; f \mid 1) = N(r, 1; g \mid 1) \leq N(r, 0; h) \leq N(r, \infty; h) + S(r, f) + S(r, g).
\]

**Proof.** Since the functions \(f\) and \(g\) have the same simple one points it can be easily verified by direct computation that the function \(h\) is zero whenever \(f - 1\) has a simple zero. This proves the lemma. \(\square\)
Lemma 2.2. Let $E_{m}(1; f) = E_{m}(1; g)$, $E_{1}(1; f) = E_{1}(1; g)$ and $h \neq 0$, where $m \geq 3$. Then

$$N(r, \infty; h) \leq \overline{N}(r, 0; f \geq 2) + \overline{N}(r, 0; g \geq 2) + \overline{N}(r, \infty; f \geq 2) + \overline{N}(r, \infty; g \geq 2)$$

$$+ \overline{N}_{L}(r, 1; f) + \overline{N}_{L}(r, 1; g) + \overline{N}_{f \geq m+1}(1; f \neq 1)$$

$$+ \overline{N}_{f \geq m+1}(1; g \neq 1) + \overline{N}_{0}(r, 0; f') + \overline{N}_{0}(r, 0; g'),$$

where $\overline{N}_{0}(r, 0; f')$ is the reduced counting function of those zeros of $f'$ which are not the zeros of $f(f - 1)$ and $\overline{N}_{0}(r, 0; g)$ is similarly defined.

Proof. We can easily verify that possible poles of $h$ occur at (i) multiple zeros of $f$ and $g$, (ii) multiple poles of $f$ and $g$, (iii) the common zeros of $f - 1$ and $g - 1$ whose multiplicities are different, (iv) those 1-points of $f$ (g) which are not the 1-points of $g$ (f), (v) zeros of $f'$ which are not the zeros of $f(f - 1)$, (vi) zeros of $g'$ which are not zeros of $g(g - 1)$.

Since all the poles of $h$ are simple, the lemma follows from above. This proves the lemma.

Lemma 2.3. Let $E_{2}(1; f) = E_{2}(1; g)$ and $h \neq 0$. Then

$$N(r, \infty; h) \leq \overline{N}(r, 0; f \geq 2) + \overline{N}(r, 0; g \geq 2) + \overline{N}(r, \infty; f \geq 2) + \overline{N}(r, \infty; g \geq 2)$$

$$+ \overline{N}_{L}(r, 1; f) + \overline{N}_{L}(r, 1; g) + \overline{N}_{f \geq 3}(r, 1; f \neq 1)$$

$$+ \overline{N}_{f \geq 3}(r, 1; g \neq 1) + \overline{N}_{0}(r, 0; f') + \overline{N}_{0}(r, 0; g').$$

Proof. We omit the proof since the proof can be carried out in the line of proof of Lemma 2.2. This proves the lemma.

Lemma 2.4 ([7]). If $N(r, 0; f^{(k)}) \mid f \neq 0$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of $f$, where a zero of $f^{(k)}$ is counted according to its multiplicity then

$$N(r, 0; f^{(k)} \mid f \neq 0) \leq k\overline{N}(r, \infty; f) + N(r, 0; f \mid k) + k\overline{N}(r, 0; f \mid \geq k) + S(r, f).$$

Lemma 2.5. Let $E_{2}(1; f) = E_{2}(1; g)$. Then

$$\overline{N}_{L}(r, 1; f) + \overline{N}_{L}(r, 1; g) + \overline{N}_{f^{2}}(r, 1; f)$$

$$+ 2\overline{N}_{f \geq 3}(r, 1; g \neq 1) - \overline{N}_{f > 1}(r, 1; g)$$

$$\leq N(r, 1; g) - \overline{N}(r, 1; g).$$

Proof. Let $z_{0}$ be a 1-point of $f$ with multiplicity $p$ and a $a$-point of $g$ with multiplicity $q$. If $q = 3$ the possible values of $p$ are as follows (i) $p = 3$ (ii) $p \geq 4$ (iii) $p = 0$. Similarly when $q = 4$ the possible values of $p$ are (i) $p = 3$ (ii) $p = 4$ (iii) $p \geq 5$ (iv) $p = 0$. If $q \geq 5$ we can similarly find the possible values of $p$. Since $E_{2}(1; f) = E_{2}(1; g)$, we note that the simple 1-points of $f$ are either simple or double 1 points of $g$ and the double 1 points of $f$ are either simple or
double 1-points of $g$. Now the lemma follows from above. This completes the proof of the lemma.

\[ \square \]

**Lemma 2.6.** Let $E_m(1; f) = E_m(1; g)$, $E_{1H}(1; f) = E_{1H}(1; g)$ and $h \neq 0$, where $m \geq 3$. Then

\[
2N_L(r, 1; f) + 2N_L(r, 1; g) + N_{E}^{(2)}(r, 1; f) + mN_{g > m+1}(r, 1; g | f \neq 1) - N_{f > 2}(r, 1; g) \leq N(r, 1; g) - N(r, 1; g).
\]

**Proof.** Since the given condition implies that the simple 1-points of $f$ and $g$ are same the proof of the lemma can be carried out in the line of proof of Lemma 2.5. This proves the lemma.

\[ \square \]

**Lemma 2.7.** Let $E_2(1; f) = E_2(1; g)$. Then

\[
\begin{align*}
N_{f > 1}(r, 1; g) + 2N_{f \geq 3}(r, 1; f | g \neq 1) & \leq N(r, 0; f) + N(r, \infty; f) - N_0(r, 0; f') + S(r, f),
\end{align*}
\]

where $N_0(r, 0; f')$ is the counting function of those zeros of $f'$ which are not the zeros of $f(f - 1)$ counting according to multiplicity.

**Proof.** We note that a 1-point of $f$ with multiplicity 2 is counted only once in the counting function $N_{f > 1}(r, 1; g)$. Also since a 1 point of $f$ with multiplicity $\geq 3$ may or may not be a 1-point of $g$, the 1-points of $f$ are counted either once or twice according as those points are counted in $N_{f > 1}(r, 1; g)$ or in $N_{f \geq 3}(r, 1; f | g \neq 1)$ respectively. In other words any 1 point of $f$ with multiplicity $\geq 3$ is counted at most twice.

Considering the above and using Lemma 2.4 we see that

\[
\begin{align*}
N_{f > 1}(r, 1; g) + 2N_{f \geq 3}(r, 1; f | g \neq 1) & \leq N(r, 0; f' | f = 1) \\
& \leq N(r, 0; f' | f \neq 0) - N_0(r, 0; f') \\
& \leq N(r, 0; f) + N(r, \infty; f) - N_0(r, 0; f') + S(r, f).
\end{align*}
\]

This proves the lemma.

\[ \square \]

**Lemma 2.8.** Let $E_2(1; f) = E_2(1; g)$. Then

(i) $N_L(r, 1; f) \leq N(r, 0; f) + N(r, \infty; f) - N_0(r, 0; f') + S(r, f)$.

(ii) $N_L(r, 1; g) \leq N(r, 0; g) + N(r, \infty; g) - N_0(r, 0; g') + S(r, g)$. 


Proof. We prove (i) because (ii) can be proved in a similar manner. Using Lemma 2.4 we obtain
\[
N_L(r, 1; f) \leq \mathcal{N}(r, 1; f \geq 2) \\
\leq N(r, 0; f' \mid f = 1) \\
\leq N(r, 0; f' \mid f \neq 0) - N_0(r, 0; f') \\
\leq \mathcal{N}(r, 0; f) + \mathcal{N}(r, \infty; f) - N_0(r, 0; f') + S(r, f).
\]
This proves the lemma. □

Lemma 2.9. Let \(E_3(1; f) = E_3(1; g)\) and \(E_4(1; f) = E_4(1; g)\). Then
\[
N_f > 2(r, 1; g) + 2 \mathcal{N}_f \geq 4(r, 1; f \mid g \neq 1) \\
\leq \frac{2}{3} \mathcal{N}(r, 0; f) + \frac{2}{3} \mathcal{N}(r, \infty; f) - \frac{2}{3} N_0(r, 0; f') + S(r, f).
\]
Proof. Using Lemma 2.4 we get
\[
N_{f > 2}(r, 1; g) + 2 \mathcal{N}_{f \geq 4}(r, 1; f \mid g \neq 1) \\
\leq \mathcal{N}(r, 1; f \geq 3; g \mid = 2) + 2 \mathcal{N}(r, 1; f \mid g \neq 1) \\
\leq \frac{2}{3} N(r, 0; f' \mid f = 1) \\
\leq \frac{2}{3} \mathcal{N}(r, 0; f) + \frac{2}{3} \mathcal{N}(r, \infty; f) - \frac{2}{3} N_0(r, 0; f') + S(r, f),
\]
where by \(\mathcal{N}(r, 1; f \geq 3; g \mid = 2)\) we mean the reduced counting function of 1 points of \(f\) with multiplicity not less than 3 which are the 1-points of \(g\) with multiplicity 2. This completes the proof of the lemma. □

Lemma 2.10. Let \(E_4(1; f) = E_4(1; g)\) and \(E_2(1; f) = E_2(1; g)\). Then
\[
N_{f > 2}(r, 1; g) + 2 \mathcal{N}_{f \geq 4}(r, 1; f \mid g \neq 1) \\
\leq \frac{1}{2} \mathcal{N}(r, 0; f) + \frac{1}{2} \mathcal{N}(r, \infty; f) - \frac{1}{2} N_0(r, 0; f') + S(r, f).
\]
Proof. We omit the proof since the lemma can be proved in the line of proof of Lemma 2.9. This proves the lemma. □

Lemma 2.11 ([16]). If \(h \equiv 0\), then \(f\) and \(g\) share 1 CM.

Lemma 2.12 ([9, 12]). If \(f\), \(g\) share 1-CM, then one of the following cases holds
\[
\text{(i)} \ T(r, f) + T(r, g) \leq 2\{N_2(r, 0; f) + N_2(r, 0; g) + N_2(r, \infty; f) + N_2(r, \infty; g)\} + S(r, f) + S(r, g) \\
\text{(ii)} \ f \equiv g \\
\text{(iii)} \ fg \equiv 1.
\]

Lemma 2.13. Let \(E_4(1; f) = E_4(1; g)\) \(E_2(1; f) = E_2(1; g)\) then the conclusion of Lemma 2.12 holds.
Proof. Let \( h \equiv 0 \). Then the result follows from Lemma 2.11 and Lemma 2.12. So we suppose that \( h \not\equiv 0 \). Then by the second fundamental theorem, Lemma 2.1 and 2.2 we get

\[
T(r, f) + T(r, g) \leq \mathcal{N}(r, 0; f) + \mathcal{N}(r, \infty; f) + \mathcal{N}(r, 0; g) + \mathcal{N}(r, \infty; g)
\]

\[
+ \mathcal{N}(r, 1; f \mid 1) + \mathcal{N}(r, 1; f \mid \geq 2) + \mathcal{N}(r, 1; g)
\]

\[
- N_0(r, 0; f') - N_0(r, 0; g') + S(r, f) + S(r, g)
\]

\[
\leq N_2(r, 0; f) + N_2(r, \infty; f) + N_2(r, 0; g) + N_2(r, \infty; g)
\]

\[
+ \mathcal{N}_L(r, 1; f) + \mathcal{N}_L(r, 1; g) + \mathcal{N}_{f \geq 5}(r, 1; f \mid f \not\equiv 1)
\]

\[
+ \mathcal{N}_{g \geq 5}(r, 1; g \mid f \neq 1) + \mathcal{N}(r, 1; f \mid \geq 2) + \mathcal{N}(r, 1; g)
\]

\[
+ S(r, f) + S(r, g).
\]

Since

\[
\mathcal{N}(r, 1; f \mid = 4; g \mid = 3) + \mathcal{N}(r, 1; f \mid = 4) \leq 2 \mathcal{N}(r, 1; f \mid = 4)
\]

and

\[
\mathcal{N}(r, 1; f \mid = 3; g \mid = 4) + \mathcal{N}(r, 1; g \mid = 4) \leq 2 \mathcal{N}(r, 1; g \mid = 4),
\]

we see that

\[
\mathcal{N}_L(r, 1; f) + \mathcal{N}_L(r, 1; g) + \mathcal{N}_{f \geq 5}(r, 1; f \mid g \neq 1) + \mathcal{N}_{g \geq 5}(r, 1; g \mid f \neq 1)
\]

\[
+ \mathcal{N}(r, 1; f \mid \geq 2) + \mathcal{N}(r, 1; g)
\]

\[
\leq \mathcal{N}(r, 1; f \mid = 4; g \mid = 3) + \mathcal{N}(r, 1; f \mid \geq 6) + \mathcal{N}(r, 1; g \mid = 4; f \mid = 3)
\]

\[
+ \mathcal{N}(r, 1; g \mid \geq 6) + \mathcal{N}(r, 1; f \mid \geq 5) + \mathcal{N}(r, 1; g \mid \geq 5) + \mathcal{N}(r, 1; f \mid = 2)
\]

\[
+ \mathcal{N}(r, 1; f \mid = 3) + \mathcal{N}(r, 1; f \mid = 4) + \mathcal{N}(r, 1; f \mid \geq 5) + \mathcal{N}(r, 1; g \mid = 1)
\]

\[
+ \mathcal{N}(r, 1; g \mid = 2) + \mathcal{N}(r, 1; g \mid = 3) + \mathcal{N}(r, 1; g \mid = 4) + \mathcal{N}(r, 1; g \mid \geq 5)
\]

\[
\leq \frac{1}{2} \mathcal{N}(r, 1; f \mid = 1) + \frac{1}{2} \mathcal{N}(r, 1; g \mid = 1) + \mathcal{N}(r, 1; f \mid = 2) + \mathcal{N}(r, 1; g \mid = 2)
\]

\[
+ \mathcal{N}(r, 1; f \mid = 3) + \mathcal{N}(r, 1; g \mid = 3) + 2 \mathcal{N}(r, 1; f \mid = 4) + 2 \mathcal{N}(r, 1; g \mid = 4)
\]

\[
+ 2 \mathcal{N}(r, 1; f \mid \geq 5) + 2 \mathcal{N}(r, 1; g \mid \geq 5) + \mathcal{N}(r, 1; f \mid \geq 6) + \mathcal{N}(r, 1; g \mid \geq 6)
\]

\[
\leq \frac{1}{2} [\mathcal{N}(r, 1; f) + \mathcal{N}(r, 1; g)]
\]

\[
\leq \frac{1}{2} [T(r, f) + T(r, g)].
\]

Now the lemma follows from (2.1). This completes the proof of the lemma. \(\square\)

**Lemma 2.14** ([1], [15]). If \( h \equiv 0 \) and

\[
\limsup_{r \to \infty} \frac{\mathcal{N}(r, 0; f) + \mathcal{N}(r, \infty; f) + \mathcal{N}(r, 0; g) + \mathcal{N}(r, \infty; g)}{T(r)} < 1,
\]

then \( f \equiv g \) or \( f.g \equiv 1 \).
Lemma 2.15 ([8], [11]). Let $f$ be a nonconstant meromorphic function and $P(f) = a_0 + a_1 f + a_2 f^2 + \cdots + a_n f^n$, where $a_0, a_1, a_2, \ldots, a_n$ are constants and $a_n \neq 0$. Then $T(r, P(f)) = nT(r, f) + O(1)$.

Lemma 2.16 ([1]). Let $f$ be a nonconstant meromorphic function and $F = \frac{f^{n+1}}{a(n+1)}$, $n$ being a positive integer. Then $T(r, F) \leq T(r, F') + N(r, 0; f) - N(r, 0; f') + S(r, f)$.

Lemma 2.17 ([1]). Let $f, g$ be two nonconstant meromorphic functions and $F = \frac{f^{n+1}}{a(n+1)}$, $G = \frac{g^{n+1}}{a(n+1)}$, where $n(> 2)$ is an integer. Then $F' \equiv G'$ implies $F \equiv G$.

Lemma 2.18 ([12]). Let $f, g$ be two nonconstant meromorphic functions and $n > 6$. If $f^n g' g'' = 1$, then $g = c_1 e^{cz}$, $f = c_2 e^{-cz}$, where $c, c_1, c_2$ are constants and $(c_1 c_2)^{n+1} e^{2cz} = -1$.

Lemma 2.19. Let $E_2(1; f) = E_2(1; g)$ and $h \neq 0$. Then

$$T(r, f) \leq N_2(r, 0; f) + N_2(r, \infty; f) + N_2(r, 0; g) + N_2(r, \infty; g) + N_L(r, 1; f) + N_L(r, 1; g) + 2 N_{f \geq 3}(r, 1; f | g \neq 1) + S(r, f) + S(r, g).$$

Proof. By the second fundamental theorem we get

$$T(r, f) + T(r, g) \leq N(r, 0; f) + N(r, \infty; f) + N(r, 0; g) + N(r, \infty; g) + N(r, 1; f) + N(r, 1; g) - N_0(r, 0; f') - N_0(r, 0; g') + S(r, f) + S(r, g).$$

If $z_0$ be a common simple zero of $f - 1$ and $g - 1$ then it is easy to verify that $h(z_0) = 0$. Since $h \neq 0$,

$$N_1^1(r, 1; f) \leq N(r, 0; h) \leq T(r, h) + O(1) \leq N(r, \infty; h) + S(r, f) + S(r, g).$$

Hence by Lemmas 2.3 and 2.5 we get

$$T(r, f) + T(r, g) \leq N(r, 1; f) + N(r, 1; g) + N_L(r, 1; f) + N_L(r, 1; g) + N_{f \geq 3}(r, 1; f | g \neq 1) + N(r, 1; g).$$
\[ \leq \mathcal{N}(r, 0; f \mid \geq 2) + \mathcal{N}(r, \infty; f \mid \geq 2) + \mathcal{N}(r, 0; g \mid \geq 2) + \mathcal{N}(r, \infty; g \mid \geq 2) \]
\[ + \mathcal{N}(r, \infty; g \mid \leq 2) + \mathcal{N}(r, 1; f) + \mathcal{N}(r, 1; g) \]
\[ + \mathcal{N}_{f \geq 3}(r, 1; f \mid g \neq 1) + \mathcal{N}_{g \geq 3}(r, 1; g \mid f \neq 1) \]
\[ + \mathcal{N}(r, 1; f) + \mathcal{N}(r, 1; g) + \mathcal{N}^2_E(r, 1; f) \]
\[ + \mathcal{N}(r, 1; f \mid g \neq 1) + T(r, g) - m(r, 1; g) \]
\[ + O(1) - \mathcal{N}_{L}(r, 1; f) - \mathcal{N}_{L}(r, 1; g) - \mathcal{N}^2_E(r, 1; f) \]
\[ - 2 \mathcal{N}_{g \geq 3}(r, 1; g \mid f \neq 1) + \mathcal{N}_{f > 1}(r, 1; g) + \mathcal{N}_0(r, 0; f') \]
\[ + \mathcal{N}_0(r, 0; g') + S(r, f) + S(r, g) \]
\[ \leq \mathcal{N}(r, 0; f \mid \geq 2) + \mathcal{N}(r, \infty; f \mid \geq 2) + \mathcal{N}(r, 0; g \mid \geq 2) \]
\[ + \mathcal{N}(r, \infty; g \mid \geq 2) + T(r, g) - m(r, 1; g) + \mathcal{N}(r, 1; f) \]
\[ + \mathcal{N}(r, 1; g) + 2 \mathcal{N}_{f \geq 3}(r, 1; f \mid g \neq 1) + \mathcal{N}_{f > 1}(r, 1; g) \]
\[ - 2 \mathcal{N}_{g \geq 3}(r, 1; g \mid f \neq 1) + \mathcal{N}_0(r, 0; f') + \mathcal{N}_0(r, 0; g') \]
\[ + S(r, f) + S(r, g) \]

From (2.2) and (2.3) the lemma follows. This proves the lemma. \(\square\)

**Lemma 2.20.** Let \(E_{3,j}(1; f) = E_{3,j}(1; g), E_{1,j}(1; f) = E_{1,j}(1; g)\) and \(h \neq 0\). Then

\[ T(r, f) \leq N_2(r, 0; f) + N_2(r, \infty; f) + N_2(r, 0; g) + N_2(r, \infty; g) \]
\[ + \mathcal{N}_{f \geq 3}(r, 1; g) + 2 \mathcal{N}_{f \geq 3}(r, 1; f \mid g \neq 1) \]
\[ - 2 \mathcal{N}_{g \geq 4}(r, 1; g \mid f \neq 1) - m(r, 1; g) \]
\[ + S(r, f) + S(r, g) \]

**Proof.** Using Lemmas 2.2 and 2.6 we note that

\[ (2.4) \quad \mathcal{N}(r, 1; f) + \mathcal{N}(r, 1; g) \]
\[ \leq N(r, 1; f \mid = 1) + \mathcal{N}_{L}(r, 1; f) + \mathcal{N}_{L}(r, 1; g) + \mathcal{N}^2_E(r, 1; f) \]
\[ + \mathcal{N}_{f \geq 3}(r, 1; f \mid g \neq 1) + \mathcal{N}(r, 1; g) \]
\[ \leq \mathcal{N}(r, 0; f \mid \geq 2) + \mathcal{N}(r, \infty; f \mid \geq 2) + \mathcal{N}(r, 0; g \mid \geq 2) \]
\[ + \mathcal{N}(r, \infty; g \mid \geq 2) + \mathcal{N}_{L}(r, 1; f) + \mathcal{N}_{L}(r, 1; g) \]
\[ + \mathcal{N}_{f \geq 3}(r, 1; f \mid g \neq 1) + \mathcal{N}_{g \geq 4}(r, 1; g \mid f \neq 1) \]
\[ + \mathcal{N}_{L}(r, 1; f) + \mathcal{N}_{L}(r, 1; g) + \mathcal{N}^2_E(r, 1; f) \]
\[ + \mathcal{N}_{f \geq 3}(r, 1; f \mid g \neq 1) + T(r, g) - m(r, 1; g) \]
\[ + O(1) - 2 \mathcal{N}_{L}(r, 1; f) - 2 \mathcal{N}_{L}(r, 1; g) - \mathcal{N}^2_E(r, 1; f) \]
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\[ -3 \mathcal{N}_{g \geq 2}(r, 1; g \mid f \neq 1) + \mathcal{N}_{f \geq 2}(r, 1; g) + \mathcal{N}_0(r, 0; f') \]
+ \[\mathcal{N}_0(r, 0; g') + S(r, f) + S(r, g)\]
\[ \leq \mathcal{N}(r, 0; f \mid g \geq 2) + \mathcal{N}(r, \infty; f \mid g \geq 2) + \mathcal{N}(r, 0; g \mid g \geq 2) \]
+ \[\mathcal{N}(r, \infty; g \mid g \geq 2) + T(r, g) - m(r, 1; g)\]
+ \[2 \mathcal{N}_{f \geq 2}(r, 1; f \mid g \neq 1) + \mathcal{N}_{f > 2}(r, 1; g)\]
- \[2 \mathcal{N}_{g \geq 2}(r, 1; g \mid f \neq 1) + \mathcal{N}_0(r, 0; f')\]
+ \[\mathcal{N}_0(r, 0; g') + S(r, f) + S(r, g)\].

From (2.2) and (2.4) the lemma follows. This proves the lemma. \(\square\)

**Lemma 2.21.** Let \(E_{41}(1; f) = E_{41}(1; g), E_{11}(1; f) = E_{11}(1; g)\) and \(h \neq 0\). Then
\[ T(r, f) \leq N_2(r, 0; f) + N_2(r, \infty; f) + N_2(r, 0; g) + N_2(r, \infty; g) \]
+ \[\mathcal{N}_{f \geq 2}(r, 1; g) + 2 \mathcal{N}_{f > 2}(r, 1; f \mid g \neq 1)\]
- \[2 \mathcal{N}_{g \geq 2}(r, 1; g \mid f \neq 1) - m(r, 1; g)\]
+ \[\mathcal{N}(r, f) + S(r, g)\].

**Proof.** We omit the proof since the proof can be carried out in the line of proof of Lemma 2.19. This completes the proof of the lemma. \(\square\)

**Lemma 2.22** ([13]). Let \(f\) be a nonconstant meromorphic function. Then
\[ N(r, 0; f^{(k)}) \leq k\mathcal{N}(r, \infty; f) + N(r, 0; f) + S(r, f). \]

3. Proofs of the theorems

**Proof of Theorem 1.1.** Let \(F = \frac{f^{n+1}}{a^{(n+1)}}\) and \(G = \frac{g^{n+1}}{a^{(n+1)}}\). Then \(F' = \frac{F'}{a}\) and \(G' = \frac{G'}{a}\). Since \(E_{21}(a; f^{a}f') = E_{21}(a; g^{a}g')\), it follows that \(E_{21}(1; F') = \frac{E_{21}(1; G')}{E_{21}(1; G')}.\) If possible, we suppose that \(H \neq 0\). Then by Lemmas 2.7, 2.8 and 2.19 we obtain
\[ (3.1) \quad T(r, F') \leq N_2(r, 0; F') + N_2(r, \infty; F') + N_2(r, 0; G') \]
+ \[2 \mathcal{N}(r, 0; F') + 2 \mathcal{N}(r, \infty; F') + \mathcal{N}(r, 0; G') + \mathcal{N}(r, \infty; G') + S(r, F') + S(r, G').\]

We see that
\[ N_2(r, 0; F') + N_2(r, \infty; F') \leq 2\mathcal{N}(r, 0; f) + N(r, 0; f') + 2\mathcal{N}(r, \infty; f), \]
\[ N_2(r, 0; G') + N_2(r, \infty; G') \leq 2\mathcal{N}(r, 0; g) + N(r, 0; g') + 2\mathcal{N}(r, \infty; g), \]
\[ 2 \mathcal{N}(r, 0; F') + 2 \mathcal{N}(r, \infty; F') \leq 2\mathcal{N}(r, 0; f) + 2N(r, 0; f') + 2\mathcal{N}(r, \infty; f) \]
and
\[ \mathcal{N}(r, 0; G') + \mathcal{N}(r, \infty; G') \leq \mathcal{N}(r, 0; g) + N(r, 0; g') + \mathcal{N}(r, \infty; g). \]
Also by Lemma 2.15 we get
\[ T(r, F') \leq 2T(r, F) + S(r, F) = 2(n + 1)T(r, f) + S(r, f) \]
and
\[ T(r, G') \leq 2T(r, G) + S(r, G) = 2(n + 1)T(r, g) + S(r, g). \]
So \( S(r, F') \) and \( S(r, G') \) can be replaced by \( S(r, f) \) and \( S(r, g) \) respectively. So by Lemmas 2.16 and 2.22 we get from (3.1) for \( \varepsilon > 0 \)
\[ T(r, F) \leq T(r, F') + N(r, 0; f') - N(r, 0; f) + S(r, f) \]
\[ \leq 4N(r, 0; f) + N(r, 0; f') + 3N(r, 0; g) + 4N(r, \infty; f) + 3N(r, \infty; g) + 2N(r, 0; g') + S(r, f) + S(r, g) \]
\[ \leq 7T(r, f) + 5T(r, g) + (6 - 6\Theta(\infty; f) + \varepsilon)T(r, f) + (5 - 5\Theta(\infty; g) + \varepsilon)T(r, g) + S(r, f) + S(r, g) \]
\[ \leq \{23 - 6\Theta(\infty; f) - 5\Theta(\infty; g) + 2\varepsilon\}T(r) + S(r). \]
So using Lemma 2.15 we get
\[ (n + 1)T(r, f) \leq \{23 - 6\Theta(\infty; f) - 5\Theta(\infty; g) + 2\varepsilon\}T(r) + S(r). \]
In a similar manner we obtain
\[ (n + 1)T(r, g) \leq \{23 - 5\Theta(\infty; f) - 6\Theta(\infty; g) + 2\varepsilon\}T(r) + S(r). \]
From (3.2) and (3.3) we obtain
\[ [n - 22 + 5\Theta(\infty; f) + 5\Theta(\infty; g) + \min\{\Theta(\infty; f), \Theta(\infty; g)\} - 2\varepsilon]T(r) \leq S(r). \]
Since \( \varepsilon > 0 \) is arbitrary, (3.4) implies a contradiction. Hence \( H \equiv 0. \)
Since
\[ N(r, 0; f') \leq T(r, f') - m(r, \frac{1}{T'}) \leq 2T(r, f) - m(r, \frac{1}{T'}) + S(r, f), \]
we note that
\[ \left. \begin{array}{l}
N(r, 0; F') + N(r, \infty; F') + N(r, 0; G') + N(r, \infty; G') \\
\leq N(r, 0; f) + N(r, \infty; f) + N(r, 0; g) + N(r, \infty; g) + N(r, 0; g') + N(r, 0; g') + S(r) \\
\leq 4T(r, f) + 4T(r, g) - m(r, 0; f') - m(r, 0; g') + S(r) \\
\end{array} \right\} \leq 8T(r) - m(r, 0; f') - m(r, 0; g') + S(r). \]
Also using Lemma 2.15 we get
\[ T(r, F') + m(r, \frac{1}{T'}) = m(r, \frac{f''}{a}) + m(r, \frac{1}{T'}) + N(r, \infty; \frac{f''}{a}) \]
\[ \geq m(r, \frac{f''}{a}) + N(r, \infty; f'') \]
\[ = T(r, f'') + O(1) \]
\[ = nT(r, f) + O(1). \]
Similarly

\begin{equation}
T(r, G') + m(r, \frac{1}{g'}) \geq nT(r, g) + O(1).
\end{equation}

From (3.6) and (3.7) we get

\begin{equation}
\max \{T(r, F'), T(r, G')\} \geq nT(r) - m(r, \frac{1}{f}) - m(r, \frac{1}{g}) + O(1).
\end{equation}

By (3.5) and (3.8) applying Lemma 2.14 we get either \( F' \equiv G' \) or \( F'G' \equiv 1 \).

If \( F' \equiv G' \), then by Lemma 2.17 we obtain \( F \equiv G \) or \( f \equiv dg \), where \( d \) is some \((n+1)\) th root of unity.

If \( F'G' \equiv 1 \) then \( f^n f'g^n g' = a^2 \). Set \( f_1 = a^{-\frac{1}{n+1}}f \) and \( g_1 = a^{-\frac{1}{n+1}}g \), then \( f_1^nf_1g_1 = 1 \). So using Lemma 2.18 we get \( g = c_1e^{cz}, f = c_2e^{-cz} \), where \( c, c_1 \) and \( c_2 \) are constants and satisfy \((c_1c_2)^{n+1}c^2 = -a^2\). This completes the proof of the theorem.

\textbf{Proof of Theorem 1.3.} Let \( F \) and \( G \) be defined as in the proof of Theorem 1.1. Since \( E_{41}(a; f^n f') = E_{41}(a; g^n g') \) and \( E_{41}(a; f^n f') = E_{41}(a; g^n g') \) it follows that \( E_{41}(1; F') = E_{41}(1; G') \) and \( E_{41}(1; F') = E_{41}(1; G') \). If possible, we suppose that \( H \neq 0 \). Then by Lemmas 2.10 and 2.21 we obtain

\begin{equation}
T(r, F') \leq N_2(r, 0; F') + N_2(r, \infty; F') + N_2(r, 0; G') + N_2(r, \infty; G') \leq N_2(r, 0; F') + N_2(r, \infty; F') + \frac{1}{2} N(r, 0; F') + \frac{1}{2} N(r, \infty; F') + S(r, F') + S(r, G').
\end{equation}

We see that

\begin{align*}
N_2(r, 0; F') + N_2(r, \infty; F') &\leq 2N(r, 0; f) + N(r, 0; f') + 2N(r, \infty; f), \\
N_2(r, 0; G') + N_2(r, \infty; G') &\leq 2N(r, 0; g) + N(r, 0; g') + 2N(r, \infty; g),
\end{align*}

and

\begin{equation}
\frac{1}{2} N(r, 0; F') + \frac{1}{2} N(r, \infty; F') \leq \frac{1}{2} [N(r, 0; f) + N(r, 0; f') + N(r, \infty; f)].
\end{equation}

Again using Lemma 2.15 and proceeding in the same way as done in the proof of Theorem 1.1 we can show that \( S(r, F') \) and \( S(r, G') \) can be replaced by \( S(r, f) \) and \( S(r, g) \) respectively. So by Lemma 2.16 and Lemma 2.22 we obtain from (3.9) for \( \varepsilon > 0 \)

\begin{align*}
T(r, F) &\leq T(r, F') + N(r, 0; f) - N(r, 0; f') + S(r, f) \\
&\leq 2N(r, 0; f) + \frac{1}{2} N(r, 0; f') + \frac{3}{2} N(r, 0; f) + 2N(r, 0; g) + N(r, 0; g) \\
&\quad + 3N(r, \infty; f) + 3N(r, \infty; g) + S(r, f) + S(r, g) \\
&\leq (7 - 3\Theta(\infty; f) + \varepsilon)T(r, f) + (6 - 3\Theta(\infty; g) + \varepsilon)T(r, g) + S(r) \\
&\leq (13 - 3\Theta(\infty; f) - 3\Theta(\infty; g) + 2\varepsilon)T(r) + S(r).
\end{align*}
So using Lemma 2.15 we get

\[(n + 1)T(r, f) \leq \{13 - 3\Theta(\infty; f) - 3\Theta(\infty; g) + 2\varepsilon\}T(r) + S(r).\]

Similarly we can obtain

\[(n + 1)T(r, g) \leq \{13 - 3\Theta(\infty; f) - 3\Theta(\infty; g) + 2\varepsilon\}T(r) + S(r).\]

From (3.10) and (3.11) we obtain

\[\frac{n - 12 + 3\Theta(\infty; f) + 3\Theta(\infty; g) - 2\varepsilon}{3} \leq S(r).\]

Since \(\varepsilon(> 0)\) is arbitrary, we get a contradiction from (3.12). Hence \(H \equiv 0\).

Now proceeding in the same way as in the proof of Theorem 1.1 we obtain either \(F' \equiv G'\) or \(F'G' \equiv 1\). Again proceeding in the same manner as in the proof of Theorem 1.1 we obtain the conclusion of Theorem 1.3. This proves the theorem.

\[\Box\]

Proof of Theorem 1.2. Let \(F\) and \(G\) be defined as in the proof of Theorem 1.1. Then by the given condition of the theorem it follows that \(E_3(1; F') = E_3(1; G')\) and \(E_1(1; F') = E_1(1; G')\). Suppose that \(H \equiv 0\). Then using Lemmas 2.9, 2.15 and 2.20 and proceeding in the same way as in the proof of Theorem 1.1 we can obtain

\[(n + 1)T(r, f) = T(r, F') \leq \frac{41}{3} - \frac{10}{3} \Theta(\infty; f) - 3 \Theta(\infty; f) + 2 \varepsilon T(r) + S(r).\]

Similarly we get

\[(n + 1)T(r, g) = T(r, G') \leq \frac{41}{3} - \frac{10}{3} \Theta(\infty; f) - 3 \Theta(\infty; f) + 2 \varepsilon T(r) + S(r).\]

From (3.13) and (3.14) we obtain

\[\frac{n - 38}{3} + 3 \Theta(\infty; f) + 3 \Theta(\infty; g) + \frac{1}{3} \{\Theta(\infty; f), \Theta(\infty; g)\} - 2\varepsilon \leq S(r).\]

Since \(\varepsilon(> 0)\) is arbitrary, (3.15) leads to a contradiction. Hence \(H \equiv 0\).

Now the theorem can be proved in the line of the proof of Theorem 1.1. This proves the theorem.

\[\Box\]

Proof of Theorem 1.4. Let \(F\) and \(G\) be defined as in the proof of Theorem 1.1. Since \(E_4(a; f^n f') = E_4(a; g^n g')\) and \(E_2(a; f^n f') = E_2(a; g^n g')\) it follows that \(E_4(1; F') = E_4(1; G')\) and \(E_2(1; F') = E_2(1; G')\). If possible let us suppose that

\[\frac{T(r, F') + T(r, G')}{2} \leq 2\{N_2(r, 0; F') + N_2(r, 0; G') + N_2(r, \infty; F') + N_2(r, \infty; G')\} + S(r, F') + S(r, G').\]

\[\text{(3.16)}\]
Then by Lemmas 2.16 and 2.22 we get from (3.16)
\[ T(r, F) + T(r, G) \]
\[ \leq T(r, F') + T(r, G') + N(r, 0; f) - N(r, 0; f') \]
\[ + N(r, 0; g) - N(r, 0; g') + S(r, f) + S(r, g) \]
\[ \leq 2N_2(r, 0; F') + 2N_2(r, 0; G') + 2N_2(r, \infty; F') \]
\[ + 2N_2(r, \infty; G') + N(r, 0; f) - N(r, 0; f') \]
\[ + N(r, 0; g) - N(r, 0; g') + S(r, f) + S(r, g) \]
\[ \leq 2\{2\overline{N}(r, 0; f) + N(r, 0; f') + 2\overline{N}(r, \infty; f) \} \]
\[ + 2\{2\overline{N}(r, 0; g) + N(r, 0; g') + 2\overline{N}(r, \infty; g) \} \]
\[ + N(r, 0; f) - N(r, 0; f') + N(r, 0; g) \]
\[ - N(r, 0; g') + S(r, f) + S(r, g) \]
\[ \leq 4\overline{N}(r, 0; f) + 2N(r, 0; f) + 5\overline{N}(r, \infty; f) \]
\[ + 4\overline{N}(r, 0; g) + 2N(r, 0; g) + 5\overline{N}(r, \infty; g) + S(r) \]
\[ \leq 6T(r, f) + 5\overline{N}(r, \infty; f) + 6T(r, g) + 5\overline{N}(r, \infty; g) + S(r). \]

So by Lemma 2.15 we get
\[ (n - 5)T(r, f) + (n - 5)T(r, g) \leq 5\overline{N}(r, \infty; f) + 5\overline{N}(r, \infty; g) + S(r) \]
\[ \leq 5T(r, f) + 5T(r, g) + S(r). \]

That is
\[ (n - 10)T(r, f) + (n - 10)T(r, g) \leq S(r) \]
which is a contradiction.

Therefore the inequality (3.16) does not hold. So from Lemma 2.13 we see that either $F' \equiv G'$ or $F'G' \equiv 1$. Again proceeding in the same manner as in the proof of Theorem 1.1 we obtain the conclusion of Theorem 1.4. This proves the theorem. $\square$

References


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