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Abstract. In this paper, we study duality in the absolute Schur algebras that were first introduced in [1] and extended in [5]. This is done in a way analogous to the classical Schatten's Theorem on the Banach space $\mathcal{B}(l_2)$ of bounded linear operators on l_2 involving the duality relation among the class of compact operators $\mathcal{K}$, the trace class $\mathcal{C}_1$ and $\mathcal{B}(l_2)$. We also study the reflexivity in such the algebras.

1. Introduction and preliminaries

Let Λ and Σ be sequence spaces in $\{c_0\} \cup \{l_p : 1 \leq p < \infty\}$. For any infinite matrix A with entries from the complex field $\mathbb{C}$, we define the non-negative extended real number $\|A\|_{\Lambda,\Sigma}$ to be the norm of the matrix transformation defined by A if it belongs to $B(\Lambda, \Sigma)$ (the Banach space of all bounded linear transformations from Λ to Σ), and to be ∞ otherwise. Let $\mathcal{B}$ be a Banach algebra with identity e; and let $\mathcal{M}(\mathcal{B})$ be the linear space of all infinite matrices with entries from $\mathcal{B}$. For any matrix $A = \left[a_{jk}\right] \in \mathcal{M}(\mathcal{B})$ and $1 \leq r < \infty$, the *absolute Schur* rth-power of A is the scalar matrix $A^{[r]} := [\|a_{jk}\|^r]$. For any two matrices $A = \left[a_{jk}\right]$ and $B = \left[b_{jk}\right]$ in $\mathcal{M}(\mathcal{B})$, the *Schur product* of A and B is the matrix $A \bullet B := \left[a_{jk} b_{jk}\right]$, where the multiplication of the entries is the multiplication of elements in $\mathcal{B}$.

In [5], J. Rakbud and P. Chaisuriya proved that the set

$$\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}) := \left\{ A \in \mathcal{M}(\mathcal{B}) : \left\| A^{[r]} \right\|_{\Lambda,\Sigma} < \infty \right\}$$

is a Banach algebra under the Schur-product multiplication and the norm $\|A\|_{\Lambda,\Sigma,r} := \left\|A^{[r]}\right\|^{1/r}_{\Lambda,\Sigma}$. For each $r \geq 1$, $\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})$ is called an absolute Schur r-algebra. The following preliminary results have been stated and proved in [5].

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Lemma 1.1. *Let* $A = [a_{jk}]$, $B = [b_{jk}]$ *be scalar matrices. If* $|a_{jk}| \le b_{jk}$ *for all j,k, then* $\|A\|_{\Lambda,\Sigma} \le \left\|A^{[1]}\right\|_{\Lambda,\Sigma} \le \|B\|_{\Lambda,\Sigma}$.

Theorem 1.2 (Hölder-type inequality)**.** *Let* $A,B \in \mathcal{M}(\mathcal{B})$. *Then*

$$\left\|(A \bullet B)^{[1]}\right\|_{\Lambda,\Sigma} \le \left\|A^{[r]}\right\|_{\Lambda,\Sigma}^{1/r} \left\|B^{[r^*]}\right\|_{\Lambda,\Sigma}^{1/r^*}$$

for $1 < r < \infty$ *and* $\frac{1}{r} + \frac{1}{r^*} = 1$.

Lemma 1.3. *For any* $A = [a_{jk}] \in \mathcal{M}(\mathbb{C})$, $|a_{jk}| \le \|A\|_{\Lambda,\Sigma}$ *for all j,k.*

The following proposition is an extension of Proposition 2.8 in [5].

Proposition 1.4. (1) *For* $1 \le r' < r < \infty$, $\mathcal{S}^{r'}_{\Lambda,\Sigma}(\mathcal{B}) \subseteq \mathcal{S}^{r}_{\Lambda,\Sigma}(\mathcal{B})$ *and* $\|A\|_{\Lambda,\Sigma,r} \le \|A\|_{\Lambda,\Sigma,r'}$ *for all* $A \in \mathcal{S}^{r'}_{\Lambda,\Sigma}(\mathcal{B})$.

(2) *If* $(\Lambda,\Sigma) \ne (l_1,c_0)$, *then* $\mathcal{S}^{r'}_{\Lambda,\Sigma}(\mathcal{B}) \subsetneq \mathcal{S}^{r}_{\Lambda,\Sigma}(\mathcal{B})$ *for all* $1 \le r' < r < \infty$.

(3) *The normed spaces* $\left(\mathcal{S}^{1}_{l_1,c_0}(\mathcal{B}), \|\cdot\|_{l_1,c_0,1}\right)$ *and* $\left(\mathcal{S}^{r}_{l_1,c_0}(\mathcal{B}), \|\cdot\|_{l_1,c_0,r}\right)$ *coincide for all* $r \ge 1$, *and for any* $A = [a_{jk}] \in \mathcal{S}^{1}_{l_1,c_0}(\mathcal{B})$, $\|A\|_{l_1,c_0,1} = \sup\limits_{j,k} \|a_{jk}\|$.

Proof. Let $A = [a_{jk}]$ be a non-zero matrix in $\mathcal{S}^{r'}_{\Lambda,\Sigma}(\mathcal{B})$. From Lemma 1.3, we have that $\|a_{jk}\| \le \|A\|_{\Lambda,\Sigma,r'}$ for all (j,k). Hence $\frac{\|a_{jk}\|}{\|A\|_{\Lambda,\Sigma,r'}} \le 1$ for all (j,k). So for each (j,k), we get that $\left(\frac{\|a_{jk}\|}{\|A\|_{\Lambda,\Sigma,r'}}\right)^r \le \left(\frac{\|a_{jk}\|}{\|A\|_{\Lambda,\Sigma,r'}}\right)^{r'}$, that is $\|a_{jk}\|^r \le \|A\|^{r-r'}_{\Lambda,\Sigma,r'} \|a_{jk}\|^{r'}$. Thus by Lemma 1.1, we obtain that

$$\left\|A^{[r]}\right\|_{\Lambda,\Sigma} \le \left\|\|A\|^{r-r'}_{\Lambda,\Sigma,r'}\left(A^{[r']}\right)\right\|_{\Lambda,\Sigma} = \|A\|^{r-r'}_{\Lambda,\Sigma,r'}\left\|A^{[r']}\right\|_{\Lambda,\Sigma}.$$

This implies that $\|A\|_{\Lambda,\Sigma,r} \le \|A\|_{\Lambda,\Sigma,r'}$, so $A \in \mathcal{S}^{r}_{\Lambda,\Sigma}(\mathcal{B})$. It follows that $\mathcal{S}^{r'}_{\Lambda,\Sigma}(\mathcal{B}) \subseteq \mathcal{S}^{r}_{\Lambda,\Sigma}(\mathcal{B})$. Next, we will show that $\mathcal{S}^{1}_{l_1,c_0}(\mathcal{B}) = \mathcal{S}^{r}_{l_1,c_0}(\mathcal{B})$ for all $r > 1$.We have from the above argument that $\mathcal{S}^{1}_{l_1,c_0}(\mathcal{B}) \subseteq \mathcal{S}^{r}_{l_1,c_0}(\mathcal{B})$. To see that $\mathcal{S}^{r}_{l_1,c_0}(\mathcal{B}) \subseteq \mathcal{S}^{1}_{l_1,c_0}(\mathcal{B})$, let $A = [a_{jk}] \in \mathcal{S}^{r}_{l_1,c_0}(\mathcal{B})$ and let $x = \{\xi_k\}_{k=1}^{\infty} \in l_1$. For each j, we have by Hölder's inequality that

$$\begin{aligned}
\sum_{k=1}^{\infty} \|a_{jk}\| \, |\xi_k| &= \sum_{k=1}^{\infty} \|a_{jk}\| \, |\xi_k|^{1/r} |\xi_k|^{1/r^*} \\
&\le \left[\sum_{k=1}^{\infty} \|a_{jk}\|^r \, |\xi_k|\right]^{1/r} \left[\sum_{k=1}^{\infty} |\xi_k|\right]^{1/r^*} \\
&= \left[\sum_{k=1}^{\infty} \|a_{jk}\|^r \, |\xi_k|\right]^{1/r} \|x\|_{l_1}^{1/r^*},
\end{aligned}$$

where $\frac{1}{r}+\frac{1}{r^*}=1$. This implies that the sequence $\left\{\sum_{k=1}^{\infty}\|a_{jk}\|\,\xi_k\right\}_{j=1}^{\infty}$ belongs to c_0. If $\|x\|\le 1$, we see that

$$\sup_j\left|\sum_{k=1}^{\infty}\|a_{jk}\|\,\xi_k\right|\le\|A\|_{l_1,c_0,r}\,.$$

It follows that $\|A\|_{l_1,c_0,1}\le\|A\|_{l_1,c_0,r}$, so $A\in\mathcal{S}^1_{l_1,c_0}(\mathcal{B})$. Therefore, $\mathcal{S}^1_{l_1,c_0}(\mathcal{B})=\mathcal{S}^r_{l_1,c_0}(\mathcal{B})$ and $\|\cdot\|_{l_1,c_0,r}=\|\cdot\|_{l_1,c_0,1}$. Let $A=[a_{jk}]\in\mathcal{S}^1_{l_1,c_0}(\mathcal{B})$. We will show that $\|A\|_{l_1,c_0,1}=\sup_{j,k}\|a_{jk}\|$. By Lemma 1.3, we have that $\|A\|_{l_1,c_0,1}\ge\sup_{j,k}\|a_{jk}\|$. For any $x=\{\xi_k\}_{k=1}^{\infty}\in l_1$ with $\|x\|\le 1$, we get that

$$\sup_j\left|\sum_{k=1}^{\infty}\|a_{jk}\|\,\xi_k\right|\le\sup_{j,k}\|a_{jk}\|\left[\sum_{k=1}^{\infty}|\xi_k|\right]\le\sup_{j,k}\|a_{jk}\|\,.$$

This implies that $\|A\|_{l_1,c_0,1}\le\sup_{j,k}\|a_{jk}\|$. Hence $\|A\|_{l_1,c_0,1}=\sup_{j,k}\|a_{jk}\|$.

For the case where $(\Lambda,\Sigma)\ne(l_1,c_0)$, the following examples show that the inclusions are proper. Let $p\ge 1$ and $1\le r'<r$.

(1) $\mathcal{S}^{r'}_{\Lambda,l_p}(\mathcal{B})\ne\mathcal{S}^{r}_{\Lambda,l_p}(\mathcal{B})$. The matrix A with the first column the sequence $\left\{\left(\frac{1}{k}\right)^{1/(pr')}e\right\}$ and all other columns 0, is in $\mathcal{S}^{r}_{\Lambda,l_p}(\mathcal{B})$ but not in $\mathcal{S}^{r'}_{\Lambda,l_p}(\mathcal{B})$.

(2) $\mathcal{S}^{r'}_{c_0,c_0}(\mathcal{B})\ne\mathcal{S}^{r}_{c_0,c_0}(\mathcal{B})$. The matrix A with the first row the sequence $\left\{\left(\frac{1}{k}\right)^{1/r'}e\right\}$ and all other rows 0, is in $\mathcal{S}^{r}_{c_0,c_0}(\mathcal{B})$ but not in $\mathcal{S}^{r'}_{c_0,c_0}(\mathcal{B})$.

(3) $\mathcal{S}^{r'}_{l_p,c_0}(\mathcal{B})\ne\mathcal{S}^{r}_{l_p,c_0}(\mathcal{B})$ for $p\ne 1$. The matrix A with the first row the sequence $\left\{\left(\frac{1}{k+1}\right)^{1/(qr')}e\right\}$, where $\frac{1}{p}+\frac{1}{q}=1$, and all other rows 0, is in $\mathcal{S}^{r}_{l_p,c_0}(\mathcal{B})$ but not in $\mathcal{S}^{r'}_{l_p,c_0}(\mathcal{B})$.

The proof is complete. □

2. Duality of absolute Schur algebras

From the results in [1], L. Livshits, S.-C. Ong and S.-W. Wang studied in [3] duality in the absolute Schur algebras $\mathcal{S}^r_{l_2,l_2}(\mathbb{C})$ by a way analogous to the classical Schatten Theorem on $\mathcal{B}(l_2)$. In this section, we extend the results in [3] to our more general setting.

Let $\mathcal{AS}$ be the linear space of all infinite matrices $A=[a_{jk}]$ over the complex field $\mathbb{C}$ such that $\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|a_{jk}|<\infty$. Since this is the space $l_1(\mathbb{N}\times\mathbb{N})$ it is a

Banach space under the norm $\|[a_{jk}]\|_{\mathcal{AS}} = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|a_{jk}|$. For $1 \le r < \infty$, we let

$$\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right) = \left\{[\varphi_{jk}] : \varphi_{jk} \in \mathcal{B}^*, [\varphi_{jk}(a_{jk})] \in \mathcal{AS}\ \forall\ [a_{jk}] \in \mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})\right\}.$$

For each $\Phi = [\varphi_{jk}] \in \mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right)$, we define a map $\widehat{\Phi} : \mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}) \to \mathcal{AS}$ by

$$\widehat{\Phi}(A) = [\varphi_{jk}(a_{jk})] \quad \text{for all } A = [a_{jk}] \in \mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}).$$

Proposition 2.1. *For any $\Phi \in \mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right)$, $\widehat{\Phi}$ is a bounded linear operator.*

Proof. The linearity of $\widehat{\Phi}$ is obvious. To show $\widehat{\Phi}$ is bounded, suppose that $A_n = \left[a^{(n)}_{jk}\right] \longrightarrow A = [a_{jk}]$ in $\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})$ and $\left[\varphi_{jk}\left(a^{(n)}_{jk}\right)\right] = \widehat{\Phi}(A_n) \longrightarrow B = [b_{jk}]$ in $\mathcal{AS}$. By Lemma 1.3, we have for any (j,k) that

$$\left\|a^{(n)}_{jk} - a_{jk}\right\| \le \|A_n - A\|_{\Lambda,\Sigma,r} \quad \text{for all } n.$$

So $a^{(n)}_{jk} \longrightarrow a_{jk}$ as $n \longrightarrow \infty$ for all (j,k). From this, we get for all (j,k) by the continuity of φ_{jk} that $\varphi_{jk}\left(a^{(n)}_{jk}\right) \longrightarrow \varphi_{jk}(a_{jk})$ as $n \longrightarrow \infty$. Since $\widehat{\Phi}(A_n) \longrightarrow B$ as $n \longrightarrow \infty$ and for each (j,k),

$$\left|\varphi_{jk}\left(a^{(n)}_{jk}\right) - b_{jk}\right| \le \left\|\widehat{\Phi}(A_n) - B\right\|_{\mathcal{AS}} \quad \text{for all } n,$$

$\varphi_{jk}\left(a^{(n)}_{jk}\right) \longrightarrow b_{jk}$ as $n \longrightarrow \infty$. Hence $\widehat{\Phi}(A) = B$. Since both $\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})$ and $\mathcal{AS}$ are Banach spaces by the Closed Graph Theorem $\widehat{\Phi}$ is bounded. □

For each $b \in \mathcal{B}$ and $(j,k) \in \mathbb{N}\times\mathbb{N}$, let $A((j,k);b)$ be the matrix whose (j,k) entry is b and all other entries 0. For each positive integer n and $A \in \mathcal{M}(\mathcal{B})$, let $A_{n\lrcorner}$ be the matrix which agrees with A on the upper left $n\times n$ block and is 0 on all other entries and let $A_{n\ulcorner} = A - A_{n\lrcorner}$. For each $z \in \mathbb{C}$, we let $\operatorname{sgn}(z) = \frac{\bar{z}}{|z|}$ if $z \neq 0$ and $\operatorname{sgn}(z) = 1$ if $z = 0$.

Proposition 2.2. *The linear space $\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right)$ equipped with the norm defined by $\|\Phi\|_{\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right)} := \left\|\widehat{\Phi}\right\|$ is a Banach space.*

Proof. First, we will show that $\|\cdot\|_{\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right)}$ is a norm on $\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right)$. Clearly, for any Φ and Φ_0 in $\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right)$ and any scalar α, $(\widehat{\Phi + \Phi_0}) = \widehat{\Phi} + \widehat{\Phi}_0$ and $\widehat{(\alpha\Phi)} = \alpha\widehat{\Phi}$. Hence $\|\Phi + \Phi_0\|_{\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right)} \le \|\Phi\|_{\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right)} + \|\Phi_0\|_{\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right)}$ and $\|\alpha\Phi\|_{\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right)} = |\alpha|\,\|\Phi\|_{\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right)}$. Suppose that $\Phi = [\varphi_{jk}] \in \mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right)$ and that $\|\Phi\|_{\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right)} = 0$. Then $\left\|\widehat{\Phi}(A)\right\|_{\mathcal{AS}} = 0$ for all $A \in \mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})$. Hence, for each $(j,k) \in \mathbb{N}\times\mathbb{N}$, we get that $|\varphi_{jk}(b)| = \left\|\widehat{\Phi}\left(A((j,k);b)\right)\right\|_{\mathcal{AS}} = 0$ for all $b \in \mathcal{B}$, so $\Phi = [\varphi_{jk}]$ is the zero matrix. Thus $\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right)$ equipped with the norm $\|\cdot\|_{\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right)}$ is a normed space. To show that it is a Banach space, let $\left\{\Phi_n = \left[\varphi^{(n)}_{jk}\right]\right\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right)$. Since for each (j,k) and $b \in \mathcal{B}$ with $\|b\| \leq 1$, $\|A((j,k);b)\|_{\Lambda,\Sigma,r} \leq 1$, for any fixed (j,k), we have for arbitrary $b \in \mathcal{B}$ with $\|b\| \leq 1$ that

$$\begin{aligned}
\left|\left(\varphi^{(n)}_{jk} - \varphi^{(m)}_{jk}\right)(b)\right| &= \left\|\widehat{\Phi_n - \Phi_m}\left(A((j,k);b)\right)\right\|_{\mathcal{AS}} \\
&\leq \left\|\widehat{\Phi_n - \Phi_m}\right\| \\
&= \|\Phi_n - \Phi_m\|_{\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right)} \quad \text{for all } n, m.
\end{aligned}$$

This gives $\left\|\varphi^{(n)}_{jk} - \varphi^{(m)}_{jk}\right\| \leq \|\Phi_n - \Phi_m\|_{\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right)}$ for all n, m. This implies that $\left\{\varphi^{(n)}_{jk}\right\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{B}^*$ for all (j,k). Thus, by the completeness of $\mathcal{B}^*$, we get for each (j,k) that there is φ_{jk} in $\mathcal{B}^*$ such that $\varphi^{(n)}_{jk} \longrightarrow \varphi_{jk}$ as $n \longrightarrow \infty$. Put $\Phi = [\varphi_{jk}]$. We will show that $\Phi \in \mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right)$ and $\Phi_n \longrightarrow \Phi$ as $n \longrightarrow \infty$. To see that $\Phi \in \mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right)$, let $A = [a_{jk}] \in \mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})$. Since $\{\Phi_n\}_{n=1}^{\infty}$ is a Cauchy sequence, there exists a positive integer M such that $\left\|\widehat{\Phi_n}(A)\right\|_{\mathcal{AS}} \leq M$ for all n. So, for any positive integers J and K,

$$\sum_{j=1}^{J}\sum_{k=1}^{K}|\varphi_{jk}(a_{jk})| \leq \sum_{j=1}^{J}\sum_{k=1}^{K}\left\|\varphi^{(n)}_{jk} - \varphi_{jk}\right\| \|a_{jk}\| + M \quad \text{for all } n.$$

Hence, by taking the limit as $n \longrightarrow \infty$, we get for all $J, K \geq 1$ that

$$\sum_{j=1}^{J}\sum_{k=1}^{K}|\varphi_{jk}(a_{jk})| \leq M.$$

Since J and K are arbitrary, we have that

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|\varphi_{jk}(a_{jk})| \leq M.$$

Therefore $\Phi \in \mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right)$. Now, for the convergence, we reason as follows. Let $\epsilon > 0$ be given. Since $\{\Phi_n\}_{n=1}^{\infty}$ is a Cauchy sequence, there exists a positive integer N such that

$$\|\Phi_n - \Phi_m\|_{\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right)} < \tfrac{\epsilon}{2} \quad \text{for all } n, m \geq N.$$

Let $A = [a_{jk}] \in \mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})$ with $\|A\|_{\Lambda,\Sigma,r} \leq 1$. Then we get for each pair of positive integers J and K that

$$\sum_{j=1}^{J}\sum_{k=1}^{K}\left|\varphi^{(n)}_{jk}(a_{jk}) - \varphi^{(m)}_{jk}(a_{jk})\right| < \frac{\epsilon}{2} \text{ for all } n, m \geq N.$$

By taking the limit as $m \longrightarrow \infty$, we have for each $n \geq N$ that

$$\sum_{j=1}^{J}\sum_{k=1}^{K}\left|\varphi^{(n)}_{jk}(a_{jk}) - \varphi_{jk}(a_{jk})\right| \leq \frac{\epsilon}{2} \text{ for all } J, K \geq 1.$$

This implies that

$$\left\|\widehat{\Phi_n - \Phi_m}(A)\right\|_{\mathcal{AS}} = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left|\varphi^{(n)}_{jk}(a_{jk}) - \varphi_{jk}(a_{jk})\right| \leq \frac{\epsilon}{2} \quad \text{for all } n \geq N.$$

It follows that $\|\Phi_n - \Phi\|_{\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right)} \leq \frac{\epsilon}{2} < \epsilon$ for all $n \geq N$, that is $\Phi_n \longrightarrow \Phi$ as $n \longrightarrow \infty$. The proof is complete. □

For $\Phi = [\varphi_{jk}] \in \mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right)$, we define a map $\widetilde{\Phi} : \mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}) \to \mathbb{C}$ by

$$\widetilde{\Phi}(A) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\varphi_{jk}(a_{jk}) \quad \text{for all } A = [a_{jk}] \in \mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}).$$

Since the series on the right-hand side is absolutely convergent,

$$\left|\widetilde{\Phi}(A)\right| \leq \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|\varphi_{jk}(a_{jk})| = \left\|\widehat{\Phi}(A)\right\|_{\mathcal{AS}} \leq \|\Phi\|_{\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right)}$$

for all $A = [a_{jk}] \in \mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})$ with $\|A\|_{\Lambda,\Sigma,r} \leq 1$. It follows that $\widetilde{\Phi} \in \left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})\right)^*$ and $\left\|\widetilde{\Phi}\right\| \leq \|\Phi\|_{\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right)}$. Let

$$\widetilde{\mathcal{M}}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right) := \left\{\widetilde{\Phi} : \Phi \in \mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right)\right\}.$$

Proposition 2.3. *For any* $\Phi \in \mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right)$, $\left\|\widetilde{\Phi}\right\| = \|\Phi\|_{\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right)}$.

Proof. Let $\Phi = [\varphi_{jk}] \in \mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right)$ and let $A = [a_{jk}] \in \mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})$ with $\|A\|_{\Lambda,\Sigma,r} \leq 1$. Put $C = [\text{sgn}(\varphi_{jk}(a_{jk}))a_{jk}]$. Then $\|C\|_{\Lambda,\Sigma,r} = \|A\|_{\Lambda,\Sigma,r} \leq 1$ and

$$\begin{aligned}
\left\|\widehat{\Phi}(A)\right\|_{\mathcal{AS}} &= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|\varphi_{jk}(a_{jk})| \\
&= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}(\text{sgn}(\varphi_{jk}(a_{jk})))\varphi_{jk}(a_{jk}) \\
&= \widetilde{\Phi}(C) \leq \left\|\widetilde{\Phi}\right\|.
\end{aligned}$$

Hence $\|\Phi\|_{\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right)} \leq \left\|\widetilde{\Phi}\right\|$. □

Corollary 2.4. *$\widetilde{\mathcal{M}}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right)$ is a closed subspace of $\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})\right)^*$.*

Proof. Since $\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right)$ is a Banach space, it follows immediately, from the above proposition, that $\widetilde{\mathcal{M}}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right)$ is a complete subspace of $\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})\right)^*$, so it is closed. □

Let $\mathcal{M}_0$ be the linear space of all infinite matrices over $\mathcal{B}$ having finitely many nonzero entries. For any $1 \leq r < \infty$, let $\mathcal{K}^r_{\Lambda,\Sigma}(\mathcal{B})$ be the closure of $\mathcal{M}_0$ in $\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})$.

For any $\Psi \in \left(\mathcal{K}^r_{\Lambda,\Sigma}(\mathcal{B})\right)^*$, we define for each $(j,k) \in \mathbb{N}\times\mathbb{N}$ a map φ_{jk} on $\mathcal{B}$ as follows

$$\varphi_{jk}(b) = \Psi\left(A((j,k);b)\right) \quad \text{for all } b \in \mathcal{B}.$$

It is easy to see that $\varphi_{jk} \in \mathcal{B}^*$ for all (j,k). Let $\Phi_\Psi = [\varphi_{jk}]$.

Proposition 2.5. *For any $\Psi \in \left(\mathcal{K}^r_{\Lambda,\Sigma}(\mathcal{B})\right)^*$,*

$$\Phi_\Psi \in \mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right), \|\Phi_\Psi\|_{\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right)} \leq \|\Psi\| \;\; \textit{and} \;\; \Psi = \widetilde{\Phi_\Psi}\mid_{\mathcal{K}^r_{\Lambda,\Sigma}(\mathcal{B})}.$$

Proof. We will first show that $\Phi_\Psi \in \mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right)$. Let $A = [a_{jk}] \in \mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})$. Put $C = [\operatorname{sgn}(\varphi_{jk}(a_{jk}))a_{jk}]$. Then $\|C\|_{\Lambda,\Sigma,r} = \|A\|_{\Lambda,\Sigma,r}$. For each (j,k), it is easy to see that $\Psi\left(A((j,k);\operatorname{sgn}(\varphi_{jk}(a_{jk}))a_{jk})\right) = |\varphi_{jk}(a_{jk})|$. So, for each positive integer n,

$$\begin{aligned}
\sum_{j=1}^{n}\sum_{k=1}^{n}|\varphi_{jk}(a_{jk})| &= \sum_{j=1}^{n}\sum_{k=1}^{n}\Psi\left(A((j,k);\operatorname{sgn}(\varphi_{jk}(a_{jk}))a_{jk})\right) \\
&= \Psi\left(C_{n\lrcorner}\right) \\
&\leq \|\Psi\|\,\|C_{n\lrcorner}\|_{\Lambda,\Sigma,r} \\
&\leq \|\Psi\|\,\|C\|_{\Lambda,\Sigma,r} \qquad \text{(by Lemma 1.1)} \\
&= \|\Psi\|\,\|A\|_{\Lambda,\Sigma,r}.
\end{aligned}$$

Thus

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|\varphi_{jk}(a_{jk})| \leq \|\Psi\|\,\|A\|_{\Lambda,\Sigma,r} < \infty.$$

So $\Phi_\Psi \in \mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right)$, and we also get that $\|\Phi_\Psi\|_{\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right)} \leq \|\Psi\|$. If $A = [a_{jk}] \in \mathcal{M}_0$, then there exists a positive integer n such that $A = A_{n\lrcorner}$. Thus

$$\Psi(A) = \sum_{j=1}^{n}\sum_{k=1}^{n}\Psi\left(A(j,k);a_{jk}\right) = \sum_{j=1}^{n}\sum_{k=1}^{n}\varphi_{jk}(a_{jk}) = \widetilde{\Phi_\Psi}(A).$$

Since Ψ and $\widetilde{\Phi_\Psi}|_{\mathcal{K}^r_{\Lambda,\Sigma}(\mathcal{B})}$ are continuous, by the definition of $\mathcal{K}^r_{\Lambda,\Sigma}(\mathcal{B})$, we obtain that $\Psi = \widetilde{\Phi_\Psi}\,|_{\mathcal{K}^r_{\Lambda,\Sigma}(\mathcal{B})}$. □

Theorem 2.6. $\left(\mathcal{K}^r_{\Lambda,\Sigma}(\mathcal{B})\right)^*$ *is isometrically isomorphic to* $\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right)$.

Proof. We will show that the map $\Gamma : \Psi \mapsto \Phi_\Psi$ is an isometric isomorphism between $\left(\mathcal{K}^r_{\Lambda,\Sigma}(\mathcal{B})\right)^*$ and $\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right)$. Clearly, Γ is linear. For any $\Phi \in \mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right)$, we have that $\Gamma\left(\widetilde{\Phi}|_{\mathcal{K}^r_{\Lambda,\Sigma}(\mathcal{B})}\right) = \Phi$. Hence Γ is surjective. By Proposition 2.3 and Proposition 2.5, we get that

$$\|\Phi_\Psi\|_{\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right)} \le \|\Psi\| = \left\|\widetilde{\Phi_\Psi}|_{\mathcal{K}^r_{\Lambda,\Sigma}(\mathcal{B})}\right\| \le \left\|\widetilde{\Phi_\Psi}\right\| = \|\Phi_\Psi\|_{\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right)}.$$

Thus $\|\Psi\| = \|\Phi_\Psi\|_{\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right)}$. Therefore, Γ is an isometric isomorphism between $\left(\mathcal{K}^r_{\Lambda,\Sigma}(\mathcal{B})\right)^*$ and $\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right)$. □

Lemma 2.7. *For any* $A, B \in \mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})$, *if* $\|A\|_{\Lambda,\Sigma,r} \le R$ *and* $\|B\|_{\Lambda,\Sigma,r} \le R$ *for some* $R > 0$, *then*

$$\left\|\left(A^{[r]} - B^{[r]}\right)^{[1]}\right\|_{\Lambda,\Sigma} \le 2rR^{r-1}\,\|A-B\|_{\Lambda,\Sigma,r}.$$

Proof. For any $x, y \ge 0$ and $r \ge 1$, we have that

$$|x^r - y^r| \le r|x-y|(x^{r-1} + y^{r-1}).$$

Suppose that $A = [a_{jk}]$ and $B = [b_{jk}]$. Then by the above fact, we have for each (j,k) that

$$\begin{aligned} |\,\|a_{jk}\|^r - \|b_{jk}\|^r\,| &\le r|\,\|a_{jk}\| - \|b_{jk}\|\,|\left(\|a_{jk}\|^{r-1} + \|b_{jk}\|^{r-1}\right) \\ &\le r\,\|a_{jk} - b_{jk}\|\left(\|a_{jk}\|^{r-1} + \|b_{jk}\|^{r-1}\right). \end{aligned}$$

If $r = 1$, the inequality clearly holds by Lemma 1.1. We now assume that $r > 1$. Let r^* be the exponent conjugate to r. Then by Lemma 1.1 and the Hölder-type inequality, we get that

$$\begin{aligned} \left\|\left(A^{[r]} - B^{[r]}\right)^{[1]}\right\|_{\Lambda,\Sigma} &\le r\left\|(A-B)^{[1]} \bullet \left(A^{[r-1]} + B^{[r-1]}\right)\right\|_{\Lambda,\Sigma} \\ &\le r\,\|A-B\|_{\Lambda,\Sigma,r}\left\|A^{[r-1]} + B^{[r-1]}\right\|_{\Lambda,\Sigma,r^*} \\ &\le r\,\|A-B\|_{\Lambda,\Sigma,r}\left(\left\|A^{[r-1]}\right\|_{\Lambda,\Sigma,r^*} + \left\|B^{[r-1]}\right\|_{\Lambda,\Sigma,r^*}\right) \\ &= r\,\|A-B\|_{\Lambda,\Sigma,r}\left(\left\|A^{[r]}\right\|_{\Lambda,\Sigma}^{1/r^*} + \left\|B^{[r]}\right\|_{\Lambda,\Sigma}^{1/r^*}\right) \\ &= r\,\|A-B\|_{\Lambda,\Sigma,r}\left(\|A\|_{\Lambda,\Sigma,r}^{r/r^*} + \|B\|_{\Lambda,\Sigma,r}^{r/r^*}\right) \\ &= r\,\|A-B\|_{\Lambda,\Sigma,r}\left(\|A\|_{\Lambda,\Sigma,r}^{r-1} + \|B\|_{\Lambda,\Sigma,r}^{r-1}\right) \\ &\le 2rR^{r-1}\,\|A-B\|_{\Lambda,\Sigma,r}. \end{aligned}$$

The proof is complete. □

Proposition 2.8. *For any $r \geq 1$, the map $A \mapsto A^{[r]}$ from $\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})$ to $\mathcal{S}^1_{\Lambda,\Sigma}(\mathbb{C})$ is continuous.*

Proof. Suppose that $A_n \longrightarrow A$ in $\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})$. Then there exists a positive integer M such that $\|A\|_{\Lambda,\Sigma,r} \leq M$ and $\|A_n\|_{\Lambda,\Sigma,r} \leq M$ for all n. So, by the previous lemma, we have that

$$\left\| A_n^{[r]} - A^{[r]} \right\|_{\Lambda,\Sigma,1} \leq 2rM^{r-1} \|A_n - A\|_{\Lambda,\Sigma,r} \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Hence the map $A \mapsto A^{[r]}$ is continuous. □

Let $\mathcal{C}^r_{\Lambda,\Sigma}(\mathcal{B})$ be the set of matrices $A \in \mathcal{M}(\mathcal{B})$ such that the linear transformation (from Λ to Σ) defined by A is compact.

Corollary 2.9. *For any $r \geq 1$, $\mathcal{K}^r_{\Lambda,\Sigma}(\mathcal{B}) \subseteq \mathcal{C}^r_{\Lambda,\Sigma}(\mathcal{B})$.*

Proof. If $A \in \mathcal{K}^r_{\Lambda,\Sigma}(\mathcal{B})$, then there exists a sequence $\{A_n\}_{n=1}^{\infty}$ in $\mathcal{M}_0$ such that $A_n \longrightarrow A$ in $\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})$. Hence, by the above proposition and Lemma 1.1, we get that

$$\left\| A_n^{[r]} - A^{[r]} \right\|_{\Lambda,\Sigma} \leq \left\| A_n^{[r]} - A^{[r]} \right\|_{\Lambda,\Sigma,1} \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Since $A_n \in \mathcal{C}^r_{\Lambda,\Sigma}(\mathcal{B})$ for all n, $A \in \mathcal{C}^r_{\Lambda,\Sigma}(\mathcal{B})$. □

Corollary 2.10. *If $\Lambda \subseteq \Sigma$, then $\mathcal{K}^r_{\Lambda,\Sigma}(\mathcal{B}) \subsetneq \mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})$.*

Proof. If $\Lambda \subseteq \Sigma$, the matrix A with the entries in the main diagonal are the identity e of $\mathcal{B}$ and all other entries 0, is in $\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})$ but not in $\mathcal{C}^r_{\Lambda,\Sigma}(\mathcal{B})$. Hence, by Corollary 2.9, $A \notin \mathcal{K}^r_{\Lambda,\Sigma}(\mathcal{B})$. □

Theorem 2.11. (1) *If $\mathcal{M}_0$ is dense in $\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})$, then*

$$\widetilde{\mathcal{M}}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right) = \left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})\right)^*.$$

If $\mathcal{K}^r_{\Lambda,\Sigma}(\mathcal{B}) \subsetneq \mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})$, we have that the annihilator $\left(\mathcal{K}^r_{\Lambda,\Sigma}(\mathcal{B})\right)^{\perp}$ of $\mathcal{K}^r_{\Lambda,\Sigma}(\mathcal{B})$ is a non-trivial closed subspace of $\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})\right)^*$ and $\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})\right)^*$, can be expressed as the non-trivial direct sum $\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})\right)^* = \widetilde{\mathcal{M}}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right) \oplus \left(\mathcal{K}^r_{\Lambda,\Sigma}(\mathcal{B})\right)^{\perp}$.

(2) *Suppose that $\mathcal{K}^r_{\Lambda,\Sigma}(\mathcal{B}) \subsetneq \mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})$ and $\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})$ satisfies the following property: for every $A \in \mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})$, $\|A\|_{\Lambda,\Sigma,r} = \max\{\|A_{n\lrcorner}\|_{\Lambda,\Sigma,r}, \|A_{n\ulcorner}\|_{\Lambda,\Sigma,r}\}$ for all $n \in \mathbb{N}$. Then, for any $\Psi \in \left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})\right)^*$, the decomposition $\Psi = \lambda + \phi$, where $\lambda \in \widetilde{\mathcal{M}}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right)$ and $\phi \in \left(\mathcal{K}^r_{\Lambda,\Sigma}(\mathcal{B})\right)^{\perp}$, satisfies $\|\Psi\| = \|\lambda\| + \|\phi\|$.*

Proof. (1) For any $\Psi \in \left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})\right)^*$, let $\Omega_\Psi = \Psi - \widetilde{\Phi_\Psi}$. Then $\Psi = \widetilde{\Phi_\Psi} + \Omega_\Psi$, and by Proposition 2.5, we have that $\Omega_\Psi \in \left(\mathcal{K}^r_{\Lambda,\Sigma}(\mathcal{B})\right)^{\perp}$. Hence $\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})\right)^* = \widetilde{\mathcal{M}}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right) + \left(\mathcal{K}^r_{\Lambda,\Sigma}(\mathcal{B})\right)^{\perp}$. If $\mathcal{M}_0$ is dense in $\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})$, then $\left(\mathcal{K}^r_{\Lambda,\Sigma}(\mathcal{B})\right)^{\perp} = \left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})\right)^{\perp} = \{0\}$. So $\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})\right)^* = \widetilde{\mathcal{M}}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right)$. Suppose that $\mathcal{K}^r_{\Lambda,\Sigma}(\mathcal{B})$

$\subsetneq \mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})$. Then, by the Hahn-Banach Extension Theorem, $\left(\mathcal{K}^r_{\Lambda,\Sigma}(\mathcal{B})\right)^{\perp}$ is a non-trivial closed subspace of $\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})\right)^*$. Assume that $\Phi \in \mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right)$ and $\widetilde{\Phi} \in \left(\mathcal{K}^r_{\Lambda,\Sigma}(\mathcal{B})\right)^{\perp}$. For any $A \in \mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})$, we have that $\lim_{n\to\infty} \widetilde{\Phi}(A_{n\lrcorner}) = \widetilde{\Phi}(A)$. Hence, by the assumption, we get that $\widetilde{\Phi}(A) = 0$ for all $A \in \mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})$. By Corollary 2.4, we have that $\widetilde{\mathcal{M}}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right)$ is a closed subspace of $\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})\right)^*$. Therefore $\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})\right)^* = \widetilde{\mathcal{M}}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right) \oplus \left(\mathcal{K}^r_{\Lambda,\Sigma}(\mathcal{B})\right)^{\perp}$.

(2) From the proof of (1), $\lambda = \widetilde{\Phi_\Psi}$ and $\phi = \Omega_\Psi$. Suppose that $\|\Psi\| < \left\|\widetilde{\Phi_\Psi}\right\| + \|\Omega_\Psi\|$. Then there exists an $A \in \mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})$ such that $\|A\|_{\Lambda,\Sigma,r} \le 1$ and $\|\Psi\| < \left|\widetilde{\Phi_\Psi}(A)\right| + \|\Omega_\Psi\|$. From this, we get that there is positive integer n_0 such that $\|\Psi\| < \left|\Psi\left(A_{n_0\lrcorner}\right)\right| + \|\Omega_\Psi\|$. Put $C = \operatorname{sgn}\left(\Psi\left(A_{n_0\lrcorner}\right)\right) A_{n_0\lrcorner}$. Then $\|C\|_{\Lambda,\Sigma,r} = \left\|A_{n_0\lrcorner}\right\|_{\Lambda,\Sigma,r} \le 1$ and $\Psi(C) = \left|\Psi\left(A_{n_0\lrcorner}\right)\right|$. Choose $B \in \mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})$ so that $\|B\|_{\Lambda,\Sigma,r} \le 1$ and $\|\Psi\| < \Psi(C) + |\Omega_\Psi(B)|$. Then there exists a positive integer $n_1 > n_0$ such that $\|\Psi\| < \Psi(C) + \left|\Psi\left(B_{n_1\ulcorner}\right)\right|$. Let $D = \operatorname{sgn}\left(\Psi\left(B_{n_1\ulcorner}\right)\right) B_{n_1\ulcorner}$. Then $\|D\|_{\Lambda,\Sigma,r} = \left\|B_{n_1\ulcorner}\right\|_{\Lambda,\Sigma,r} \le 1$ and $\Psi(D) = \left|\Psi\left(B_{n_1\ulcorner}\right)\right|$. It follows that $\|\Psi\| < \Psi(C + D)$. By the assumption, we have that $\|C + D\|_{\Lambda,\Sigma,r} = \max\left\{\|C\|_{\Lambda,\Sigma,r}, \|D\|_{\Lambda,\Sigma,r}\right\} \le 1$. So we get a contradiction, therefore $\|\Psi\| = \left\|\widetilde{\Phi_\Psi}\right\| + \|\Omega_\Psi\|$. □

Example 2.12. If (Λ, Σ) is either (l_2, l_2) or (l_1, c_0), by Corollary 2.10, we obtain that $\mathcal{K}^r_{\Lambda,\Sigma}(\mathcal{B}) \subsetneq \mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})$. From Proposition 1.4(3), we have that $\mathcal{S}^r_{l_1,c_0}(\mathcal{B})$ satisfies the property given in (2) of the above theorem. For $\mathcal{S}^r_{l_2,l_2}(\mathcal{B})$, that property is inherited from $B(l_2, l_2)$.

3. Preduality

In this section, we investigate the preduality of $\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})$.

For $\varphi \in \mathcal{B}^*$ and $\Phi \in \mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right)$, let $A((j,k);\varphi)$ and $\Phi_{n\lrcorner}$ be the matrices having the same meaning as the corresponding ones defined over $\mathcal{B}$.

Theorem 3.1. *$\mathcal{S}^1_{l_1,c_0}(\mathcal{B})$ can not be the dual space of a normed space.*

Proof. Let $A = [a_{jk}] \in \mathcal{S}^1_{l_1,c_0}(\mathcal{B})$ with $\|A\|_{l_1,c_0,1} = 1$. It is easy to see that $x = \{a_{j1}\}_{j=1}^\infty$ belongs to the closed unit ball of the Banach space $c_0(\mathcal{B})$ of all sequences in $\mathcal{B}$ converging to 0. It is well-known that the closed unit ball of $c_0(\mathcal{B})$ has no extreme points. Hence there exists $0 < \alpha < 1$, and $y = \{y_j\}_{j=1}^\infty$ and $z = \{z_j\}_{j=1}^\infty$ in the closed unit ball of $c_0(\mathcal{B})$ such that $x \ne y$, $x \ne z$ and $x = \alpha y + (1-\alpha)z$. Let B and C be matrices obtained by replacing in the first column of the matrix A with the sequences y and z respectively. Then $A \ne B$, $A \ne C$ and $A = \alpha B + (1-\alpha)C$, and by Proposition 1.4(3), we see that

$\|B\|_{l_1,c_0,1} \le 1$ and $\|C\|_{l_1,c_0,1} \le 1$. So A is not an extreme point of the closed unit ball of $\mathcal{S}^1_{l_1,c_0}(\mathcal{B})$. Thus the closed unit ball of $\mathcal{S}^1_{l_1,c_0}(\mathcal{B})$ has no extreme points. If $\mathcal{S}^1_{l_1,c_0}(\mathcal{B})$ was isometrically isomorphic to the dual space of a normed space, by Alaoglu's Theorem and Krine Milman's Theorem, the closed unit ball of $\mathcal{S}^1_{l_1,c_0}(\mathcal{B})$ would have to contain at least one extreme point. This is a contradiction. □

The following lemma is inherited from $B(\Lambda, \Sigma)$.

Lemma 3.2. (1) *If $A \in \mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})$, then $\|A_{n\lrcorner}\|_{\Lambda,\Sigma,r} \nearrow \|A\|_{\Lambda,\Sigma,r}$.*

(2) *If the set $\left\{\|A_{n\lrcorner}\|_{\Lambda,l_p,r} : n = 1, 2, 3, \dots\right\}$ is bounded, then $A \in \mathcal{S}^r_{\Lambda,l_p}(\mathcal{B})$.*

Remark 3.3. The assertion (2) is not generally true for the case $\Sigma = c_0$, for example, the matrix A whose the entries in the first column are e and all other entries 0 does not belong to $\mathcal{S}^r_{\Lambda,c_0}(\mathcal{B})$, but $\|A_{n\lrcorner}\|_{\Lambda,c_0,r} = 1$ for all n.

Let $\overline{\mathcal{M}_0}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right)$ be the closure, in $\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right)$, of the set of matrices over $\mathcal{B}^*$ having finitely many nonzero entries.

Proposition 3.4. $\Phi \in \overline{\mathcal{M}_0}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right)$ *if and only if*

$$\|\Phi_{n\lrcorner} - \Phi\|_{\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right)} \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Proof. Suppose that $\Phi \in \overline{\mathcal{M}_0}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right)$. Let $\epsilon > 0$. Then there exists a matrix Φ' over $\mathcal{B}^*$ having finitely many nonzero entries such that

$$\|\Phi' - \Phi\|_{\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right)} < \frac{\epsilon}{2}.$$

Let N be a positive integer such that $\Phi' = \Phi'_{N\lrcorner}$. Then, for $n \ge N$, $\Phi' - \Phi_{n\lrcorner} = (\Phi' - \Phi)_{n\lrcorner}$. Thus if $n \ge N$, we get that

$$\begin{aligned}
&\|\Phi_{n\lrcorner} - \Phi\|_{\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right)} \\
\le\ &\|\Phi' - \Phi_{n\lrcorner}\|_{\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right)} + \|\Phi' - \Phi\|_{\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right)} \\
=\ &\|(\Phi' - \Phi)_{n\lrcorner}\|_{\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right)} + \|\Phi' - \Phi\|_{\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right)} \\
\le\ &2\,\|\Phi' - \Phi\|_{\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right)} < \epsilon.
\end{aligned}$$

The converse is obvious. The proof is complete. □

For $A \in \mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})$, we define a linear map $\lambda_A : \overline{\mathcal{M}_0}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right) \longrightarrow \mathbb{C}$ by

$$\lambda_A(\Phi) = \widetilde{\Phi}(A) \quad \text{for all } \Phi \in \overline{\mathcal{M}_0}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right).$$

It is clear that $\lambda_A \in \overline{\mathcal{M}_0}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right)^*$ and $\|\lambda_A\| \le \|A\|_{\Lambda,\Sigma,r}$.

Proposition 3.5. *For any $A \in \mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})$, $\|A\|_{\Lambda,\Sigma,r} = \|\lambda_A\|$.*

Proof. Let $A \in \mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})$. Then by the Hahn-Banach Extension Theorem and Theorem 2.11, we get for every n that

$$\begin{aligned}
\|A_{n\lrcorner}\|_{\Lambda,\Sigma,r} &= \sup\{|\Psi(A_{n\lrcorner})| : \Psi \in \left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})\right)^*, \|\Psi\| \le 1\} \\
&= \sup\left\{\left|\widetilde{\Phi_\Psi}(A_{n\lrcorner})\right| : \Psi \in \left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})\right)^*, \|\Psi\| \le 1\right\} \\
&= \sup\left\{\left|\widetilde{\Phi}(A_{n\lrcorner})\right| : \Phi \in \mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right), \|\Phi\|_{\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right)} \le 1\right\} \\
&= \sup\left\{\left|\widetilde{\Phi_{n\lrcorner}}(A)\right| : \Phi \in \mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right), \|\Phi\|_{\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right)} \le 1\right\} \\
&= \sup\left\{|\lambda_A(\Phi_{n\lrcorner})| : \Phi \in \mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right), \|\Phi\|_{\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right)} \le 1\right\} \\
&\le \|\lambda_A\|.
\end{aligned}$$

Hence, by the above lemma, we get that $\|A\|_{\Lambda,\Sigma,r} = \|\lambda_A\|$. □

Proposition 3.6. *If the map $A \mapsto \lambda_A$ from $\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})$ to $\left(\overline{\mathcal{M}_0}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right)\right)^*$ is onto, then $\mathcal{B}$ is reflexive.*

Proof. Let $g \in \mathcal{B}^{**}$. Put $\Psi_g(\Phi) = g(\varphi_{11})$ for all $\Phi = [\varphi_{jk}] \in \overline{\mathcal{M}_0}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right)$. Then $\Psi_g \in \left(\overline{\mathcal{M}_0}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right)\right)^*$. So, by the assumption, we get that there exists $A = [a_{jk}] \in \mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})$ such that $\lambda_A = \Psi_g$. Hence $\varphi(a_{11}) = \lambda_A(A((1,1);\varphi)) = \Psi_g(A((1,1);\varphi)) = g(\varphi)$ for all $\varphi \in \mathcal{B}^*$. It follows that $\mathcal{B}$ is reflexive. □

From the above proposition, we have that the reflexivity of $\mathcal{B}$ is a necessary condition for the map $A \mapsto \lambda_A$ to be onto. For the case of $\Sigma = l_p$, we also have it is sufficient.

Theorem 3.7. *The map $A \mapsto \lambda_A$ is an isometric isomorphism from $\mathcal{S}^r_{\Lambda,l_p}(\mathcal{B})$ onto $\left(\overline{\mathcal{M}_0}\left(\mathcal{S}^r_{\Lambda,l_p}(\mathcal{B}), \mathcal{AS}\right)\right)^*$ if and only if $\mathcal{B}$ is reflexive.*

Proof. Suppose that $\mathcal{B}$ is reflexive. We will show that the map $A \mapsto \lambda_A$ is onto. Let $\mathcal{S}_0 = \left\{\lambda_A : A \in \mathcal{S}^r_{\Lambda,l_p}(\mathcal{B}), \|A\|_{\Lambda,l_p,r} \le 1\right\}$ and $\mathcal{S}$ be the closed unit ball of $\left(\overline{\mathcal{M}_0}\left(\mathcal{S}^r_{\Lambda,l_p}(\mathcal{B}), \mathcal{AS}\right)\right)^*$. It is clear that $\mathcal{S}_0 \subseteq \mathcal{S}$. Let σ be the weak* topology on $\left(\overline{\mathcal{M}_0}\left(\mathcal{S}^r_{\Lambda,l_p}(\mathcal{B}), \mathcal{AS}\right)\right)^*$. Now, we want to show that $\mathcal{S}_0$ is closed in $\left(\left(\overline{\mathcal{M}_0}\left(\mathcal{S}^r_{\Lambda,l_p}(\mathcal{B}), \mathcal{AS}\right)\right)^*, \sigma\right)$. To see this, let $\{\lambda_{A_\alpha}\}_\alpha$, where $A_\alpha = \left[a^{(\alpha)}_{jk}\right]$, be a Cauchy net in $\left(\left(\overline{\mathcal{M}_0}\left(\mathcal{S}^r_{\Lambda,l_p}(\mathcal{B}), \mathcal{AS}\right)\right)^*, \sigma\right)$ which is contained in $\mathcal{S}_0$. Then $\{\lambda_{A_\alpha}(\Phi)\}_\alpha$ is a Cauchy net in $\mathbb{C}$ for all $\Phi \in \overline{\mathcal{M}_0}\left(\mathcal{S}^r_{\Lambda,l_p}(\mathcal{B}), \mathcal{AS}\right)$. From this, we get for each $(j,k) \in \mathbb{N} \times \mathbb{N}$ that $\left\{\varphi\left(a^{(\alpha)}_{jk}\right)\right\}_\alpha$ is a Cauchy net in $\mathbb{C}$ for all $\varphi \in \mathcal{B}^*$. This implies that $\left\{a^{(\alpha)}_{jk}\right\}_\alpha$ is a Cauchy net in $\mathcal{B}$ equipped with the weak topology, for all (j,k). It is easy to see that for each (j,k), $\left\{a^{(\alpha)}_{jk}\right\}_\alpha$ is contained in the closed unit ball of $\mathcal{B}$. Hence, by reflexivity of $\mathcal{B}$, we get for each (j,k) that there exists an a_{jk} in $\mathcal{B}$ such that $w-\lim_\alpha a_{jk}^{(\alpha)} = a_{jk}$. Put $A=[a_{jk}]$, we will show that $\|A\|_{\Lambda,l_p,r}\leq 1$ and $w^*-\lim_\alpha \lambda_{A_\alpha}=\lambda_A$. Let $\Phi=[\varphi_{jk}]\in\overline{\mathcal{M}_0}\left(\mathcal{S}^r_{\Lambda,l_p}(\mathcal{B}),\mathcal{AS}\right)$. Then $\varphi_{jk}\left(a_{jk}^{(\alpha)}\right)\longrightarrow\varphi_{jk}(a_{jk})$ for all (j,k). For each $n\in\mathbb{N}$, we have that

$$\left|\lambda_{(A_\alpha)_{n\lrcorner}}(\Phi)-\lambda_{A_{n\lrcorner}}(\Phi)\right|\leq\sum_{j=1}^{n}\sum_{k=1}^{n}\left|\varphi_{jk}\left(a_{jk}^{(\alpha)}\right)-\varphi_{jk}(a_{jk})\right|\quad\text{for all }\alpha.$$

Thus $\lambda_{(A_\alpha)_{n\lrcorner}}(\Phi)\longrightarrow\lambda_{A_{n\lrcorner}}(\Phi)$ for all n. This implies that $w^*-\lim_\alpha\lambda_{(A_\alpha)_{n\lrcorner}}=\lambda_{A_{n\lrcorner}}$ and $\|A_{n\lrcorner}\|_{\Lambda,l_p,r}=\left\|\lambda_{A_{n\lrcorner}}\right\|\leq 1$ for all n . So, by Lemma 3.2, we obtain that $\|A\|_{\Lambda,l_p,r}\leq 1$. To see that $w^*-\lim_\alpha\lambda_{A_\alpha}=\lambda_A$, let $\epsilon>0$ and $\Phi\in\overline{\mathcal{M}_0}\left(\mathcal{S}^r_{\Lambda,l_p}(\mathcal{B}),\mathcal{AS}\right)$. Then there exists γ such that

$$\left|\lambda_{A_\alpha}(\Phi)-\lambda_{A_\beta}(\Phi)\right|<\tfrac{\epsilon}{4}\quad\text{for all }\alpha,\beta\succeq\gamma.$$

Since $\|\Phi_{n\lrcorner}-\Phi\|_{\mathcal{M}\left(\mathcal{S}^r_{\Lambda,l_p}(\mathcal{B}),\mathcal{AS}\right)}\longrightarrow 0$ as $n\longrightarrow\infty$. There exists a positive integer N such that $\|\Phi_{n\lrcorner}-\Phi\|_{\mathcal{M}\left(\mathcal{S}^r_{\Lambda,l_p}(\mathcal{B}),\mathcal{AS}\right)}\leq\frac{\epsilon}{8}$ for all $n\geq\mathbb{N}$. So, for every $n\geq N$ and $\alpha,\beta\succeq\gamma$,

$$\begin{aligned}\left|\lambda_{(A_\alpha)_{n\lrcorner}}(\Phi)-\lambda_{(A_\beta)_{n\lrcorner}}(\Phi)\right| &=\left|(\lambda_{A_\alpha}-\lambda_{A_\beta})(\Phi_{n\lrcorner})\right|\\ &\leq\left|(\lambda_{A_\alpha}-\lambda_{A_\beta})(\Phi_{n\lrcorner}-\Phi)\right|+\left|(\lambda_{A_\alpha}-\lambda_{A_\beta})(\Phi)\right|\\ &\leq\left\|\lambda_{A_\alpha}-\lambda_{A_\beta}\right\|\|\Phi_{n\lrcorner}-\Phi\|_{\mathcal{M}\left(\mathcal{S}^r_{\Lambda,l_p}(\mathcal{B}),\mathcal{AS}\right)}+\frac{\epsilon}{4}\\ &\leq\|A_\alpha-A_\beta\|_{\Lambda,l_p,r}\frac{\epsilon}{8}+\frac{\epsilon}{4}<\frac{\epsilon}{2}.\end{aligned}$$

Taking limit in β we have for each $\alpha\succeq\gamma$ that

$$\left|\lambda_{(A_\alpha)_{n\lrcorner}}(\Phi)-\lambda_{A_{n\lrcorner}}(\Phi)\right|\leq\tfrac{\epsilon}{2}\quad\text{for all }n\geq N.$$

Hence, by taking the limit as $n\longrightarrow\infty$, we get that

$$|\lambda_{A_\alpha}(\Phi)-\lambda_A(\Phi)|\leq\tfrac{\epsilon}{2}<\epsilon\quad\text{for all }\alpha\succeq\gamma.$$

Therefore $w^*-\lim_\alpha\lambda_{A_\alpha}=\lambda_A$. It follows that $\mathcal{S}_0$ is a complete subset of $\left(\left(\overline{\mathcal{M}_0}\left(\mathcal{S}^r_{\Lambda,l_p}(\mathcal{B}),\mathcal{AS}\right)\right)^*,\sigma\right)$, so it is closed in $\left(\left(\overline{\mathcal{M}_0}\left(\mathcal{S}^r_{\Lambda,l_p}(\mathcal{B}),\mathcal{AS}\right)\right)^*,\sigma\right)$. If there exists $\Omega\in\mathcal{S}\setminus\mathcal{S}_0$, then by Theorem V.2.10 in [2], there exist constants c and $\epsilon>0$, and $\Phi\in\overline{\mathcal{M}_0}\left(\mathcal{S}^r_{\Lambda,l_p}(\mathcal{B}),\mathcal{AS}\right)$ such that $\mathcal{R}e\,(\lambda_A(\Phi))\leq c-\epsilon<c\leq\mathcal{R}e(\Omega(\Phi))$ for all $A\in\mathcal{S}^r_{\Lambda,l_p}(\mathcal{B})$ with $\|A\|_{\Lambda,l_p,r}\leq 1$. For $A\in\mathcal{S}^r_{\Lambda,l_p}(\mathcal{B})$, we let $\widetilde{A}:=\operatorname{sgn}(\lambda_A(\Phi))\,A$, it is obvious that $\left\|\widetilde{A}\right\|_{\Lambda,l_p,r}=\|A\|_{\Lambda,l_p,r}$. From this, we

obtain that

$$\begin{aligned}
\|\Phi\|_{\mathcal{M}\left(\mathcal{S}^r_{\Lambda,l_p}(\mathcal{B}),\mathcal{AS}\right)} &= \left\|\widetilde{\Phi}\right\| \\
&= \sup\left\{\left|\widetilde{\Phi}(A)\right| : A\in\mathcal{S}^r_{\Lambda,l_p}(\mathcal{B}), \|A\|_{\Lambda,l_p,r}\le 1\right\} \\
&= \sup\left\{|\lambda_A(\Phi)| : A\in\mathcal{S}^r_{\Lambda,l_p}(\mathcal{B}), \|A\|_{\Lambda,l_p,r}\le 1\right\} \\
&= \sup\left\{\lambda_{\widetilde{A}}(\Phi) : A\in\mathcal{S}^r_{\Lambda,l_p}(\mathcal{B}), \|A\|_{\Lambda,l_p,r}\le 1\right\} \\
&\le c-\epsilon < c \le \mathcal{R}e(\Omega(\Phi)).
\end{aligned}$$

Since $\Omega\in\mathcal{S}$, by Hahn-Banach Extension Theorem, we have that

$$\mathcal{R}e(\Omega(\Phi)) \le |\Omega(\Phi)| \le \sup_{\Psi\in\mathcal{S}}|\Psi(\Phi)| = \|\Phi\|_{\mathcal{M}\left(\mathcal{S}^r_{\Lambda,l_p}(\mathcal{B}),\mathcal{AS}\right)}.$$

So we get a contradiction, therefore $\mathcal{S}_0=\mathcal{S}$. This implies that the map $A\mapsto\lambda_A$ is onto. □

Remark 3.8. In [3], the duality of $\mathcal{S}^r_{l_2,l_2}(\mathbb{C})$ was studied. We summarize some results as follows.

Let $\mathcal{M}^r$ denote the linear space of all matrices $B\in\mathcal{M}(\mathbb{C})$ such that $A\bullet B\in\mathcal{AS}$ for all $A\in\mathcal{S}^r_{l_2,l_2}(\mathbb{C})$. For any $B\in\mathcal{M}^r$, the linear map $\Psi_B:\mathcal{S}^r_{l_2,l_2}(\mathbb{C})\to\mathcal{AS}$ defined by $B\mapsto A\bullet B$ is bounded. Define the norm $\|\cdot\|_{\mathcal{M}^r}$ on $\mathcal{M}^r$ by $\|B\|_{\mathcal{M}^r}=\|\Psi_B\|$. Let $\sigma\mathcal{M}^r=\{\sigma\circ\Psi_B : B\in\mathcal{M}^r\}$, where $\sigma([b_{jk}]) := \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}b_{jk}$ for all $[b_{jk}]\in\mathcal{AS}$.

(1) $\mathcal{K}^r_{l_2,l_2}(\mathbb{C})=\mathcal{C}^r_{l_2,l_2}(\mathbb{C})$.
(2) $\mathcal{M}^r$ equipped with the norm $\|\cdot\|_{\mathcal{M}^r}$ is a Banach space.
(3) $\left(\mathcal{K}^r_{l_2,l_2}(\mathbb{C})\right)^*$ is isometrically isomorphic to $\mathcal{M}^r$.
(4) $\left(\mathcal{K}^r_{l_2,l_2}(\mathbb{C})\right)^\perp$ is a non-trivial closed subspace of $\left(\mathcal{S}^r_{l_2,l_2}(\mathbb{C})\right)^*$, and
$$\left(\mathcal{S}^r_{l_2,l_2}(\mathbb{C})\right)^* = \sigma\mathcal{M}^r\oplus\left(\mathcal{K}^r_{l_2,l_2}(\mathbb{C})\right)^\perp.$$
(5) For any $\varphi\in\left(\mathcal{S}^r_{l_2,l_2}(\mathbb{C})\right)^*$, the decomposition $\varphi=\psi+\lambda$, where $\psi\in\sigma\mathcal{M}^r$ and $\lambda\in\left(\mathcal{K}^r_{l_2,l_2}(\mathbb{C})\right)^\perp$, satisfies $\|\varphi\|=\|\psi\|+\|\lambda\|$.
(6) $(\mathcal{M}^r)^*$ is isometrically isomorphic to $\mathcal{S}^r_{l_2,l_2}(\mathbb{C})$.

It is easy to see that $\mathcal{M}\left(\mathcal{S}^r_{l_2,l_2}(\mathbb{C}),\mathcal{AS}\right)$ and $\mathcal{M}^r$ are isometrically isomorphic. It was also shown in [3] that $\overline{\mathcal{M}_0}\left(\mathcal{S}^r_{l_2,l_2}(\mathbb{C}),\mathcal{AS}\right)=\mathcal{M}\left(\mathcal{S}^r_{l_2,l_2}(\mathbb{C}),\mathcal{AS}\right)$. So our results generalize the results in [3].

4. Reflexivity

We now investigate the reflexivity of $\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})$ and $\mathcal{K}^r_{\Lambda,\Sigma}(\mathcal{B})$.

For $A \in \mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})$, we have that the linear functional $\widetilde{\lambda}_A$ on $\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathbb{C}), \mathcal{AS}\right)$ defined by $A \mapsto \widetilde{\Phi}(A)$ is also bounded and $\left\|\widetilde{\lambda}_A\right\| \leq \|A\|_{\Lambda,\Sigma,r}$. Obviously, $\widetilde{\lambda}_A(\Phi) = \lambda_A(\Phi)$ for all $\Phi \in \overline{\mathcal{M}_0}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathbb{C}), \mathcal{AS}\right)$.

Lemma 4.1. *For any $A \in \mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})$, $\left\|\widetilde{\lambda}_{A_{n\lrcorner}}\right\| \nearrow \left\|\widetilde{\lambda}_A\right\|$.*

Proof. Let $A = [a_{jk}] \in \mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})$ and let $\Phi = [\varphi_{jk}] \in \mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right)$ with $\|\Phi\|_{\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right)} \leq 1$. Put $\Phi' = [\operatorname{sgn}(\varphi_{jk}(a_{jk}))\varphi_{jk}]$. It is clear that $\Phi' \in \mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right)$ and $\|\Phi'\|_{\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right)} = \|\Phi\|_{\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right)} \leq 1$. For each $n \in \mathbb{N}$, we have that

$$\begin{aligned}
\left|\widetilde{\Phi}(A_{n\lrcorner})\right| &\leq \sum_{j=1}^{n}\sum_{k=1}^{n}|\varphi_{jk}(a_{jk})| \\
&\leq \sum_{j=1}^{n+1}\sum_{k=1}^{n+1}\operatorname{sgn}(\varphi_{jk}(a_{jk}))\varphi_{jk}(a_{jk}) \quad \left(= \widetilde{\Phi'}(A_{n+1\lrcorner})\right) \\
&\leq \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\operatorname{sgn}(\varphi_{jk}(a_{jk}))\varphi_{jk}(a_{jk}) \quad \left(= \widetilde{\Phi'}(A)\right).
\end{aligned}$$

It follows that $\left\|\widetilde{\lambda}_{A_{n\lrcorner}}\right\| \leq \left\|\widetilde{\lambda}_{A_{n+1\lrcorner}}\right\| \leq \left\|\widetilde{\lambda}_A\right\|$ for all n. Hence $\left\|\widetilde{\lambda}_{A_{n\lrcorner}}\right\| \nearrow \sup_n \left\|\widetilde{\lambda}_{A_{n\lrcorner}}\right\|$ and $\sup_n \left\|\widetilde{\lambda}_{A_{n\lrcorner}}\right\| \leq \left\|\widetilde{\lambda}_A\right\|$. Let $\epsilon > 0$. Then there exists $\Phi \in \mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right)$ such that $\|\Phi\|_{\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right)} \leq 1$ and $\left\|\widetilde{\lambda}_A\right\| < \left|\widetilde{\Phi}(A)\right| + \epsilon$. From this, we get that there exists a positive integer n_0 such that

$$\begin{aligned}
\left\|\widetilde{\lambda}_A\right\| &< \left|\widetilde{\Phi}(A_{n_0\lrcorner})\right| + \epsilon \\
&\leq \left\|\widetilde{\lambda}_{A_{n_0\lrcorner}}\right\| + \epsilon \\
&\leq \sup_n \left\|\widetilde{\lambda}_{A_{n\lrcorner}}\right\| + \epsilon.
\end{aligned}$$

Since ϵ is arbitrary, $\left\|\widetilde{\lambda}_A\right\| \leq \sup_n \left\|\widetilde{\lambda}_{A_{n\lrcorner}}\right\|$. The proof is complete. □

Proposition 4.2. *For any $A \in \mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})$, $\|A\|_{\Lambda,\Sigma,r} = \left\|\widetilde{\lambda}_A\right\|$.*

Proof. By the above lemma and Lemma 3.2(1), it is sufficient to show that $\|A\|_{\Lambda,\Sigma,r} = \left\|\widetilde{\lambda}_A\right\|$ for all $A \in \mathcal{M}_0$. To this end, let $A \in \mathcal{M}_0$. Then by Hahn-Banach Extension Theorem and Theorem 2.11, we get that

$$\begin{aligned}
\|A\|_{\Lambda,\Sigma,r} &= \sup\{|\Psi(A)| : \Psi \in \left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})\right)^*, \|\Psi\| \le 1\} \\
&= \sup\left\{\left|\widetilde{\Phi_\Psi}\right| : \Psi \in \left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})\right)^*, \|\Psi\| \le 1\right\} \\
&= \sup\left\{\left|\widetilde{\Phi}(A)\right| : \Phi \in \mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right), \|\Phi\|_{\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}),\mathcal{AS}\right)} \le 1\right\} \\
&= \left\|\widetilde{\lambda}_A\right\|.
\end{aligned}$$

The proof is complete. □

From the above proposition, we have that the map R sending λ_A to $\widetilde{\lambda}_A$ is an isometric isomorphism from $\{\lambda_A : A \in \mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})\}$ into $\left(\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right)\right)^*$.

Proposition 4.3. *We denote the isometric isomorphisms* $A \mapsto \lambda_A$ *from* $\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})$ *into* $\left(\overline{\mathcal{M}_0}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right)\right)^*$ *and* $\Psi \mapsto \Phi_\Psi$ *from* $\left(\mathcal{K}^r_{\Lambda,\Sigma}(\mathcal{B})\right)^*$ *onto* $\mathcal{M}\left(\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B}), \mathcal{AS}\right)$ *by* T *and* W *respectively. Let* $Q : \mathcal{K}^r_{\Lambda,\Sigma}(\mathcal{B}) \to \left(\mathcal{K}^r_{\Lambda,\Sigma}(\mathcal{B})\right)^{**}$ *be the natural map. Then* $W^*RT(A) = Q(A)$ *for all* $A \in \mathcal{K}^r_{\Lambda,\Sigma}(\mathcal{B})$*, where* W^* *is the adjoint of* W*.*

Proof. Let $A = [a_{jk}] \in \mathcal{M}_0$. Then there is a positive integer n such that $A_{n\lrcorner} = A$. It is easy to see that $W^*RT(A) = \widetilde{\lambda}_A W$. Let $\Psi \in \left(\mathcal{K}^r_{\Lambda,\Sigma}(\mathcal{B})\right)^*$. Then $\widetilde{\lambda}_A W(\Psi) = \widetilde{\lambda}_A(W(\Psi)) = \widetilde{\lambda}_A\left(\Phi_\Psi\right) = \widetilde{\Phi_\Psi}(A) = \sum_{j=1}^{n}\sum_{k=1}^{n}\Psi(A((j,k);a_{jk})) = \Psi(A) = Q(A)(\Psi)$. So $W^*RT(A) = Q(A)$ for all $A \in \mathcal{M}_0$. Since $\mathcal{M}_0$ is dense in $\mathcal{K}^r_{\Lambda,\Sigma}(\mathcal{B})$, $W^*RT = Q$ on $\mathcal{K}^r_{\Lambda,\Sigma}(\mathcal{B})$. □

Corollary 4.4. (1) *If either* $\mathcal{K}^r_{\Lambda,\Sigma}(\mathcal{B}) \subsetneq \mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})$ *or* $\overline{\mathcal{M}_0}\left(\mathcal{S}^r_{\Lambda,l_p}(\mathcal{B}), \mathcal{AS}\right) \subsetneq \mathcal{M}\left(\mathcal{S}^r_{\Lambda,l_p}(\mathcal{B}), \mathcal{AS}\right)$*, then both* $\mathcal{K}^r_{\Lambda,\Sigma}(\mathcal{B})$ *and* $\mathcal{S}^r_{\Lambda,\Sigma}(\mathcal{B})$ *are not reflexive.*

(2) $\mathcal{S}^r_{\Lambda,l_p}(\mathcal{B})$ *is reflexive if and only if* $\mathcal{B}$ *is reflexive,* $\overline{\mathcal{M}_0}\left(\mathcal{S}^r_{\Lambda,l_p}(\mathcal{B}), \mathcal{AS}\right) = \mathcal{M}\left(\mathcal{S}^r_{\Lambda,l_p}(\mathcal{B}), \mathcal{AS}\right)$ *and* $\mathcal{K}^r_{\Lambda,l_p}(\mathcal{B}) = \mathcal{S}^r_{\Lambda,l_p}(\mathcal{B})$.

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