ANALYTIC FUNCTIONS SHARING THREE VALUES DM IN ONE ANGULAR DOMAIN

TING-BIN CAO AND HONG-XUN YI

Reprinted from the
Journal of the Korean Mathematical Society
Vol. 45, No. 6, November 2008

©2008 The Korean Mathematical Society
ANALYTIC FUNCTIONS SHARING THREE VALUES DM IN ONE ANGULAR DOMAIN

TING-BIN CAO AND HONG-XUN YI

Abstract. We investigate the uniqueness of transcendental analytic functions that share three values DM in one angular domain instead of the whole complex plane.

1. Introduction and main results

In this paper, a transcendental meromorphic (analytic) function is meromorphic (analytic) in the whole complex plane $\mathbb{C}$ and not rational. We assume that the reader is familiar with the Nevanlinna’s theory of meromorphic functions and the standard notations such as $m(r, f)$, $T(r, f)$. For references, see [2]. We say that two meromorphic functions $f$ and $g$ share the value $a$ ($a \in \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$) in $X \subseteq \mathbb{C}$ provided that in $X$, we have $f(z) = a$ if and only if $g(z) = a$. We will state whether a shared value is by DM (differential multiplicities), or by IM (ignoring multiplicities). R. Nevanlinna (see [4]) proved that if two meromorphic functions $f$ and $g$ have five distinct IM shared values in $X = \mathbb{C}$, then $f(z) \equiv g(z)$. After his very work, the uniqueness of meromorphic functions with shared values in the whole complex plane attracted many investigations (for references, see [7]). E. Mues consider DM shared values and proved the following theorem.

**Theorem A** ([3]). There are no two distinct nonconstant analytic functions $f$ and $g$ that share three distinct values DM in $X = \mathbb{C}$.

In [8], Zheng took into account of the uniqueness dealing with five shared values in some angular domains of $\mathbb{C}$. It is an interesting topic to investigate the uniqueness with shared values in the remaining part of the complex plane removing an unbounded closed set. In [9], Zheng continued to investigate this subject and obtain some results on uniqueness of meromorphic functions with five or four shared values in one angular domain.

Received March 30, 2006; Revised March 11, 2008.
2000 Mathematics Subject Classification. Primary 30D35.
Key words and phrases. meromorphic function, uniqueness theorem, shared values, angular domain.

This work was supported by the NNSF of China (No. 10771121) and the Research Foundation of Doctor Points of China (No. 20060422049).
We may ask: What can be said to an analogous result as Theorem A in one angular domain?

Nevanlinna’s theory on angular domain (see [1]) will play a key role in this paper. Let $f$ be a meromorphic function on the angular domain $\Omega(\alpha, \beta) = \{ z : \alpha \leq \arg z \leq \beta \}$, where $0 < \beta - \alpha \leq 2\pi$. Following Nevanlinna define

\[
A_{\alpha, \beta}(r, f) = \frac{\omega}{\pi} \int_1^r \left( \frac{1}{t^2} - \frac{t^\omega}{r^{2\omega}} \right) \{ \log^+ |f(te^{i\alpha})| + \log^+ |f(te^{i\beta})| \} \frac{dt}{t},
\]

\[
B_{\alpha, \beta}(r, f) = \frac{2\omega}{\pi r^\omega} \int_0^\beta \log^+ |f(re^{i\theta})| \sin \omega(\theta - \alpha) \, d\theta,
\]

\[
C_{\alpha, \beta}(r, f) = 2 \sum_{1 < |b_n| < r} \left( \frac{1}{|b_n|^\omega} - \frac{|b_n|\omega}{r^{2\omega}} \right) \sin \omega(\theta_n - \alpha),
\]

\[
D_{\alpha, \beta}(r, f) = A_{\alpha, \beta}(r, f) + B_{\alpha, \beta}(r, f),
\]

where $\omega = \frac{\beta - \alpha}{\pi}$, $1 \leq r < \infty$ and $b_n = |b_n|e^{i\theta_n}$ are the poles of $f$ on $\Omega(\alpha, \beta)$ appearing according to their multiplicities. If we only consider the distinct poles of $f$, we denote the corresponding angular counting function by $C_{\alpha, \beta}(r, f)$.

Nevanlinna’s angular characteristic is defined as follows:

\[
S_{\alpha, \beta}(r, f) = A_{\alpha, \beta}(r, f) + B_{\alpha, \beta}(r, f) + C_{\alpha, \beta}(r, f).
\]

Throughout, we denote respectively by $R(r, \ast)$ and $R_{\alpha, \beta}(r, \ast)$ quantities satisfying

\[
R(r, \ast) = O \left( \log \left( rT(r, \ast) \right) \right), \quad r \notin E,
\]

\[
R_{\alpha, \beta}(r, \ast) = O \left( \log \left( rS_{\alpha, \beta}(r, \ast) \right) \right), \quad r \notin E,
\]

where $E$ denotes a set of positive real numbers with finite linear measure. The notation $E$ is not necessarily the same for its every time occurrence in the context.

Now we show our main result which can answer the above question.

**Theorem 1.** There are no two distinct transcendental analytic functions $f$ and $g$ that share three distinct values $a_1, a_2, a_3$ DM in one angular domain $X = \{ z : \alpha < \arg z < \beta \}$ with $0 \leq \alpha < \beta \leq 2\pi$, provided that

\[
\lim_{r \to \infty} \frac{S_{\alpha, \beta}(r, f)}{\log \left( rT(r, f) \right)} = \infty, \quad (r \notin E).
\]

2. Lemmas

**Lemma A** ([5], [6], [10]). Suppose that $g(z)$ is a non-constant meromorphic function in the plane and that $\Omega(\alpha, \beta)$ is an angular domain, where $0 < \beta - \alpha \leq 2\pi$. Then

(i) ([1], Chap. 1) for any complex number $a \neq \infty$,

\[
S_{\alpha, \beta} \left( r, \frac{1}{g - a} \right) = S_{\alpha, \beta}(r, g) + \varepsilon(r, a),
\]
where \( \varepsilon(r,a) = O(1) \) \( (r \to \infty) \);

(ii) ([1], p. 138) for any \( 1 \leq r < R \),

\[
A_{\alpha,\beta} \left( r, \frac{g'}{g} \right) \leq K \left\{ \left( \frac{R}{r} \right)^{\omega} \int_{1}^{R} \frac{\log^{+} T(t, g)}{t^{1+\omega}} \frac{dt}{t} + \log^{+} \frac{r}{R-r} + \log \frac{R}{r} + 1 \right\}
\]

and

\[
B_{\alpha,\beta} \left( r, \frac{g'}{g} \right) \leq \frac{4\omega}{r} \log \left( r, \frac{g'}{g} \right),
\]

where \( \omega = \frac{\pi}{\beta-\alpha} \) and \( K \) is a positive constant not depending on \( r \) and \( R \).

Remark. It follows from Lemma A(ii) that

\[
D_{\alpha,\beta}(r, g) = A_{\alpha,\beta}(r, g) + B_{\alpha,\beta}(r, g) = R_{\alpha,\beta}(r, g) = R(r, g).
\]

**Lemma B** ([9]). Suppose that \( f(z) \) is a non-constant meromorphic function in the plane and that \( \Omega(\alpha, \beta) \) is an angular domain, where \( 0 < \beta - \alpha \leq 2\pi \). Then for arbitrary \( q \) distinct \( a_j \in \mathbb{C} \) \( (1 \leq j \leq q) \), we have

\[
(q-2)S_{\alpha,\beta}(r, f) \leq \sum_{j=1}^{q} C_{\alpha,\beta} \left( r, \frac{1}{f-a_j} \right) + R_{\alpha,\beta}(r, f)
\]

\[
= \sum_{j=1}^{q} C_{\alpha,\beta} \left( r, \frac{1}{f-a_j} \right) + R(r, f),
\]

where the term \( C_{\alpha,\beta} \left( r, \frac{1}{f-a_j} \right) \) will be replaced by \( C_{\alpha,\beta}(r, f) \) when some \( a_j = \infty \).

**Lemma 1.** Suppose that \( f(z) \) is a non-constant meromorphic function in the plane and that \( \Omega(\alpha, \beta) \) is an angular domain, where \( 0 < \beta - \alpha \leq 2\pi \). Let \( P(f) = a_0 f^m + a_1 f^{m-1} + \cdots + a_m \) \( (a_0 \neq 0) \) be a polynomial in \( f \) with degree \( m \), where the coefficients \( a_j \) \( (j = 0, 1, \ldots, m) \) are constants, and let \( b_j \) \( (j = 1, 2, \ldots, q) \) \( (q > m) \) be \( q \) distinct finite complex numbers. Then

\[
D_{\alpha,\beta} \left( r, \frac{P(f)}{f-b_1}(f-b_2)\cdots(f-b_q) \right) = R(r, f).
\]

**Proof.** One can deduce that

\[
\frac{P(f)}{(f-b_1)(f-b_2)\cdots(f-b_q)} = \sum_{j=1}^{q} \frac{A_j}{f-b_j}
\]

holds, where \( A_j \) are nonzero constants. Hence we deduce by Lemma A(ii) and the lemma of logarithmic derivative of meromorphic function in the complex.
plane,

\[
\begin{align*}
D_{\alpha, \beta}(r, \frac{P(f) \cdot f'}{f(b_1)(f(b_2) \cdots (f(b_q))}} &\leq D_{\alpha, \beta}(r, \sum_{j=1}^{q} \frac{A_j \cdot f'}{f - b_j}) + \sum_{j=1}^{q} D_{\alpha, \beta}(r, A_j) + O(1) \\
&= R(r, f).
\end{align*}
\]

\[\Box\]

**Lemma 2.** Let \(f\) and \(g\) be two distinct transcendental meromorphic functions that share four distinct values \(a_1, a_2, a_3, a_4\) in \(X = \{ z : \alpha < \arg z < \beta \}\) with \(0 \leq \alpha < \beta \leq 2\pi\). Then

(i) \(S_{\alpha, \beta}(r, f) = S_{\alpha, \beta}(r, g) + R(r, f), S_{\alpha, \beta}(r, g) = S_{\alpha, \beta}(r, f) + R(r, g)\);

(ii) \(\sum_{j=1}^{4} C_{\alpha, \beta}(r, \frac{1}{f - a_j}) = 2S_{\alpha, \beta}(r, f) + R(r, f)\);

(iii) \(C_{\alpha, \beta}(r, \frac{1}{f}) = S_{\alpha, \beta}(r, f) + R(r, f), C_{\alpha, \beta}(r, \frac{1}{g}) = S_{\alpha, \beta}(r, g) + R(r, g), \) where \(b \neq a_j (j = 1, 2, 3, 4)\);

(iv) \(C^{*\alpha, \beta}_{\alpha, \beta}(r, \frac{1}{f}) = R(r, f), C^{*\alpha, \beta}_{\alpha, \beta}(r, \frac{1}{g}) = R(r, g), \) where \(C^{*\alpha, \beta}_{\alpha, \beta}(r, \frac{1}{f})\) and \(C^{*\alpha, \beta}_{\alpha, \beta}(r, \frac{1}{g})\) are respectively the counting functions of the zeros of \(f'\) that are not zeros of \(f - a_j (j = 1, 2, 3, 4)\), and the zeros of \(g'\) that are not zeros of \(g - a_j (j = 1, 2, 3, 4)\);

(v) \(\sum_{j=1}^{4} C^{**\alpha, \beta}_{\alpha, \beta}(r, f(z) = a_j = g(z)) = R(r, f), \) where \(C^{**\alpha, \beta}_{\alpha, \beta}(r, f(z) = a_j = g(z))\) is the counting function for common multiple zeros of \(f - a_j\) and \(g - a_j (j = 1, 2, 3, 4)\), counting the smaller one of the two multiplicities at each of the points.

**Proof.** From Lemma B we have

\[
2S_{\alpha, \beta}(r, f) \leq \sum_{j=1}^{4} C_{\alpha, \beta}(r, \frac{1}{f - a_j}) + R(r, f)
\leq C_{\alpha, \beta}(r, \frac{1}{f - g}) + R(r, f)
\leq S_{\alpha, \beta}(r, f) + S_{\alpha, \beta}(r, g) + R(r, f),
\]

and by interchanging \(f\) and \(g\) we obtain (i) and (ii).
Again by Lemma B and (ii) we have
\[ 3S_{\alpha,\beta}(r, f) \leq \sum_{j=1}^{4} C_{\alpha,\beta}(r, f - a_j) + \overline{C}_{\alpha,\beta}(r, f - b) + R(r, f) \]
\[ = 2S_{\alpha,\beta}(r, f) + \overline{C}_{\alpha,\beta}(r, f - b) + R(r, f), \]
i.e.,
\[ \overline{C}_{\alpha,\beta}(r, f - b) = S_{\alpha,\beta}(r, f) + R(r, f). \]

By interchanging \( f \) and \( g \), we get
\[ \overline{C}_{\alpha,\beta}(r, g - b) = S_{\alpha,\beta}(r, g) + R(r, g). \]

Thus we obtain (iii).

Without loss of generality we will assume that \( a_4 = \infty \). This is allowed because if all the shared values are finite, then we can consider
\[ F = (f - a_4)^{-1} \quad \text{and} \quad G = (g - a_4)^{-1}. \]

Set
\[ (6) \quad \Psi = \frac{f'g'(f - g)^2}{(f - a_1)(f - a_2)(f - a_3)(g - a_1)(g - a_2)(g - a_3)}. \]

It is easy to see from Lemma 1 and (6) that
\[ D_{\alpha,\beta}(r, \Psi) = R(r, f) + R(r, g). \]

If \( z_0 \in X \) is a point such that \( f(z_0) = g(z_0) = a_j \) for some \( j = 1, 2, 3, 4 \), then from (6) we see that \( \Psi \) will be analytic at \( z_0 \). Thus we can deduce that
\[ C_{\alpha,\beta}(r, \frac{1}{f}) + C^*_\alpha,\beta(r, \frac{1}{g'}) + \sum_{j=1}^{4} C^{**}_{\alpha,\beta}(r, f(z) = a_j = g(z)) \]
\[ \leq C_{\alpha,\beta}(r, \frac{1}{\Psi}) \]
\[ \leq S_{\alpha,\beta}(r, \frac{1}{\Psi}) \]
\[ = S_{\alpha,\beta}(r, \Psi) + O(1) \]
\[ = D_{\alpha,\beta}(r, \Psi) + C_{\alpha,\beta}(r, \Psi) + O(1) \]
\[ = R(r, f) + R(r, g). \]

From (i) we see that \( R(r, f) = R(r, g) \). Therefore we obtain (v) and (vi). \( \square \)
Lemma 3. Let \( f \) and \( g \) be two distinct transcendental meromorphic functions that share four distinct values \( a_1, a_2, a_3, a_4 \) in \( X = \{ z : \alpha < \arg z < \beta \} \) with \( 0 \leq \alpha < \beta \leq 2\pi \). Then

\[
D_{\alpha,\beta} \left( r, \frac{1}{f-g} \right) = R(r, f).
\]

Proof. We assume that \( a_j (j = 1, 2, 3, 4) \) are finite, then it follows from Lemma A and Lemma B that

\[
2S_{\alpha,\beta}(r, f) \leq \sum_{j=1}^{4} C_{\alpha,\beta} \left( r, \frac{1}{f-a_j} \right) + R(r, f)
\]

\[
\leq C_{\alpha,\beta} \left( r, \frac{1}{f-g} \right) + R(r, f)
\]

\[
\leq S_{\alpha,\beta} \left( r, \frac{1}{f-g} \right) + R(r, f)
\]

\[
\leq S_{\alpha,\beta}(r, f) + S_{\alpha,\beta}(r, g) + R(r, f),
\]

namely,

\[
S_{\alpha,\beta}(r, f) \leq S_{\alpha,\beta}(r, g) + R(r, f).
\]

Similarly, we have

\[
S_{\alpha,\beta}(r, g) \leq S_{\alpha,\beta}(r, f) + R(r, g).
\]

It implies from above discussion that

\[
S_{\alpha,\beta} \left( r, \frac{1}{f-g} \right) = C_{\alpha,\beta} \left( r, \frac{1}{f-g} \right) + R(r, f).
\]

Hence we have

\[
D_{\alpha,\beta} \left( r, \frac{1}{f-g} \right) = R(r, f).
\]

We now assume \( a_4 = \infty \). Let \( b \neq a_j (j = 1, 2, 3, 4) \), \( F(z) = \frac{1}{f(z)-b} \), and \( G(z) = \frac{1}{g(z)-b} \). Then \( b_j = \frac{1}{a_j-b} (j = 1, 2, 3) \) and \( b_4 = 0 \) are IM shared values of \( F(z) \) and \( G(z) \) in \( X \). From the discussion above, we have

\[
D_{\alpha,\beta} \left( r, \frac{1}{F-G} \right) = R(r, f).
\]

From Lemma 2(iii) we have

\[
C_{\alpha,\beta} \left( r, \frac{1}{f-b} \right) = S_{\alpha,\beta}(r, f) + R(r, f).
\]

This implies from Lemma A that

\[
C_{\alpha,\beta}(r, F) = S_{\alpha,\beta}(r, F) + R(r, F).
\]

Hence

\[
D_{\alpha,\beta}(r, F) = R(r, F).
\]
Similarly, we have
\begin{equation}
D_{\alpha,\beta}(r, G) = R(r, G).
\end{equation}
From the equalities (7), (8), and (9), we get
\begin{align*}
D_{\alpha,\beta}\left(r, \frac{1}{f-g}\right) &= D_{\alpha,\beta}\left(r, \frac{FG}{F-G}\right) \\
&\leq D_{\alpha,\beta}\left(r, \frac{1}{F-G}\right) + D_{\alpha,\beta}(r, F) + D_{\alpha,\beta}(r, G) \\
&= R(r, f).
\end{align*}
This completes the proof of the lemma. \qed

**Lemma 4.** Let \( f \) and \( g \) be two distinct transcendental meromorphic functions that share four distinct values \( a_1, a_2, a_3, a_4 \) IM in \( X = \{z : \alpha < \arg z < \beta\} \) with \( 0 \leq \alpha < \beta \leq 2\pi \). Then
\begin{align*}
D_{\alpha,\beta}\left(r, \frac{f}{f-g}\right) + D_{\alpha,\beta}\left(r, \frac{g}{f-g}\right) &+ D_{\alpha,\beta}\left(r, \frac{f'}{f-g}\right) + D_{\alpha,\beta}\left(r, \frac{g'}{f-g}\right) \\
&+ D_{\alpha,\beta}\left(r, \frac{fg'}{f-g}\right) = R(r, f).
\end{align*}

**Proof.** Set
\begin{align*}
F &= \frac{1}{f}, & G &= \frac{1}{g}.
\end{align*}
Then \( b_j = \frac{1}{a_j} (j = 1, 2, 3, 4) \) are IM shared values of \( F \) and \( G \) in \( X \). From Lemma 3 we have
\begin{align*}
D_{\alpha,\beta}\left(r, \frac{1}{f-g}\right) &= R(r, f) \\
D_{\alpha,\beta}\left(r, \frac{fg}{f-g}\right) &= D_{\alpha,\beta}\left(r, \frac{1}{F-G}\right) = R(r, F) = R(r, f).
\end{align*}
Since
\begin{align*}
\left(\frac{f}{f-g}\right)^2 &= \frac{f}{f-g} + \frac{fg}{(f-g)^2},
\end{align*}
we have
\begin{align*}
2D_{\alpha,\beta}\left(r, \frac{f}{f-g}\right) &\leq D_{\alpha,\beta}\left(r, \frac{f}{f-g}\right) \\
&+ D_{\alpha,\beta}\left(r, \frac{fg}{f-g}\right) + D_{\alpha,\beta}\left(r, \frac{1}{f-g}\right) + O(1) \\
&\leq D_{\alpha,\beta}\left(r, \frac{f}{f-g}\right) + R(r, f).
\end{align*}
Hence
\begin{align*}
D_{\alpha,\beta}\left(r, \frac{f}{f-g}\right) &= R(r, f).
\end{align*}
Similarly, we have
\[ D_{\alpha,\beta} \left( r, \frac{g}{f - g} \right) = R(r, f). \]
Furthermore, we have
\[ D_{\alpha,\beta} \left( r, \frac{f'}{f - g} \right) \leq D_{\alpha,\beta} \left( r, \frac{f'}{f} \right) + D_{\alpha,\beta} \left( r, \frac{f}{f - g} \right) = R(r, f), \]
\[ D_{\alpha,\beta} \left( r, \frac{g'}{f - g} \right) \leq D_{\alpha,\beta} \left( r, \frac{g'}{g} \right) + D_{\alpha,\beta} \left( r, \frac{g}{f - g} \right) = R(r, f), \]
\[ D_{\alpha,\beta} \left( r, \frac{f'g'}{f - g} \right) \leq D_{\alpha,\beta} \left( r, \frac{f'}{f} \right) + D_{\alpha,\beta} \left( r, \frac{g'}{g} \right) + D_{\alpha,\beta} \left( r, \frac{fg}{f - g} \right) = R(r, f). \]
\[ \square \]

3. Proof of Theorem 1

We assume that the conclusion of Theorem 1 is not true, namely, there exist two distinct transcendental analytic functions \( f(z) \) and \( g(z) \) that share three distinct values \( a_1, a_2, a_3 \) in \( X \), provided that
\[ \lim_{r \to \infty} \frac{S_{\alpha,\beta}(r, f)}{\log (rT(r, f))} = \infty \quad (r \notin E). \]
Without loss of generality, we assume \( a_1 = 0, a_2 = 1, a_3 = c. \) Set
\[ \Phi = \frac{(f')^2(g')^2(f - g)}{f(f - 1)(f - c)g(g - 1)(g - c)}. \]
If \( z_0 \in X \) is a zero of both \( f \) and \( g \), with multiplicities \( p \) and \( q \) respectively. Since 0 is DM shared values of \( f \) and \( g \) in \( X \), we have \( p \neq q \). Then by computation, we have
\[ \Phi(z) = O \left( (z - z_0)^t \right), \]
where \( t = p + q - 4 + \min\{p, q\} \geq 0 \). Hence zeros of \( f \) in \( X \) are not poles of \( \Phi \) in \( X \). Similarly, zeros of \( f - 1 \) or \( f - c \) in \( X \) are not poles of \( \Phi \) in \( X \). Therefore we get that \( \Phi \) is analytic in \( X \). Obviously,
\[ \Phi = \frac{f'g'}{f - g} \cdot \Psi, \]
where \( \Psi \) is the function defined in (6). From Lemma 4 we have
\[ S_{\alpha,\beta}(r, \Phi) = D_{\alpha,\beta} \left( r, \frac{f'g'}{f - g} \right) + D_{\alpha,\beta}(r, \Psi) = R(r, f). \]
We denote by \( C_{\alpha,\beta}(r) \) the counting function of zeros of \( f, f - 1, f - c \) (or \( g, g - 1, g - c \)) with multiplicities more than 2, each point counts only once. Then
\[ C_{\alpha,\beta}(r) \leq C_{\alpha,\beta} \left( r, \frac{1}{\Phi} \right) \leq S_{\alpha,\beta}(r, \Phi) + O(1) = R(r, f). \]
Since $0, 1, c$ are DM shared values of $f$ and $g$ in $X$, from Lemma 2(iv) we have

$$\overline{C}_{\alpha,\beta} \left( r, \frac{1}{f} \right) + \overline{C}_{\alpha,\beta} \left( r, \frac{1}{f-1} \right) + \overline{C}_{\alpha,\beta} \left( r, \frac{1}{f-c} \right)$$

$$= C_{\alpha,\beta} \left( r, \frac{1}{f} \right) + C_{\alpha,\beta} \left( r, \frac{1}{g} \right) + R(r, f).$$

Since $f$ and $g$ are analytic in $X$, from the above equality, Lemma A, Lemma B, Lemma 2(i) and Lemma 2(ii) we have

$$2S_{\alpha,\beta}(r, f) = \overline{C}_{\alpha,\beta} \left( r, \frac{1}{f} \right) + \overline{C}_{\alpha,\beta} \left( r, \frac{1}{f-1} \right) + \overline{C}_{\alpha,\beta} \left( r, \frac{1}{f-c} \right) + R(r, f)$$

$$= C_{\alpha,\beta} \left( r, \frac{1}{f} \right) + C_{\alpha,\beta} \left( r, \frac{1}{g} \right) + R(r, f)$$

$$= S_{\alpha,\beta} \left( r, \frac{1}{f} \right) - D_{\alpha,\beta} \left( r, \frac{1}{f} \right) + S_{\alpha,\beta} \left( r, \frac{1}{g} \right) - D_{\alpha,\beta} \left( r, \frac{1}{g} \right) + R(r, f)$$

$$= S_{\alpha,\beta}(r, f') - D_{\alpha,\beta} \left( r, \frac{1}{f} \right) + D_{\alpha,\beta}(r, g') - D_{\alpha,\beta} \left( r, \frac{1}{g} \right) + R(r, f)$$

$$\leq D_{\alpha,\beta}(r, f) - D_{\alpha,\beta} \left( r, \frac{1}{f} \right) + D_{\alpha,\beta}(r, g) - D_{\alpha,\beta} \left( r, \frac{1}{g} \right) + R(r, f)$$

$$= 2S_{\alpha,\beta}(r, f) - D_{\alpha,\beta} \left( r, \frac{1}{f} \right) - D_{\alpha,\beta} \left( r, \frac{1}{g} \right) + R(r, f),$$

namely,

$$D_{\alpha,\beta} \left( r, \frac{1}{f} \right) + D_{\alpha,\beta} \left( r, \frac{1}{g} \right) = R(r, f).$$

Hence we get from Lemma A that

$$D_{\alpha,\beta} \left( r, \frac{1}{f} \right) + D_{\alpha,\beta} \left( r, \frac{1}{f-1} \right) + D_{\alpha,\beta} \left( r, \frac{1}{f-c} \right)$$

$$\leq D_{\alpha,\beta} \left( r, \frac{1}{f} \right) + R(r, f) = R(r, f).$$

If

$$C_{\alpha,\beta} \left( r, \frac{1}{f} \right) - \overline{C}_{\alpha,\beta} \left( r, \frac{1}{f} \right) = R(r, f),$$

then we have

$$S_{\alpha,\beta}(r, f) = C_{\alpha,\beta} \left( r, \frac{1}{f} \right) + D_{\alpha,\beta} \left( r, \frac{1}{f} \right) + O(1)$$

$$= C_{\alpha,\beta} \left( r, \frac{1}{f} \right) + R(r, f)$$
\[ \begin{align*}
&= C_{\alpha,\beta}(r, \frac{1}{f}) + R(r, f) \\
&\leq \frac{1}{2} C_{\alpha,\beta}(r, \frac{1}{f}) + R(r, f) \\
&= \frac{1}{2} S_{\alpha,\beta}(r, \frac{1}{f}) + R(r, f),
\end{align*} \]

namely,
\[ S_{\alpha,\beta}(r, f) = R(r, f). \]

This contradicts the condition of Theorem 1. Hence we have
\[ C_{\alpha,\beta}(r, \frac{1}{f}) = C_{\alpha,\beta}(r, \frac{1}{f}) \neq R(r, f). \]

Similarly, we have
\[ C_{\alpha,\beta}(r, \frac{1}{f-1}) = C_{\alpha,\beta}(r, \frac{1}{f-1}) \neq R(r, f), \]
\[ C_{\alpha,\beta}(r, \frac{1}{f-c}) = C_{\alpha,\beta}(r, \frac{1}{f-c}) \neq R(r, f). \]

Set
\[ \Omega = 2 f'' f - 3 \left( \frac{f'}{f} + \frac{f'}{f-1} + \frac{f'}{f-c} \right) - 2 g'' g' \]
\[ = 2 \left( \frac{g'}{g} + \frac{g'}{g-1} + \frac{g'}{g-c} \right) + \frac{f'-2g'}{f-g}. \]

Then from Lemma A, Lemma 4 we have
\[ (12) \quad D_{\alpha,\beta}(r, \Omega) \leq R(r, f). \]

If \( z_1 \in X \) is a zero of both \( f \) and \( g \), (or both \( f-1 \) and \( g-1 \); or both \( f-c \) and \( g-c \), with multiplicities 2 and 1, respectively; and if \( z_2 \in X \) is a zero of both \( f \) and \( g \), (or both \( f-1 \) and \( g-1 \); or both \( f-c \) and \( g-c \), with multiplicities 1 and 2, respectively. Then by simple computation, we get that both \( z_1 \) and \( z_2 \) are not poles of \( \Omega(z) \) in \( X \). Hence we can deduce by Lemma 2(iv) and (10) that
\[ (13) \quad C_{\alpha,\beta}(r, \Omega) \leq R(r, f). \]

Hence combining (12) and (13), we have
\[ S_{\alpha,\beta}(r, \Omega) \leq R(r, f). \]

If \( z_1 \in X \) is a zero of both \( f \) and \( g \), with multiplicities 2 and 1, respectively. Then by computation we have
\[ \Psi(z_1) = \frac{2}{c^2} (g'(z_1))^2, \]
\[ \Omega(z_1) = - \left( 1 + \frac{1}{c} \right) g'(z_1). \]
Hence we get
\[ \frac{\Omega^2(z_1)}{2\Psi(z_1)} = (c + 1)^2. \]

If
\[ \frac{\Omega^2(z)}{2\Psi(z)} = (c + 1)^2 \neq 0, \]
then
\[ C_{\alpha,\beta} \left( r, \frac{1}{f} \right) - C_{\alpha,\beta} \left( r, \frac{1}{f} \right) - R(r, f) \]
\[ \leq C_{\alpha,\beta} \left( r, \frac{\Omega}{2\Psi} - (c + 1)^2 \right) \]
\[ \leq 2S_{\alpha,\beta}(r, \Omega) + S_{\alpha,\beta}(r, \Psi) + O(1) \]
\[ = R(r, f). \]
This contradicts to (11). Hence
\[ \frac{\Omega^2(z)}{2\Psi(z)} = (c + 1)^2 \equiv 0. \]

If \( z_3 \in X \) is a zero of both \( f - 1 \) and \( g - 1 \), with multiplicities 2 and 1, respectively. Then by computation we have
\[ \frac{\Omega^2(z_3)}{2\Psi(z_3)} = (2 - c)^2. \]
Hence
\[ (c + 1)^2 = (2 - c)^2, \]
namely, \( c = \frac{1}{2} \).

If \( z_4 \in X \) is a zero of both \( f - c \) and \( g - c \), with multiplicities 2 and 1, respectively. Then by computation we have
\[ \frac{\Omega^2(z_4)}{2\Psi(z_4)} = (2c - 1)^2. \]
Hence
\[ (c + 1)^2 = (2c - 1)^2. \]
This implies \( c = 2 \). We obtain a contradiction. Therefore, we complete the proof of Theorem 1.

Acknowledgements. The authors would like to thank the referee for making valuable suggestions and comments to improve the present paper.
References


Ting-Bin Cao
Department of Mathematics
Nanchang University
Nanchang 330031, P. R. China
E-mail address: ctb970163.com or tbcao@ncu.edu.cn

Hong-Xun Yi
Department of Mathematics
Shandong University
Jinan 250100, P. R. China
E-mail address: hxy1@sdu.edu.cn